Starshapedness of Level Sets for Solutions of Nonlinear Parabolic Equations

ELISA FRANCINI (*)

SOMMARIO. - *Data una soluzione u di un'equazione parabolica non lineare, si studia la proprietà degli insiemi di livello di $u(\cdot,t)$ di essere stellati rispetto ad un punto preso come origine. Si misura la suddetta proprietà considerando l'angolo w tra la direzione normale alla superficie di livello e la direzione radiale. Si mostra che per w vale un principio di massimo, perciò la stellarità degli insiemi di livello dei dati iniziali si conserva per tutti i tempi positivi.*

SUMMARY. - *Given a solution u of a nonlinear parabolic equation, we study the starshapedness of level sets of $u(\cdot,t)$ with respect to a point which we take as the origin. We measure the starshapedness of level sets considering the angle w between the normal direction to the level surface and the radial direction. We show that a maximum principle for w holds, hence the starshapedness of level sets of the initial data is preserved for all positive times.*

1. Introduction

Let Ω_0 and Ω_1 be two bounded, simply connected open sets in $\mathbb{R}^n$ with C^1 boundaries and $\Omega_0 \supset \overline{\Omega}_1$. We say that $\Omega = \Omega_0 \setminus \overline{\Omega}_1$ is a *starshaped ring with respect to* $x_0 \in \mathbb{R}^n$ if both Ω_0 and Ω_1 are

(*) Indirizzo dell'Autore: Istituto di Analisi Globale e Applicazioni, Via S. Marta 13a, 50139 Firenze (Italy).

starshaped with respect to x_0. In the following we will simply call *starshaped* sets and rings which are starshaped with respect to the origin. If T is a positive constant, set $\Gamma = \Omega \times (0, T]$ and $\partial_P \Gamma = \{\partial\Omega \times (0, T)\} \cup \{\overline{\Omega} \times \{0\}\}$. We denote with $x = (x_1, \ldots, x_n)$ the space variable in Ω and with t the time variable in $(0, T]$.

Consider an initial-boundary value problem of parabolic type

$$\begin{cases} -u_t + F(D^2u) + f(r, u, |Du|^2) = 0 \ \text{ in } \Gamma, \\ \qquad u = 0 \quad \text{on } \partial\Omega_0 \times (0, T], \\ \qquad u = 1 \quad \text{in } \overline{\Omega}_1 \times (0, T], \\ \qquad u = u_0 \quad \text{in } \Omega_0 \times \{0\}, \end{cases} \tag{1}$$

where $Du = (u_1, \ldots, u_n)$ and $D^2u = (u_{ij})_{i,j=1,\ldots,n}$ are the gradient and the Hessian matrix of a function u with respect to the space variable x, $u_t = \dfrac{\partial u}{\partial t}$, $r = |x|$, $u_0 \in C(\Omega_0)$ and $u_0 \equiv 1$ in Ω_1, $f \in C^1(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+)$ and F is an elliptic rotationally invariant operator defined on the space of symmetric $n \times n$ matrices. From now on we will denote $s = |Du|^2$.

Let S be a simply connected open set in $\mathbb{R}^n$, with C^1 boundary, and consider for each point $x \in \partial S$ the angle $w(x)$ between the outer normal to ∂S at x and the radial direction x. S is *starshaped* if $w(x) \le \pi/2$ for every $x \in \partial S$ and we will say that S is *properly starshaped* if $w(x) < \pi/2$ for every $x \in \partial S$. At a maximum point of w the normal direction is as far as possible from the radial direction. We will say that at such a point we have a minimum for the starshapedness.

If S is a level set of a function, the normal direction to ∂S at x coincides with the direction of the gradient at x. Given a function $u(x, t)$ defined in Γ, we denote with $w(x, t)$ the angle between $-Du(x, t)$ and the radial direction x; w is well defined if $Du(x, t) \ne 0$.

The main goal of this article is to show that, under suitable hypotheses, if u_0 has starshaped level sets $\{x \in \Omega_0 : u_0(x) \ge c\}$, then, for every positive t, the level sets $\{x \in \Omega_0 : u(x, t) \ge c\}$ are starshaped. We also prove that the starshapedness attains its minimum on $\partial_P \Gamma$.

