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Optimal withdrawal strategies in GLWB variable annuities

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ABSTRACT[1]

The aim of this paper is to construct a dynamic programming algorithm for pricing variable annuities with GLWB under a stochastic mortality framework. Although our set-up is very general and only requires the Markovian property for the mortality intensity and the asset price processes, in the numerical implementation of the algorithm we model the former as a non mean reverting square root process, and the latter as an exponential Lévy process. In this way we get a tractable and flexible stochastic model for efficient pricing and risk management of the GLWB. Another contribution of our paper is the verification, through backward induction, of the bang-bang condition for the set of discrete withdrawal strategies of the model. This result is particularly remarkable as in the insurance literature either the existence of optimal bang-bang controls is assumed or it requires suitable conditions. We present extensive numerical examples and compare the results obtained for different market parameters and policyholder behaviours.

KEYWORDS: GLWB, Dynamic withdrawals, Bang-bang condition, Lévy processes, Stochastic mortality

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1. Introduction

Variable annuities (VAs) are very flexible life insurance investment products that package living and death benefits endowed with a number of possible guarantees in respect of financial or biometric risks. Usually, a lump-sum premium is paid when the product is bought, and is invested in well-diversified mutual funds chosen by the policyholder among a range of alternative opportunities. This initial investment establishes a reference portfolio (*policy account*), and all guarantees are financed through periodical deductions from the policy account value.

The market for variable annuities has shown phenomenal growth since the 1990s: according to Chopra et al. (2009), the annual rate of growth in the U.S. VA market during that decade was 21%, and the level of total assets reached almost USD 1 trillion by 2001. A good overview of VA products and the development of their market can be found, e.g., in Bauer et al. (2008) and Ledlie et al. (2008).

A possible rider that can be included in a VA contract in order to provide a post-retirement income is the Guaranteed Lifelong Withdrawal Benefit (GLWB), that offers a lifelong withdrawal guarantee. Therefore, there is no explicit limit on the total amount that can be withdrawn over the term of the policy because, if the policy account is depleted while the policyholder is still alive, she continues to receive the guaranteed amount until death. Any remaining account value at the time of death is paid to the beneficiary as a death benefit.

The guaranteed withdrawal is computed by applying a fixed percentage (*withdrawal rate*) to the so called *benefit base* (also referred to as *base amount*, or *withdrawal base*), that is initially set equal to the value of the investment portfolio and then evolves according to the contractually stated rules. A plain GLWB can be enriched by adding elements such as a *ratchet* (or *step-up*) feature and a *roll-up* (or *bonus*) feature. Under the ratchet provision the benefit base is increased to the policy account value on predetermined (ratchet) dates if the latter exceeds the previous benefit base recorded on the last withdrawal date. Under the roll-up provision, the benefit base may also be increased by a proportional amount if the policyholder chooses not to withdraw on a withdrawal date. Therefore, roll-ups are commonly used as a disincentive to withdraw during the first years of contract. Moreover, complete surrender refers to the withdrawal of the whole policy account, if its value exceeds the guaranteed withdrawal amount. The actual withdrawal can be lower than the guaranteed amount, or even exceed it, but in this second case the net amount received by the policyholder is subject to a penalty, that can be accompanied by further disincentives. There are also deferred versions of the contract, as opposed to immediate withdrawals, where the policyholder can only withdraw money after the deferred period. Details on additional features embedded in GLWB can be found, e.g., in Piscopo and Haberman (2011).

There has been a number of papers dealing with pricing of the VA products. Most of them are focused on pricing VA guarantees under the *static* policyholder behaviour in withdrawal and surrender (see e.g., Milevsky and Salisbury (2006)), meaning that the policyholder always withdraws exactly the guaranteed amount, and never surrenders the contract. Some studies include pricing under the *dynamic* approach, when the policyholder optimally decides the amount to withdraw at each withdrawal date depending on the information available at that date (see, e.g., Steinorth and Mitchell (2015)). According to whether withdrawals are assumed to occur continuously or discretely, the optimal withdrawal problem under the dynamic approach is usually solved using, respectively, stochastic control (e.g. Chen and Forsyth (2008) and Dai et al. (2008)) and dynamic programming (Bacinello et al. (2016) and Alonso-García et al. (2018)). A range of numerical methods are employed, including standard and regression-based Monte Carlo methods (Bacinello et al. (2011) and Huang and Kwok (2016)), Partial Differential Equation (PDE) and direct integration methods (Chen and Forsyth (2008), Dai et al. (2008), Luo and Shevchenko (2015a), Luo and Shevchenko (2015b), Forsyth and Vetzal (2014), Shevchenko and Luo (2017)). A comprehensive overview of numerical methods for the pricing of VA guarantees is provided in Shevchenko and Luo (2016).

In particular, the authors in Bacinello et al. (2016) propose a dynamic programming algorithm for the valuation of a Guaranteed Minimum Withdrawal Benefit (GMWB)[2] in a VA contract using exponential Lévy processes to model the evolution of the reference asset price and incorporating as special cases, within the dynamic algorithm, alternative policyholder behaviours. Following the same approach, our aim is to construct a dynamic programming algorithm for pricing variable annuities with GLWB under a stochastic mortality framework that allows to capture systematic longevity improvements as well as mortality shocks due, e.g., to pandemics. Although our set-up is very general and only requires the Markovian property for the mortality intensity and the asset price processes, in the numerical implementation of the algorithm we model the former as an affine diffusion, namely a non mean reverting square root process (see e.g., Fung et al. (2014) and Dacorogna and Apicella (2016)), and the latter, like in Bacinello et al. (2016), as an exponential Lévy process. In this way we get a tractable and flexible stochastic model for efficient pricing and risk management of the GLWB. Also our algorithm allows for different withdrawal behaviours of the policyholder. In fact we present a lot of numerical examples where we compare the results obtained for different parameters and policyholder behaviours.