The minimum principle for starshapedness of level sets, for elliptic equations, has been studied in [5] for nonlinear Poisson equation and in [6] for fully nonlinear elliptic equations, while starshapedness of level sets was considered in [1] for nonlinear Poisson equation and in [4] for the p-Laplace operator in the elliptic and parabolic case.

In the following section we state the main result (Theorem 1), that is a minimum principle for starshapedness of level sets for solutions of problem (1) when F is a rotationally invariant and homogeneous (in a way that will be defined later) elliptic operator.

The third section is devoted to the proof of Theorem 1. We start proving a preliminary lemma which is true for a more general class of equations. This will allow us to indicate some generalization of the minimum principle for starshapedness (Section 4). In particular, in Theorem 2, we prove the result for the p-Laplace operator.

2. The main result

Before stating the main theorem, let us point out something about rotationally invariant operators.

A rotationally invariant operator F defined on the space of symmetric $n \times n$ matrices, can be written in the form

$$F(A) = g\left(\lambda_1(A), \ldots, \lambda_n(A)\right),$$

where $\lambda_1(A) \leq \ldots \leq \lambda_n(A)$ are the eigenvalues of A. If $g \in C^1(\mathbb{R}^n)$, F is Lipschitz continuous with respect to the matrix A and

$$F' = \left(\frac{\partial F}{\partial a_{ij}}(A)\right)_{i,j=1,\ldots,n},$$

where it is defined, is a symmetric matrix with the same eigenvectors of A and eigenvalues $\dfrac{\partial g}{\partial \lambda_i}$ for $i = 1, \ldots, n$ (see, for example [2]).

$$F(D^2u) = g\left(\lambda_1\left(D^2u\right), \ldots, \lambda_n\left(D^2u\right)\right) \tag{2}$$

is a second order differential operator which is strictly elliptic at u if there exists a positive constant σ_0 such that

$$\frac{\partial g}{\partial \lambda_i}\left(\lambda_1\left(D^2u\right), \ldots, \lambda_n\left(D^2u\right)\right) > \sigma_0 \quad \text{for every} \quad i = 1, \ldots, n.$$

THEOREM 1. *Let $\Omega = \Omega_0 \backslash \overline{\Omega}_1$ be a starshaped ring and $\Gamma = \Omega \times (0, T]$. Let $u \in C^3(\Gamma) \cap C^1(\Gamma) \cap C(\overline{\Omega}_0 \times [0, T])$ a solution of problem (1), where $f(r, u, s) \in C^1(I\!R^+ \times I\!R \times I\!R^+)$ and $F(D^2u) = g(\lambda_1\left(D^2u\right), \ldots, \lambda_n\left(D^2\right.$*

u)) with $g \in C^1(\mathbb{R}^n)$ *a homogeneus function of order* k.
Suppose that F *is strictly elliptic at* u,

$$0 \le u \le 1 \quad in \quad \Gamma, \tag{3}$$

$$\frac{\partial f}{\partial u} \le 0 \quad in \quad \Gamma, \tag{4}$$

and

$$2kf + r\frac{\partial f}{\partial r} - 2s\frac{\partial f}{\partial s} \le 0 \quad in \quad \Gamma, \tag{5}$$

where f *and its derivatives are taken at* $(r, u, |Du|^2)$.
Assume that $u_0 \in C(\overline{\Omega}_0) \cap C^2(\Omega)$,

$$0 \le u_0 \le 1 \quad in \quad \Omega \quad and \quad u_0 \equiv 1 \quad in \quad \Omega_1, \tag{6}$$

$$u_0 \quad has\ starshaped\ level\ sets\ , \tag{7}$$

and

$$F(D^2 u_0) + f(r, u_0, |Du_0|^2) \ge 0 \quad in \quad \Omega. \tag{8}$$

Then, $u(\cdot, t)$ *has properly starshaped level sets and* $Du(x,t) \ne 0$ *in* Γ. *Moreover, unless* Ω_0 *and* Ω_1 *are concentric balls and* u_0 *is radial, starshapedness does not attain its minimum in* Γ, *that is the angle* $w(x,t)$ *does not attain its maximum in* Γ.

There are several differential operators of the form (2), since each invariant of the Hessian matrix is a homogeneous function of the eigenvalues.

Let us see some examples

Example 1. Theorem 1 holds for solutions of

$$\begin{cases} -u_t + \Delta u + f(r, u, |Du|^2) = 0 & \text{in } \Gamma, \\ u = 0 & \text{on } \partial\Omega_0 \times (0, T], \\ u = 1 & \text{in } \overline{\Omega}_1 \times (0, T], \\ u = u_0 & \text{in } \Omega_0 \times \{0\}, \end{cases} \tag{9}$$

where

$$\frac{\partial f}{\partial u} \le 0 \quad \text{and} \quad 2f + r\frac{\partial f}{\partial r} - 2s\frac{\partial f}{\partial s} \le 0 \quad \text{in} \quad \Gamma.$$

In this case, if $f(r, 0, 0) = 0$ for every r, assumption (3) is not necessary, since it is a consequence of (6).

EXAMPLE 2. Theorem 1 can be applied to maximal operators (see [8]) which can be written in the form

$$M_{\alpha,-\beta,\gamma}[u] = \alpha\Delta u + (1-n\alpha)\lambda_n\left(D^2u\right) - \beta|Du| - \gamma\frac{u-|u|}{2},$$

and

$$m_{\alpha,\beta,\gamma}[u] = \alpha\Delta u + (1-n\alpha)\lambda_1\left(D^2u\right) - \beta|Du| - \gamma\frac{u+|u|}{2},$$

with α, β and γ positive constants such that $\alpha \leq \dfrac{1}{n}$.

REMARK 1. If $\dfrac{\partial f}{\partial r} \equiv 0$, we can consider the starshapedness of level sets with respect to any other point of Ω_1. This gives more informations about the shape of level sets of u. If Ω_0, Ω_1 and the level sets of u_0 are starshaped with respect to each point of a set $K \subset \Omega_1$, the level sets of $u(x,t)$ will still be starshaped with respect to each point of K. In this case we can consider for each $y \in K$ the function $w_y(x,t)$ which represents the angle between $-Du(x,t)$ and the direction $x-y$. Let $M(y)$ be the maximum of w_y on $\overline{\Omega} \times \{0\}$ and call $C(y)$ the cone in $\mathbb{R}^n$ with vertex y, axis $x-y$ and angle $M(y)$ with this axis.
For a fixed point $(x,t) \in \Gamma$,

$$-Du(x,t) \in \bigcap_{y\in K} C(y).$$

REMARK 2. The minimum principle for starshapedness usually gives more informations than the starshapedness with respect to each point in Ω_1 (See remark 4 in [6]).
If Ω_0 and Ω_1 are balls with centre at the origin, u_0 is radial and u satisfies the assumptions of Theorem 1, since $w \equiv 0$ on $\partial_P\Gamma$ then $u(x,t)$ is radial for every positive t.

REMARK 3. The assumption that F is rotationally invariant cannot be removed. Suppose that Ω_0 and Ω_1 are balls in $\mathbb{R}^2$ with centre at the origin and u_0 is a radial function. Consider the following problem

$$\begin{cases} a\dfrac{\partial^2 u}{\partial x^2} + b\dfrac{\partial^2 u}{\partial y^2} = u_t & \text{in} \quad \Gamma, \\ u = 1 & \text{in} \quad \partial\overline{\Omega}_1 \times (0,T), \\ u = 0 & \text{on} \quad \partial\Omega_0 \times (0,T), \\ u = u_0 & \text{in} \quad \overline{\Omega} \times \{0\}, \end{cases}$$

where a and b are positive constants with $a \neq b$. If u_0 is smooth, radial and $a\,(u_0)_{xx} + b\,(u_0)_{yy} \geq 0$, there exists a solution and $u_t \geq 0$ in Γ.
Since $w = 0$ on $\partial_P \Gamma$, a minimum principle for starshapedness would give $w \equiv 0$ in Γ and the solution would be radial. However, it can easily be seen that no radial solution exists for $a \neq b$.