Note that the pricing models of GLWB can be considered as extensions of those of GMWB with the inclusion of mortality risk. For example the authors in Holz et al. (2012) price the GLWB for different product designs and model parameters under the Geometric Brownian Motion (GBM) dynamics of the underlying fund value process. They consider various forms of policyholder withdrawal behaviour, and establish that the optimal strategy for a GLWB contract without the ratchet and roll-up feature consists only of either a withdrawal of the guaranteed amount or complete surrender. In this context they use the Monte Carlo simulation method to approximate the optimal strategy. Also the authors in Bacinello et al. (2012) value the GLWB under the *mixed* approach,[3] by using the Least Squares Monte Carlo (LSMC) technique implemented with a very sophisticated model framework.

Another important contribution of our paper is the verification, through backward

[2]It differs from the GLWB in that withdrawals take place for a fixed duration, regardless of the policyholder's survival.

[3]That is withdrawal of the guaranteed amount or complete surrender.

induction, of the bang-bang condition for the set of discrete withdrawal strategies of the GLWB model. This means that the set of the optimal withdrawals consists of three choices only: zero withdrawal, withdrawal at the contractual amount, complete surrender. This result, proven in our discrete time framework for the evolution of the policy account value, is particularly remarkable as in the insurance literature either the existence of optimal bang-bang controls is assumed or it requires suitable conditions. For example, in Azimzadeh and Forsyth (2015) a GBM for the reference asset price, as well as the convexity and monotonicity of the contract value function for all times, are assumed to ensure the existence of optimal bang-bang strategies. A regression-based Monte Carlo method for pricing GLWB under the bang-bang strategy in the case of stochastic volatility is also developed in Huang and Kwok (2016).

The bang-bang condition, beyond drastically reducing the computational time needed to search the optimal withdrawal in the backward recursive step of our dynamic algorithm, allows also to clearly separate the various contract components: the first one is the *basic GLWB contract* value, obtained under the static approach, the second is the value of the *surrender option*, and the last component is the value of the *roll-up option*.

The paper is structured as follows. In Section 2, we describe the structure of the variable annuity contract. In Section 3, we introduce our valuation framework and define the optimal withdrawal problem. In Section 4, we first define the dynamic programming equations that allow to solve the problem, then we verify the validity of the bang-bang condition for the strategy space of discrete withdrawal policies of the GLWB model, after we outline our valuation algorithm and finally present the contract decomposition. In Section 5, we perform a sensitivity analysis comparing the numerical results obtained for different parameters and policyholder behaviours. Finally, Section 6 concludes the paper.

2. The contract structure

In this section we describe the GLWB rider in our variable annuity contract. At time 0 (contract inception), the policyholder, aged x, pays a single premium P which is entirely invested in a well-diversified and non-dividend paying mutual fund of her own choice. We denote by S_t the market price at time t of each unit of this fund, that drives the return on the investment portfolio built up with the policyholder's payment. The value at time t of such portfolio, that is called *personal account* (or *policy account*) is denoted by W_t.

The GLWB rider gives the policyholder the right to make periodical withdrawals from her personal account at some specified dates for the whole life, even if the account value is reduced to zero. The cost of the guarantee is financed by periodical proportional deductions from the policy account value (*insurance fees*). The guaranteed withdrawal amount is calculated as a fixed proportion g (*withdrawal rate*) of the *benefit base* (or *withdrawal base*, or *base amount*), denoted by A_t. The benefit base is initially set equal to the single premium and is adjusted upward via the *roll-up* (or *bonus*) feature, that applies when no withdrawal is made on a specified withdrawal date.

Complete surrender of the policy refers to the withdrawal of the whole personal account, if it exceeds the guaranteed amount. As a result, the variable annuity contract terminates. Another event that causes the closure of the contract is the policyholder's death. The value that remains in the policy account when the policyholder dies is paid to the beneficiary as a death benefit. In particular, from now on we assume that: (i) withdrawals are allowed on a predetermined set of equidistant dates and, without loss of generality, we take the distance between two consecutive withdrawal dates as unit of measurement of time; (ii) the death benefit is paid to the beneficiary on the next upcoming withdrawal date.

We now formalize what just described. Let τ denote the time of death of the policyholder, so that withdrawals are allowed only at times $i = 1, 2, \ldots$, provided that $\tau > i$. The guaranteed amount that can be withdrawn at time i is equal to gA_i, $i = 1, 2, \ldots$, and the return on the reference fund over the interval $[i-1, i]$ is

$$R_i = \frac{S_i}{S_{i-1}} - 1, \quad i = 1, 2, \ldots \ . \tag{1}$$

As already mentioned in Section 1, the policyholder is not obliged to withdraw the guaranteed amount, but she can decide to withdraw less than this amount, or even more if not exceeding the policy account value. We denote by y_i the actual withdrawal made by the policyholder at time i. Hence, under our dynamic approach, we assume that the set of possible withdrawals at this time is given by the interval $[0, \max\{gA_i, W_i\}]$.[4] If the policyholder does not withdraw anything at time i, the benefit base is proportionally increased according to the roll-up rate, that we denote by b_i (with $0 < b_i < 1$), while, if the withdrawal exceeds gA_i, it is proportionally reduced according to the so called 'pro-rata' adjustment rule. Then the benefit base evolves as follows:

$$A_{i+1} = f^A_{i+1}(W_i, A_i, y_i) = \begin{cases} A_i(1+b_i) & \text{if} \quad y_i = 0, \\ A_i & \text{if} \quad 0 < y_i \le gA_i, \\ A_i \dfrac{W_i - y_i}{W_i - gA_i} & \text{if} \quad gA_i < y_i \le W_i \end{cases}, \quad i = 1, 2, \ldots, \tag{2}$$

with $A_1 = A_0 = P$.[5]