3. Proof of Theorem 1

We start with a preliminary lemma which will be used for the proof of Theorem 1. This lemma is stated for solutions of a more general class of equations: this will allow us to study some kind of equations which do not fit the form of problem (1) (see section 4).

Consider a rotationally invariant nonlinear equation of the form

$$-u_t + G\Big(r, u, |Du|^2, \sum_{i,j=1}^{n} u_i u_j u_{ij}, \lambda_1\left(D^2 u\right), \ldots, \lambda_n\left(D^2 u\right)\Big) = 0, \tag{10}$$

where $\lambda_1\left(D^2u\right) \leq \ldots \leq \lambda_n\left(D^2u\right)$ are the eigenvalues of the symmetric matrix D^2u and $G \in C^1(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^{n+1})$.

We denote with $s = |Du|^2$, $q = \sum_{i,j=1}^n u_i u_j u_{ij}$ and $[u] = (r, u, s, q, \lambda_1, \ldots, \lambda_n)$. G is a *strictly elliptic* operator at u if there exists a positive constant σ_0 such that

$$\sum_{i,j=1}^{n} \frac{\partial G}{\partial u_{ij}}[u]\mu_i\mu_j \geq \sigma_0 |\mu|^2 \text{ in } \Gamma \text{ for every } \mu \in \mathbb{R}^n. \tag{11}$$

If $u \in C^2(\Gamma)$, the linear differential operator

$$L^u = \sum_{i,j=1}^{n} \frac{\partial G}{\partial u_{ij}}[u] \frac{\partial^2}{\partial x_i \partial x_j} \tag{12}$$

is strictly elliptic if G is strictly elliptic at u.

Remark 4. The part of G which depends on the eigenvalues of D^2u has the same behaviour of a rotationally invariant operator on the space of $n \times n$ symmetric matrices. If we denote with

$$G' = \left(\sum_{k=1}^{n} \frac{\partial G}{\partial \lambda_k}[u] \frac{\partial \lambda_k}{\partial u_{ij}} \right)_{i,j=1,\ldots,n},$$

and let B be the orthonormal matrix such that

$$B^T D^2 u B = diag\left(\lambda_1\left(D^2 u\right), \ldots, \lambda_n\left(D^2 u\right)\right),$$

then

$$G' = B\ diag\left(\frac{\partial G}{\partial \lambda_1}[u], \ldots, \frac{\partial G}{\partial \lambda_n}[u]\right) B^T, \tag{13}$$

and

$$G' D^2 u = B\ diag\left(\frac{\partial G}{\partial \lambda_1}[u]\lambda_1\left(D^2 u\right), \ldots, \frac{\partial G}{\partial \lambda_n}[u]\lambda_n\left(D^2 u\right)\right) B^T. \tag{14}$$

Notice that

$$\frac{\partial G}{\partial u_{ij}}[u] = \frac{\partial G}{\partial q}[u] u_i u_j + (G')_{ij}. \tag{15}$$

Define the following functions related to the angle $w(x,t)$:

$$v(x,t) = \sum_{k=1}^{n} x_k u_k(x,t) = -\|x\|\|Du(x,t)\| \cos w(x,t) \tag{16}$$

and

$$\Phi(x,t) = \tan w(x,t) = -\frac{h}{v}, \tag{17}$$

where

$$h = \left[\frac{1}{2}\sum_{k,l=1}^{n}(x_k u_l - x_l u_k)^2\right]^{1/2}.$$

Notice that, if $u \in C^1(\Gamma)$, v is defined in the whole Γ, while Φ is defined in $\{(x,t) \in \Gamma : v(x,t) \neq 0\}$.