Moreover, in case of withdrawals exceeding the guaranteed amount, there is also a proportional penalization on the surplus according to a penalty rate, that we denote by k_i (such that $0 < k_i < 1$). Therefore, the net amount (cash-flow) received by the policyholder at time i is given by

$$\begin{aligned} B^{(s)}_i = f^{(s)}_i(y_i, A_i) &= \begin{cases} y_i & \text{if} \quad 0 \le y_i \le gA_i \\ gA_i + (1-k_i)(y_i - gA_i) & \text{if} \quad y_i > gA_i \end{cases} \\ &= y_i - k_i \max\{y_i - gA_i, 0\}, \quad i = 1, 2, \ldots \ . \end{aligned} \tag{3}$$

The policy account value evolves according to the following equation:

$$W_{i+1} = f^W_{i+1}(W_i, R_{i+1}, y_i) = \max\{W_i - y_i, 0\}(1 + R_{i+1})(1 - \varphi), \quad i = 0, 1, \ldots, \tag{4}$$

[4] Under the static approach this set is the singleton $\{gA_i\}$ while, under the mixed approach, it is again the singleton $\{gA_i\}$ if $gA_i \le W_i$, and the two-points set $\{gA_i, W_i\}$ otherwise; see also Bacinello et al. (2016).

[5] Note that A_0 is only a fictitious value because we assume that no withdrawals can be made at contract inception and that the roll-up feature does not apply between times 0 and 1.

where φ (such that $0 < \varphi < 1$) is the insurance fee rate, $W_0 = P$ and $y_0 = 0$. Note that 0 is an absorbent barrier for W because, once it becomes null, it remains so for ever. The contract, however, continues while $A_t > 0$ (and the insured is still alive).

Finally, in case of death in the time interval $(i-1, i]$, the death benefit, paid at time i, is

$$B_i^{(d)} = W_i, \quad i-1 < \tau \leq i\ , \quad i = 1, 2, \ldots\ . \tag{5}$$

We notice that, in case of surrender at time i, i.e., when $y_i = W_i > gA_i$, the contract is automatically closed because Equations (2) and (4) imply $A_t = W_t = 0$ for all $t > i$, hence no further withdrawals are admitted, nor a death benefit will be paid.

3. The valuation framework

Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, Q)$ supporting all sources of financial and biometric uncertainty, where all random variables and processes are defined. The filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, that represents the flow of information available to the insurer and the policyholder over time, satisfies the usual conditions of right continuity and completeness, and is such that $\mathcal{F}_0$ is Q-trivial. Q is a risk-neutral probability measure selected by the insurer, for pricing purposes, among the infinitely many equivalent martingale measures existing in incomplete arbitrage-free markets.

In this setting, the residual lifetime of the policyholder τ is a stochastic $\mathbb{F}$-stopping time. Let $\mu_t := \mu_{x+t}(t)$ be the mortality intensity which determines the probability of death at time t conditional on survival for the policyholder aged x at time 0. Letting moreover

$$\mathcal{G}_t^S = \sigma(S_u, u \leq t), \quad \mathcal{G}_t^\tau = \sigma(\mathbf{1}_{\{\tau \leq u\}}, u \leq t), \quad \mathcal{G}_t^\mu = \sigma(\mu_u, u \leq t)$$

the natural filtrations associated to the fund value process S_t, the death indicator process $\mathbf{1}_{\{\tau \leq t\}}$ and the mortality process μ_t, respectively, we take the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ as

$$\mathcal{F}_t = \mathcal{G}_t^S \vee \mathcal{G}_t^\tau \vee \mathcal{G}_t^\mu,$$

with $\mathcal{G}_t^S$ independent of $\mathcal{G}_t^\tau \vee \mathcal{G}_t^\mu$. In other words, there is independence between financial- and biometric-related variables. Therefore, we can define

$$Q\left(\tau > s | \tau > t, \mathcal{G}_t^\mu\right) = \mathbb{E}^Q\left[\mathrm{e}^{-\int_t^s \mu_u \,\mathrm{d}u} | \mathcal{G}_t^\mu\right], \quad 0 \leq t < s \tag{6}$$

as the probability of surviving $s-t$ periods for the policyholder aged $x+t$ at time t, considering all the information about the mortality until this time. With the usual actuarial notation this probability should be indicated with ${}_{s-t}p_{x+t}(t)$. However, with specific reference to the one-period death and survival probabilities, we use slightly different symbols. In particular, assuming for μ a Markovian process, we denote by

$$p_i(\mu_i) = Q\left(\tau > i+1 \,|\, \tau > i, \mu_i\right) = \mathbb{E}^Q\left[\mathrm{e}^{-\int_i^{i+1} \mu_u \,\mathrm{d}u} | \mu_i\right], \quad i = 0, 1, \ldots, \tag{7}$$

the probability of survival up to $i+1$ for the policyholder aged $x+i$ given the mortality intensity's values up to i. Consequently, $q_i(\mu_i) = 1 - p_i(\mu_i)$ is the probability of death before $i+1$ conditional on survival at time i.

Our framework is general enough to allow for any (reasonable) Markovian process μ_t. However, in the numerical implementation, we model it as an affine diffusion, following the non mean-reverting square root process

$$d\mu_t = (\alpha + \theta\mu_t)dt + \sigma_\mu\sqrt{\mu_t}\, dB_t, \tag{8}$$

where $\alpha > 0$, $\theta > 0$, B is a Brownian motion and $\mu_0 > 0$ is given. Although this model is very simple (it collapses to the well-known deterministic Gompertz force of mortality when $\alpha = \sigma_\mu = 0$), it has the desirable property of producing strictly positive paths with probability 1, provided that $2\alpha > \sigma_\mu^2$. Moreover, it allows to get a closed-form formula for the survival probability given in (7) (see, e.g., Fung et al. (2014) and Dacorogna and Apicella (2016)). Hence we have

$$p_i(\mu_i) = c_1(1)e^{-c_2(1)\mu_i}, \quad i = 0, 1, \ldots, \tag{9}$$

where, for every $\zeta > 0$,

$$c_1(\zeta) = \left(\frac{2\gamma e^{1/2(\gamma-\theta)\zeta}}{(\gamma-\theta)(e^{\gamma\zeta}-1)+2\gamma}\right)^{2\alpha/\sigma_\mu^2}, \quad c_2(\zeta) = \frac{2(e^{\gamma\zeta}-1)}{(\gamma-\theta)(e^{\gamma\zeta}-1)+2\gamma}, \ \gamma = \sqrt{\theta^2 + 2\sigma_\mu^2}\,.$$

The conditional law of μ_s given μ_t (with $t < s$) features a noncentral chi-squared distribution. In particular, letting $Q(\mu_s \le z|\mu_t)$ the conditional cumulative distribution function of μ_s, the corresponding density function is

$$f_{\mu_s|\mu_t}(z) = K f_{\chi^2_{\nu,\lambda}}(Kz), \tag{10}$$

where

$$K = \frac{4\theta}{\sigma_\mu^2(e^{\theta(s-t)}-1)}, \quad \nu = \frac{4\alpha}{\sigma_\mu^2}, \quad \lambda = K\mu_t\, e^{\theta(s-t)},$$

and $f_{\chi^2_{\nu,\lambda}}$ denotes the probability density function of a noncentral chi-squared distribution with ν degrees of freedom and non-centrality parameter λ. More in detail:

$$f_{\chi^2_{\nu,\lambda}}(h) = \sum_{j=0}^{\infty} \frac{e^{-\frac{\lambda+h}{2}}\lambda^j h^{j+\frac{\nu}{2}-1}}{2^{2j+\frac{\nu}{2}}\Gamma(j+\frac{\nu}{2})},$$

where $\Gamma(z) = \int_0^\infty y^{z-1}e^{-y}dy$ is the gamma function (see also Brigo and Mercurio (2006) for further details).

In order to get the contract value (which we will consider in the next part of the paper) we need to compute the joint density function of μ_s and $\int_t^s \mu_u du$, given μ_t (with $t < s$).[6] Such a density can be characterized by its Laplace transform (see Lamberton and Lapeyre (1995)):

$$\mathcal{L}_{s-t}^{(\mu_t)}(\beta_1, \beta_2) := \mathbb{E}^Q\left[e^{-\beta_1\mu_s}\, e^{-\beta_2\int_t^s \mu_u du}|\mu_t\right] = e^{-\alpha\Psi_1}\, e^{-\mu_t\Psi_2}, \tag{11}$$

[6]For the (marginal) density function of $\int_t^s \mu_u du$ we refer the reader to Dufresne (2001).

for any $\beta_1, \beta_2 \geq 0$, where

$$\begin{aligned}
\Psi_1 &= -\frac{2}{\sigma_\mu^2} \ln\left(\frac{2\tilde{\gamma}\, \mathrm{e}^{(s-t)(\tilde{\gamma}-\theta)/2}}{\sigma_\mu^2 \beta_1(\mathrm{e}^{\tilde{\gamma}(s-t)} - 1) + \tilde{\gamma} + \theta + \mathrm{e}^{\tilde{\gamma}(s-t)}(\tilde{\gamma}-\theta)} \right), \\
\Psi_2 &= \frac{\beta_1(\tilde{\gamma} - \theta + \mathrm{e}^{\tilde{\gamma}(s-t)}(\tilde{\gamma}+\theta)) + 2\beta_2(\mathrm{e}^{\tilde{\gamma}(s-t)} - 1)}{\sigma_\mu^2 \beta_1(\mathrm{e}^{\tilde{\gamma}(s-t)} - 1) + \tilde{\gamma} + \theta + \mathrm{e}^{\tilde{\gamma}(s-t)}(\tilde{\gamma}-\theta)}, \\
\tilde{\gamma} &= \sqrt{\theta^2 + 2\sigma_\mu^2 \beta_2}\,.
\end{aligned}$$

Therefore we can obtain the conditional joint density of μ_s and $\int_t^s \mu_u \mathrm{d}u$, that we denote by $g_{s-t}^{(\mu_t)}$, through (numerical) inversion of the Laplace transform. Indeed, if $g_{s-t}^{(\mu_t)}$ possesses first order partial derivatives $\partial g_{s-t}^{(\mu_t)}/\partial x_1$ and $\partial g_{s-t}^{(\mu_t)}/\partial x_2$ and second order derivative $\partial^2 g_{s-t}^{(\mu_t)}/\partial x_1 \partial x_2$, and there exist positive constants M, γ_1, γ_2 such that, for all strictly positive numbers x_1, x_2, it is

$$|g_{s-t}^{(\mu_t)}(x_1, x_2)| < M \mathrm{e}^{\gamma_1 x_1 + \gamma_2 x_2}, \qquad \left| \frac{\partial^2 g_{s-t}^{(\mu_t)}}{\partial x_1 \partial x_2} \right| < M \mathrm{e}^{\gamma_1 x_1 + \gamma_2 x_2},$$

then

$$g_{s-t}^{(\mu_t)}(x_1, x_2) = \frac{1}{(2\pi i)^2} \int_{d_1 - i\infty}^{d_1 + i\infty} \int_{d_2 - i\infty}^{d_2 + i\infty} \mathrm{e}^{\beta_1 x_1 + \beta_2 x_2} \mathcal{L}_{s-t}^{(\mu_t)}(\beta_1, \beta_2) \mathrm{d}\beta_1\, \mathrm{d}\beta_2, \tag{12}$$

where $d_1 > \gamma_1$, $d_2 > \gamma_2$ and i is the unit imaginary number (see Cohen (2007)).