LEMMA 1. *Let $G \in C^1(I\!R^+ \times I\!R \times I\!R^+ \times I\!R^{n+1})$ and $u \in C^3(\Gamma)$ be a solution of (10) such that G is strictly elliptic at u. Then*

$$-v_t + L^u v + 2\sum_{i=1}^{n} b_i v_i + \frac{\partial G}{\partial u}[u] v = d \quad in \quad \Gamma, \tag{18}$$

where

$$b_i = \frac{\partial G}{\partial s}[u] u_i + \frac{\partial G}{\partial q}[u] \sum_{j=1}^{n} u_j u_{ij},$$

and

$$d = 2\sum_{h=1}^{n} \frac{\partial G}{\partial \lambda_h}[u]\lambda_h + 4q\frac{\partial G}{\partial q}[u] + 2s\frac{\partial G}{\partial s}[u] - r\frac{\partial G}{\partial r}[u]. \qquad (19)$$

Moreover, if $v < 0$ *in* Γ, *then*

$$-\Phi_t + L^u\Phi + 2\sum_{i=1}^{n} c_i\Phi_i + \frac{d}{v}\Phi \geq 0 \quad in \quad \Gamma, \qquad (20)$$

where d *is as before and*

$$c_i = \frac{1}{v}\sum_{j=1}^{n} \frac{\partial G}{\partial u_{ij}}[u]v_j + \frac{\partial G}{\partial s}[u]u_i + \frac{\partial G}{\partial q}[u]\sum_{j=1}^{n} u_j u_{ij}.$$

Proof of Lemma 1. Directly from (16) and by the regularity assumption on u, we get

$$L^u v = 2\sum_{i,j=1}^{n} \frac{\partial G}{\partial u_{ij}}[u]u_{ij} + \sum_{k=1}^{n} x_k \sum_{i,j=1}^{n} \frac{\partial G}{\partial u_{ij}}[u]u_{ijk}. \qquad (21)$$

From (15) and (13) we have

$$\begin{aligned}\sum_{i,j=1}^{n} \frac{\partial G}{\partial u_{ij}}[u]u_{ij} &= q\frac{\partial G}{\partial q}[u] + tr\left(G'D^2u\right) \qquad (22)\\ &= q\frac{\partial G}{\partial q}[u] + \sum_{h=1}^{n} \frac{\partial G}{\partial \lambda_h}[u]\lambda_h\left(D^2u\right).\end{aligned}$$

Differentiating the equation (10) with respect to the variable x_k, we obtain

$$\begin{aligned}\sum_{i,j=1}^{n} \frac{\partial G}{\partial u_{ij}}[u]u_{ijk} &= u_{tk} - \left(\frac{x_k}{r}\frac{\partial G}{\partial r}[u] + 2\frac{\partial G}{\partial s}[u]\sum_{i=1}^{n} u_i u_{ik} + \right. \\ &\left. + u_k\frac{\partial G}{\partial u}[u] + 2\frac{\partial G}{\partial q}[u]\sum_{i,j=1}^{n} u_i u_{ij} u_{jk}\right), \qquad (23)\end{aligned}$$

for every $(x,t) \in \Gamma$. Since

$$\sum_{k=1}^{n} x_k u_{ki} = v_i - u_i$$

and combining (23), (22) and (21) we finally get (18).
Assume now that $v < 0$ in Γ; this mean that w is defined and less than $\frac{\pi}{2}$ in Γ. Φ is defined in the whole Γ, is positive, differentiable in $\Gamma' = \Gamma \setminus \{(x,t) \in \Gamma : h(x,t) = 0\}$,

$$\Phi_i = -\frac{h_i}{v} - \frac{\Phi}{v} v_i,$$

and

$$\Phi_{ij} = -\frac{h_{ij}}{v} - \frac{\Phi}{v} v_{ij} - \frac{1}{v}(v_j \Phi_i + v_i \Phi_j),$$

hence

$$L^u \Phi + \frac{2}{v} \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \frac{\partial G}{\partial u_{ij}}[u] v_i \right) \Phi_j = -\frac{1}{v} L^u h - \frac{\Phi}{v} L^u v. \qquad (24)$$