Concerning the financial uncertainty, in order to keep the curse of dimensionality of our valuation algorithm manageable, we assume the instantaneous interest rate to be deterministic and constant, and denote it by r.

The reference price S, instead, in principle could be any Markovian process whose discounted value is a martingale under Q. However, in the numerical implementation of the model, we restrict ourselves to the flexible and wide class of exponential Lévy processes, i.e., we assume that

$$S_t = S_0 \mathrm{e}^{(r+d)t + X_t}, \tag{13}$$

where $(X_t)_{t \geq 0}$, with $X_0 = 0$, is a Lévy process,[7] and d represents an adjustment so that $S_t\, \mathrm{e}^{-rt}$ is a martingale under Q. Hence X_t has right-continuous with left limits paths, $X_s - X_t$ is independent of $(X_u)_{0 \leq u \leq t}$ and is distributed as X_{s-t}, for $0 \leq t < s$. Moreover, X_t can be obtained as a combination of a linear drift, a Brownian motion, and a jump component, and is completely determined by its characteristic function

$$\Phi_t(u) := \mathbb{E}^Q \left[\mathrm{e}^{iuX_t} \right] = [\Phi_1(u)]^t. \tag{14}$$

Then the drift adjustment d is given by

$$d = -\frac{1}{t} \ln \Phi_t(-i) = -\ln \Phi_1(-i). \tag{15}$$

[7] For a comprehensive description of Lévy processes, their properties and applications we refer to Tankov and Cont (2003) and Schoutens (2003).

All moments of X_t can be numerically recovered from the knowledge of Φ_t (when not available in closed form), and if Φ_t is integrable, then X_t has density given by:

$$f_t(\phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-iz\phi} \Phi_t(z) \mathrm{d}z. \tag{16}$$

In particular, in our numerical experiments we use the Carr, Geman, Madan, Yor (CGMY) process whose characteristic function is

$$\Phi_t(u) = \exp\left(C\,\Gamma(-\mathcal{Y})t\left[(M-iu)^{\mathcal{Y}} - M^{\mathcal{Y}} + (G+iu)^{\mathcal{Y}} - g^{\mathcal{Y}}\right]\right), \tag{17}$$

with G, $M \geq 0$, $C > 0$, $\mathcal{Y} < 2$ and Γ is the gamma function.
Such a process, introduced in Carr et al. (2002), allows for both a diffusion and a jump component. Moreover, it can be parametrized in order to capture finite or infinite variation.

Consider now a withdrawal strategy $y = (y_i)_{i\in\mathbb{N}^+}$, where y_i denotes the actual withdrawal made at time i (in case of survival). This is a stochastic process, adapted to the filtration $\mathbb{F}$, because at each withdrawal date the policyholder takes her withdrawal decision once she knows the values of all state variables. This strategy is *admissible* if it belongs to the set of admissible withdrawal strategies $Y = (Y_i)_{i\in\mathbb{N}^+}$ that, as already mentioned, is the sequence of intervals $Y_i = [0,\ \max\{W_i,\ gA_i\}]$. Then we define the initial value of the GLWB variable annuity as the solution of the following optimization problem:

$$V_0 = \sup_{y\in Y} \mathbb{E}^Q \left[\sum_{i=1}^{\infty} \mathrm{e}^{-ri} \left(\mathbf{1}_{\{\tau>i\}} f_i^{(s)}(y_i, A_i) + \mathbf{1}_{\{i-1<\tau\leq i\}} W_i\right)\right], \tag{18}$$

where the account value and the benefit base satisfy Equations (4) and (2) respectively.

Hence the policyholder is assumed to act in such a way to maximize the present expected value, under Q, of all the future cash-flows generated by the variable annuity contract. Note that the solution of this problem does not necessarily lead to an optimal withdrawal strategy for the policyholder, but leads to the worst-case scenario from the viewpoint of the insurer, that fixes the pricing measure Q. In particular, if the single premium is chosen in such a way that $V_0 = P$,[8] the insurer puts itself on the safety side.

4. Dynamic programming

In order to solve (18), in this section we implement a dynamic programming algorithm for discrete stochastic control problems (see, e.g., Bertsekas (2005) and Seierstad (2009)). As we act in a Markovian framework, for each $i = 0, 1, \ldots$ and each value of policy account W_i, benefit base A_i and mortality intensity μ_i, we denote by $V_i(W_i, A_i, \mu_i)$ the contract value at time i (before the periodic withdrawal) and by $v_i(W_i, A_i, \mu_i)$ the contract value at the same time when, moreover, the policyholder is then alive. Clearly $V_i(W_i, A_i, \mu_i) = \mathbf{1}_{\{\tau>i\}} v_i(W_i, A_i, \mu_i)$ and $V_0 = V_0(P, P, \mu_0) = v_0(P, P, \mu_0)$.

[8]This implies a suitable choice of the insurance fee rate φ.

Since the algorithm proceeds backward, we need a starting point. To this end, as is common in life insurance mathematics, we assume that there is an ultimate age for the policyholder beyond which her survival probability is null. We denote by ω this age, that typically is in the range 110-120 years, and let $n = \lfloor \omega - x \rfloor$, hence $n \le \omega - x < n+1$.[9] In particular, it must be understood that the strictly positive (conditional) survival probabilities produced by Equation (9) are valid only for $i < n$, while for $i \ge n$ they are equal to 0. Then Problem (18) can be rewritten as

$$V_0 = \sup_{y \in Y} \mathbb{E}^Q \left[\sum_{i=1}^{n} \mathrm{e}^{-ri} \left(\mathbf{1}_{\{\tau > i\}} f_i^{(s)}(y_i, A_i) + \mathbf{1}_{\{i-1 < \tau \le i\}} W_i \right) + \mathrm{e}^{-r(n+1)} \mathbf{1}_{\{\tau > n\}} W_{n+1} \right]. \tag{19}$$

We take $n+1$ as starting point of our backward dynamic algorithm, and define the following terminal condition:

$$v_{n+1}(W_i, A_i, \mu_i) \equiv 0. \tag{20}$$

Note that the particular value of v_{n+1} is quite irrelevant because it will be multiplied for the survival indicator $\mathbf{1}_{\{\tau > n+1\}}$, almost surely $= 0$, in order to get the contract value V_{n+1}.

Then, to proceed backward, we define the Bellman recursive equation of the problem as follows:

$$\begin{aligned} v_i(W_i, A_i, \mu_i) = \sup_{y_i \in Y_i} \Bigg(& f_i^{(s)}(y_i, A_i) \; + \; \mathbb{E}^Q \Big[\mathbf{1}_{\{\tau \le i+1\}} f_{i+1}^W(W_i, R_{i+1}, y_i) \mathrm{e}^{-r} | W_i, A_i, \mu_i, \tau > i \Big] \\ & + \; \mathbb{E}^Q \Big[V_{i+1} \left(f_{i+1}^W(W_i, R_{i+1}, y_i), f_{i+1}^A(W_i, A_i, y_i), \mu_{i+1} \right) \mathrm{e}^{-r} | W_i, A_i, \mu_i, \tau > i \Big] \Bigg), \\ & i = n, n-1, \ldots, 1. \end{aligned} \tag{21}$$

In the first expectation of (21) we can exploit the stochastic independence between financial and demographic factors, the definition of the policy account value in (4), and the martingale property according to which $\mathbb{E}^Q\left[(1 + R_{i+1})\mathrm{e}^{-r}\right] = 1$, to get

$$\mathbb{E}^Q \Big[\mathbf{1}_{\{\tau \le i+1\}} f_{i+1}^W(W_i, R_{i+1}, y_i) \mathrm{e}^{-r} | W_i, A_i, \mu_i, \tau > i \Big] = q_i(\mu_i) \max\{W_i - y_i, \, 0\}(1 - \varphi). \tag{22}$$

In the second expectation we first replace $V_{i+1}(\dots)$ with $\mathbf{1}_{\{\tau > i+1\}} v_{i+1}(\dots)$, then we further condition on μ_u, $i < u \le i+1$, so that we can factorize the conditional Q-expectation inside, and after apply the tower property, obtaining

$$\begin{aligned} & \mathbb{E}^Q \Big[V_{i+1} \left(f_{i+1}^W(W_i, R_{i+1}, y_i), f_{i+1}^A(W_i, A_i, y_i), \mu_{i+1} \right) \mathrm{e}^{-r} | W_i, A_i, \mu_i, \tau > i \Big] \\ & = \mathbb{E}^Q \Big[\mathrm{e}^{-\int_i^{i+1} \mu_u \, \mathrm{d}u} v_{i+1} \left(f_{i+1}^W(W_i, R_{i+1}, y_i), f_{i+1}^A(W_i, A_i, y_i), \mu_{i+1} \right) \mathrm{e}^{-r} | W_i, A_i, \mu_i \Big]. \end{aligned} \tag{23}$$

Finally, the initial contract value is given by

$$\begin{aligned} v_0(P, P, \mu_0) &= q_0(\mu_0) \mathbb{E}^Q \Big[f_1^W(P, R_1, 0) \mathrm{e}^{-r} \Big] + \mathbb{E}^Q \Big[\mathrm{e}^{-\int_0^1 \mu_u \, \mathrm{d}u} v_1(f_1^W(P, R_1, 0), P, \mu_1) e^{-r} \Big] \\ &= q_0(\mu_0) P (1 - \varphi) + \mathbb{E}^Q \Big[\mathrm{e}^{-\int_0^1 \mu_u \, \mathrm{d}u} v_1(P(1 + R_1)(1 - \varphi), P, \mu_1) e^{-r} \Big]. \end{aligned} \tag{24}$$

[9] Recall that our unit of measurement of time is the common distance between two consecutive withdrawal dates, so that also x and ω are expressed according to this measure.

a. Bang-bang analysis

The Bellman recursive equation (21) requires, at each time step $i = n, n-1, \dots 1$, to solve a real-valued optimization problem where the domain of the single variable y_i is the interval $Y_i = [0, \max\{W_i,\, gA_i\}]$. Even if withdrawals can take place only once a year, for, e.g., a 65-years old policyholder this would imply around 50 time steps. Moreover, at each time step the problem must be solved for every possible triplet of state variables (W_i, A_i, μ_i). Then the computational effort could be substantial. A property that would drastically reduce this effort is the *bang-bang* condition, which states that the set of the optimal withdrawals consists of three choices only: zero withdrawal, withdrawal at the contractual amount, complete surrender, i.e., the research of the maximum can be restricted to the subset $\{0, gA_i, W_i\}$ of Y_i.

The intuition behind this condition is clear: due to the roll-up feature it may be convenient to withdraw nothing ($y_i = 0$) in order to increase the benefit base for future withdrawals, especially if the expected residual lifetime of the policyholder is sufficiently long, but it is never convenient to withdraw something less than the guaranteed amount, i.e., $0 < y_i < gA_i$. In fact, if $y_i > 0$, the policyholder loses the roll-up incentive without taking full advantage of the guarantee ($y_i < gA_i$): if she does not need the entire amount gA_i and prefers to benefit from the investment of what exceeds her needs, it would be better to invest this money in a risk-free asset because the policy account, on average, yields less than the risk-free rate r due to the charge of the insurance fees. On the other hand, if $W_i > gA_i$, it is possible to withdraw an amount $y_i > gA_i$, but it is subject to the withdrawal penalty $k_i(y_i - gA_i)$. If, moreover, $gA_i < y_i < W_i$, there is a second penalization due to the reduction of the benefit base according to the pro-rata rule: this suggests that, rather than suffering the second penalization as well, it would be more convenient to withdraw the entire account value W_i, i.e., to surrender the contract. The following result states that this intuition is right.