By the ellipticity of G, $\left(\frac{\partial G}{\partial u_{ij}}[u] \right)_{i,j=1,\dots,n}$ is positive definite, then applying the Schwarz inequality,

$$\begin{aligned} 2hL^u h \quad &\geq \quad \sum_{i,j=1}^{n} \frac{\partial G}{\partial u_{ij}}[u] \sum_{k,l=1}^{n} (x_k u_l - x_l u_k) \frac{\partial^2}{\partial x_i \partial x_j} (x_k u_l - x_l u_k) \\ &= \quad 4 \sum_{i,j,l=1}^{n} \frac{\partial G}{\partial u_{ij}}[u] u_{lj} (x_i u_l - x_l u_i) + \qquad (25) \\ &\quad + \sum_{k,l=1}^{n} (x_k u_l - x_l u_k) \times \\ &\quad\quad \times \left(x_k \sum_{i,j=1}^{n} \frac{\partial G}{\partial u_{ij}}[u] u_{ijl} - x_l \sum_{i,j=1}^{n} \frac{\partial G}{\partial u_{ij}}[u] u_{ijk} \right). \end{aligned}$$

$G' D^2 u$ is a symmetric matrix, (see (14)), while $(x_i u_l - x_l u_i)_{i,l=1,\dots,n}$ is antisymmetric, hence

$$\sum_{i,j,l=1}^{n} \frac{\partial G}{\partial u_{ij}}[u] u_{lj} (x_i u_l - x_l u_i) = \frac{\partial G}{\partial q}[u] \sum_{i,j,l=1}^{n} u_i u_j u_{lj} (x_i u_l - x_l u_i). \qquad (26)$$

From (25), (26) and (23) and after some calculations, we can conclude that

$$2hL^u h \geq -2h^2 \frac{\partial G}{\partial u} - 2hh_t \frac{\partial G}{\partial u_t} - 4h \sum_{j=1}^{n} \left(u_j \frac{\partial G}{\partial s} + \sum_{i=1}^{n} u_i u_{ij} \frac{\partial G}{\partial q} \right) h_j. \tag{27}$$

From (24), (27), (18) and $v < 0$ we finally get (20). $\diamondsuit$

Proof of Theorem 1. In the proof we will strongly use the maximum principle for linear parabolic equations. (See, for example, [7], pp. 173–175). We start showing that

$$u_t \geq 0 \quad \text{in} \quad \Gamma. \tag{28}$$

From (8) and from the boundary conditions for u, follows that $u_t \geq 0$ on $\partial_P \Gamma$. Differentiating the equation in (1) with respect to the variable t, we see that u_t satisfies in Γ the linear parabolic equation

$$-(u_t)_t + \sum_{i,j=1}^{n} \frac{\partial F}{\partial u_{ij}}[u](u_t)_{ij} + \frac{\partial f}{\partial u}[u]u_t + 2\frac{\partial f}{\partial s}[u] \sum_{i=1}^{n} u_i (u_t)_i = 0,$$

with $\dfrac{\partial f}{\partial u}[u] \leq 0$. By the maximum principle u_t does not assume its negative minimum in Γ, that is (28).

From (3), the boundary conditions for u and from (7) we also get $v \leq 0$ on $\partial_P \Gamma$. By Lemma 1, v satisfies in Γ the linear parabolic equation (18) where

$$L^u = \sum_{i,j=1}^{n} \frac{\partial F}{\partial u_{ij}}[u] \frac{\partial^2}{\partial x_i \partial x_j}, \qquad b_i = \frac{\partial f}{\partial s}[u]u_i,$$

$$\frac{\partial G}{\partial u}[u] = \frac{\partial f}{\partial u}[u] \leq 0 \quad \textit{and} \quad d = 2\sum_{h=1}^{n} \frac{\partial g}{\partial \lambda_h}\lambda_h + 2s\frac{\partial f}{\partial s} - r\frac{\partial f}{\partial r}.$$

Since g is homogeneous of order k,

$$\sum_{h=1}^{n} \frac{\partial g}{\partial \lambda_h}\lambda_h = kg,$$

hence,

$$d = 2ku_t - (2kf + r\frac{\partial f}{\partial r} - 2s\frac{\partial f}{\partial s}) \geq 0 \quad \text{in} \quad \Gamma, \tag{29}$$