Proposition 1 *The optimal solution of Problem* (21) *is $y_i = 0$, or $y_i = gA_i$, or $y_i = W_i$.*

Proof: We prove this result by backward induction, starting from the case $i = n$.

Initial step. Taking into account that $\mathbf{1}_{\{\tau \le n+1\}} = 1$ almost surely, and replacing Equations (3) and (4) in Problem (21), we get

$$v_n(W_n, A_n, \mu_n) = \sup_{0 \le y_n \le \max\{gA_n,\, W_n\}} F_n(y_n; W_n, A_n, \mu_n),$$

where

$$F_n(y_n; W_n, A_n, \mu_n) = y_n - k_n \max\{y_n - gA_n,\, 0\} + \max\{W_n - y_n,\, 0\}(1 - \varphi).$$

- Then, if $gA_n < W_n$,

$$F_n(y_n; W_n, A_n, \mu_n) = \begin{cases} \varphi y_n + (1-\varphi) W_n & \text{if } 0 \le y_n \le gA_n \\ (\varphi - k_n) y_n + k_n g A_n + (1-\varphi) W_n & \text{if } gA_n < y_n \le W_n \end{cases},$$

so that the maximizer of F_n is (see Figure 1)

i. $y_n^* = W_n$ if $\varphi > k_n$,

ii. any element of the interval $[gA_n,\, W_n]$ (hence both $y_n^* = W_n$ and $y_n^* = gA_n$ are OK) if $\varphi = k_n$,

iii. $y_n^* = gA_n$ if $\varphi < k_n$.

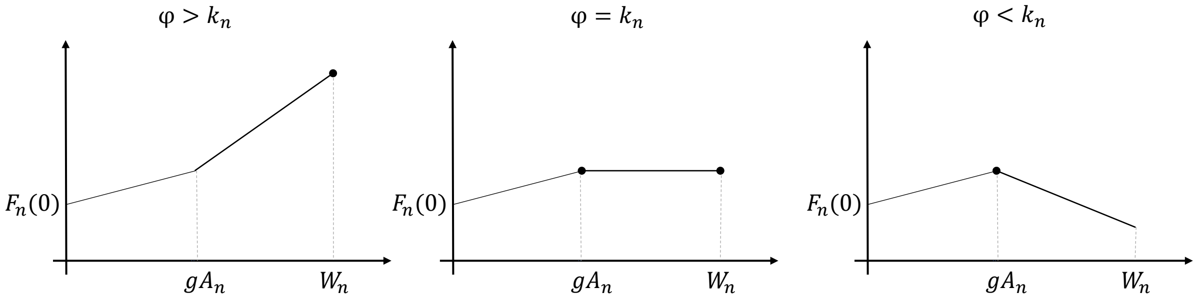

FIG. 1. The maximizer $y_n^* \in \{gA_n, W_n\}$

- If $gA_n \geq W_n$, then

$$F_n(y_n; W_n, A_n, \mu_n) = \begin{cases} \varphi y_n + (1-\varphi)W_n & \text{if } 0 \leq y_n \leq W_n \\ y_n & \text{if } W_n < y_n \leq gA_n \end{cases},$$

so that the maximizer of F_n is $y_n^* = gA_n$ (see Figure 2).

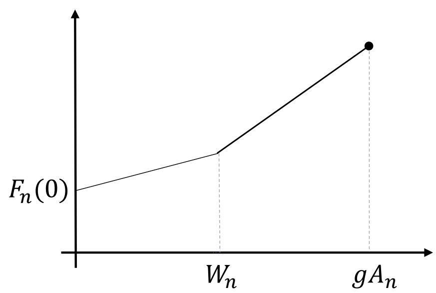

FIG. 2. The maximizer $y_n^* = gA_n$

Summing up, replacing the maximizers in F_n, we have the following three alternatives:

$$v_n(W_n, A_n, \mu_n) = \begin{cases} (1-k_n)W_n + k_n gA_n & \text{if } gA_n < W_n \text{ and } \varphi > k_n \\ (1-\varphi)W_n + \varphi gA_n & \text{if } gA_n < W_n \text{ and } \varphi \leq k_n \\ gA_n & \text{if } gA_n \geq W_n \end{cases}.$$

Then v_n is linear in both W_n and A_n, and independent of μ_n. More generally, we can express it as

$$v_n(W_n, A_n, \mu_n) = C_n(\mu_n)W_n + D_n(\mu_n)gA_n,$$

where the two (constant) functions C_n and D_n are such that $0 \leq C_n(\mu_n) < 1$ and $D_n(\mu_n) > 0$ since the penalty and fee rates k_n and φ are strictly between 0 and 1.

Iterative step. Assume now that $v_{i+1}(W_{i+1}, A_{i+1}, \mu_{i+1}) = C_{i+1}(\mu_{i+1})W_{i+1} + D_{i+1}(\mu_{i+1})gA_{i+1}$, where the two functions C_{i+1} and D_{i+1} are such that $0 \leq C_{i+1}(\mu_{i+1}) < 1$ and $D_{i+1}(\mu_{i+1}) > 0$ (almost surely), $i = n-1, n-2, \ldots, 1$. Then we have

$$v_i(W_i, A_i, \mu_i) = \sup_{0 \leq y_i \leq \max\{gA_i,\, W_i\}} F_i(y_i; W_i, A_i, \mu_i),$$

where

$$\begin{aligned} F_i(y_i; W_i, A_i, \mu_i) &= y_i - k_i \max\{y_i - gA_i,\, 0\} + q_i(\mu_i)\max\{W_i - y_i,\, 0\}(1-\varphi) \\ &+ \mathbb{E}^Q\left[\mathrm{e}^{-\int_i^{i+1}\mu_u\,\mathrm{d}u} C_{i+1}(\mu_{i+1})\mathrm{e}^{-r}\max\{W_i - y_i,\, 0\}(1+R_{i+1})(1-\varphi)|W_i, A_i, \mu_i\right] \\ &+ \mathbb{E}^Q\left[\mathrm{e}^{-\int_i^{i+1}\mu_u\,\mathrm{d}u} D_{i+1}(\mu_{i+1})\mathrm{e}^{-r} g f^A_{i+1}(W_i, A_i, y_i)|W_i, A_i, \mu_i\right]. \end{aligned} \tag{25}$$