(from (28) and (5)).
Then, we can apply to v the strong maximum principle and assert that

$$v < 0 \quad \text{in} \quad \Gamma. \tag{30}$$

(If $v = 0$ at some point in Γ, then, by the strong maximum principle, $v \equiv 0$ in $\Omega_0 \times \{0\}$, but this cannot be true since $u_0 = 0$ on $\partial\Omega_0$ and $u_0 = 1$ on $\partial\Omega_1$).
This means that $Du(x,t) \neq 0$ and $w(x,t)$ is strictly less than $\frac{\pi}{2}$ in Γ, that is, level sets of $u(\cdot,t)$ are properly starshaped for each $t \in (0,T]$.
Once we have proved that $v < 0$ in Γ, Φ is defined in the whole Γ and differentiable in $\Gamma' = \Gamma \setminus \{(x,t) \in \Gamma : h(x,t) = 0\}$. Notice that, since $0 \leq w < \frac{\pi}{2}$, in order to prove that w achieves its maximum value on $\partial_P\Gamma$, it is enough to prove that $\Phi = \tan w$ achieves its maximum value on $\partial_P\Gamma$.
By Lemma 1, Φ satisfies (20) in Γ', and, since $\frac{d}{v} \leq 0$ (see (29) and (30)), by the maximum principle, Φ attains its maximum value on $\partial_P\Gamma$. (Notice that on $\Gamma \setminus \Gamma'$ we have $\Phi \equiv 0$). We can also say that Φ does not assume its maximum value in Γ unless u_0 is radial: if $\Phi(x_1,t_1) = max\{\Phi(x,t) : (x,t) \in \partial_P\Gamma\}$ for $(x_1,t_1) \in \Gamma$, by the strong maximum principle $\Phi(x,0) \equiv \Phi(x_1,t_1)$ in Ω. Since the level sets of u_0 are starshaped, the only admissible constant for the angle w, and hence for Φ, is zero, which means that u_0 is radial and Ω_0 and Ω_1 are concentric balls. ◇

Remark 5. In the proof of Theorem 1 we show that

$$u_t \geq 0 \quad \text{and} \quad < Du(x,t), x > \leq 0 \quad \text{in} \quad \Gamma.$$

Thus the level sets $\{(x,t) \in \Gamma : u(x,t) \geq c\}$ are starshaped with respect to $(0,T) \in \mathbb{R}^n \times \mathbb{R}$.

4. Other results

Lemma 1 has been proved for a class of equations wider that the one treated in Theorem 1. From the proof of Theorem 1, one easily sees that in order to have a minimum principle for starshapedness, it is enough to know that

$$d \geq 0 \quad \text{in} \quad \Gamma$$

where d is as in (19).

Unfortunately, d depends upon u and, generally, it is difficult to estabilish its sign without knowing u. Still there are equations that can be treated although they do not fit the form of problem (1).

In the following we will assume $u \in C^3(\Gamma) \cap C^1(\Gamma) \cap C(\overline{\Omega}_0 \times [0, T])$ solution of (10) with the same initial-boundary value conditions as in Theorem 1 and such that $0 \leq u \leq 1$, $\frac{\partial G}{\partial u}[u] \leq 0$ and $G[u_0] \geq 0$. As we showed at the beginning of Theorem 1, $u_t \geq 0$ in Γ.

If we drop the dependence from q and consider solutions of

$$-u_t + G(r, u, |Du|^2, \lambda_1, \ldots, \lambda_n)) = 0,$$

with G a homogeneous function of order k in $\lambda_1, \ldots, \lambda_n$, we have

$$d = 2kG + 2s\frac{\partial G}{\partial s}[u] - r\frac{\partial G}{\partial r}[u].$$

Since $G = u_t \geq 0$, if

$$2s\frac{\partial G}{\partial s}[u] - r\frac{\partial G}{\partial r}[u] \geq 0,$$

the minimum principle for starshapedness holds.

However, a lot of interesting equations depend upon q, for example the equations in divergence form

$$-u_t + div(a(|Du|^2)Du) + f(r, u, |Du|^2) = 0$$

that is

$$-u_t + a(s)\Delta u + 2a'(s)q + f(r, u, s) = 0 \tag{31}$$

which are strictly parabolic if $a(|Du|^2) > 0$ and $a(|Du|^2) + 2|Du|^2 a'$ $(|Du|^2) > 0$.

Recalling that $u_t \geq 0$, it can easily be checked that if

$$4q\left(sa'' + a' - \frac{s(a')^2}{a}\right) - 2\left(1 + \frac{sa'}{a}\right)f - r\frac{\partial f}{\partial r} + 2s\frac{\partial f}{\partial s}[u] \geq 0, \tag{32}$$

the minimum principle for starshapedness holds.

Sometimes this can be assured with some additional informations on u.

EXAMPLE 3. Consider a solution $u \in C^3(\Gamma) \cap C(\Omega_0 \times [0,T])$ of

$$-u_t + div\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0$$

with the same initial-boundary value conditions upon u, $0 < u < 1$ and u_0 such that $div\left(\frac{Du_0}{\sqrt{1+|Du_0|^2}}\right) \geq 0$. Since $a(s) = (1+s)^{-1/2}$, $d \geq 0$ if

$$q \leq 0$$

which is true, for example, if u is concave in direction of Du.

Another operator in divergence form which can be considered is the p-Laplace operator

$$\Delta_p u = div(a(|Du|^2)Du)$$

with $a(s) = s^{\frac{p-2}{2}}$ and $p \geq 2$. For $p = 2$ we have exactly example 1 while for $p > 2$ the equation is degenerate when $|Du| = 0$.

THEOREM 2. *Let $\Omega = \Omega_0 \setminus \overline{\Omega}_1$ be a properly starshaped ring with the interior sphere condition, $\Gamma = \Omega \times (0,T]$, $p \geq 2$ and $f \in C^1(\mathbb{R}^+ \times \mathbb{R})$ such that $f(r,0) = 0$, $\frac{\partial f}{\partial u} \leq 0$ and $pf + r\frac{\partial f}{\partial r} \leq 0$.*
Let $u_0 \in C(\overline{\Omega}_0) \cap C^2(\Omega)$ and suppose $0 \leq u_0 \leq 1$ in Ω_0, $u_0 \equiv 1$ on Ω_1, the level sets of u_0 are starshaped and $\Delta_p u_0 + f(r,u_0) \geq 0$ in Ω.
If $u \in C^3(\Gamma) \cap C^1(\Gamma) \cap C(\overline{\Omega}_0 \times [0,T])$ is the solution of problem

$$\begin{cases} -u_t + \Delta_p u + f(r,u) = 0 \ \ in\ \Gamma, \\ u = 0 \ \ on\ \ \partial\Omega_0 \times (0,T], \\ u = 1 \ \ in\ \ \overline{\Omega}_1 \times (0,T], \\ u = u_0 \ \ in\ \ \Omega_0 \times \{0\}, \end{cases} \tag{33}$$

then the level sets $\{x \in \Omega : u(x,t) \geq c\}$ of u are starshaped for every $t \in [0,T]$, $Du(t,x) \neq 0$ and the starshapedness attains its minimum on $\partial_P\Gamma$.

Proof of Theorem 2. The maximum principle holds for (33) (see [3]) hence we conclude that $0 \leq u \leq 1$ in $\Omega_0 \times (0,T)$ and $u_t \geq 0$ in Γ. From Lemma 1, we know that $v = \sum_{k=1}^n x_k u_k$ satisfies (18) where

$$d = \frac{p}{2|Du|}u_t - pf - r\frac{\partial f}{\partial r} \geq 0.$$

Since the points at which the equation degenerate are those at which $|Du| = 0$ (and hence $v = 0$), we conclude that $v \leq 0$ in Γ.
In [4] it was shown with a barrier argument that

$$v < 0 \quad \text{in} \quad \Gamma,$$

(the proof in [4] is carried on with Ω_0 and Ω_1 convex, but it is enough to assume that they are properly starshaped with the interior sphere condition).
We can apply the second part of Lemma 1 and see that the minimum principle for starshapedness holds.

$\diamondsuit$

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