Exploiting independence between financial and demographic factors in the first expectation of (25), as well as the martingale property, we get

$$\begin{aligned} F_i(y_i; W_i, A_i, \mu_i) &= y_i - k_i \max\{y_i - gA_i,\, 0\} + q_i(\mu_i)\max\{W_i - y_i,\, 0\}(1-\varphi) \\ &\quad + G_i(\mu_i)\max\{W_i - y_i,\, 0\}(1-\varphi) + H_i(\mu_i)\mathrm{e}^{-r} g f^A_{i+1}(W_i, A_i, y_i), \end{aligned}$$

where

$$G_i(\mu_i) = \mathbb{E}^Q\left[\mathrm{e}^{-\int_i^{i+1}\mu_u\,\mathrm{d}u} C_{i+1}(\mu_{i+1})|\mu_i\right], \quad H_i(\mu_i) = \mathbb{E}^Q\left[\mathrm{e}^{-\int_i^{i+1}\mu_u\,\mathrm{d}u} D_{i+1}(\mu_{i+1})|\mu_i\right].$$

Note that $0 \leq C_{i+1}(\mu_{i+1}) < 1$ and $D_{i+1}(\mu_{i+1}) > 0$ imply $0 \leq G_i(\mu_i) < p_i(\mu_i)$ and $H_i(\mu_i) > 0$.

- Then, if $gA_i < W_i$,

$$F_i(y_i; W_i, A_i, \mu_i) = \begin{cases} [q_i(\mu_i) + G_i(\mu_i)]\, W_i(1-\varphi) + H_i(\mu_i)\mathrm{e}^{-r} gA_i(1+b_i) \\ \qquad\qquad\qquad\qquad \text{if } y_i = 0 \\ \{1 - [q_i(\mu_i) + G_i(\mu_i)]\,(1-\varphi)\}\, y_i + [q_i(\mu_i) + G_i(\mu_i)]\, W_i(1-\varphi) \\ +H_i(\mu_i)\mathrm{e}^{-r} gA_i \qquad\qquad \text{if } 0 < y_i \leq gA_i \\ \{1 - k_i - [q_i(\mu_i) + G_i(\mu_i)]\,(1-\varphi) - H_i(\mu_i)\mathrm{e}^{-r}\frac{1}{W_i - gA_i} gA_i\} y_i \\ +k_i gA_i + [q_i(\mu_i) + G_i(\mu_i)]\, W_i(1-\varphi) + H_i(\mu_i)\mathrm{e}^{-r}\frac{W_i}{W_i - gA_i} gA_i \\ \qquad\qquad\qquad\qquad \text{if } gA_i < y_i \leq W_i \end{cases} \tag{26}$$

  Note that, being $G_i(\mu_i) < p_i(\mu_i)$, in the second line of (26) the slope belongs to $(0, 1)$ (hence F_i is increasing with y_i) while the intercept is lower than $F_i(0; W_i, A_i, \mu_i)$ as $b_i > 0$. It is not clear, instead, which is the sign of the slope in the third line. Hence, we can conclude that the maximizers of (26) belong to $\{0, gA_i, W_i\}$ (see Figure 3).

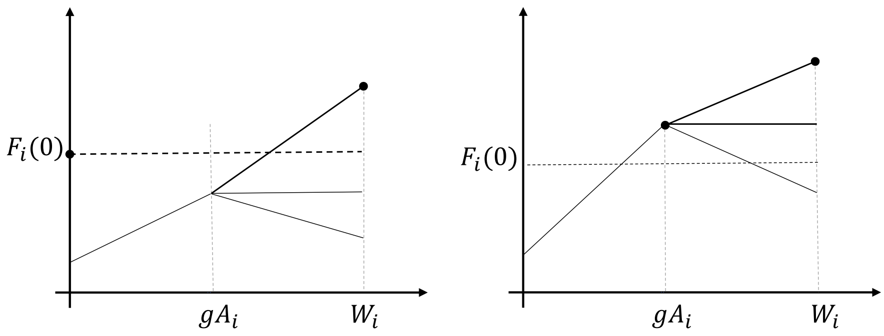

FIG. 3. The maximizer $y_i^* \in \{0, gA_i, W_i\}$

- If instead $gA_i \geq W_i$, then

$$F_i(y_i; W_i, A_i, \mu_i) = \begin{cases} [q_i(\mu_i) + G_i(\mu_i)]\, W_i(1-\varphi) + H_i(\mu_i)\mathrm{e}^{-r} gA_i(1+b_i) \\ \hfill \text{if } y_i = 0 \\ \{1 - [q_i(\mu_i) + G_i(\mu_i)]\,(1-\varphi)\}\, y_i + [q_i(\mu_i) + G_i(\mu_i)]\, W_i(1-\varphi) \\ +H_i(\mu_i)\mathrm{e}^{-r} gA_i \hfill \text{if } 0 < y_i \leq W_i \\ y_i + H_i(\mu_i)\mathrm{e}^{-r} gA_i \hfill \text{if } W_i < y_i \leq gA_i \end{cases} \tag{27}$$

The first two lines in (27) are the same as in (26), while the slope in the third line (= 1) is greater than in the second one. Hence now we can conclude that the maximizers of (27) belong to $\{0, gA_i\}$ (see Figure 4).

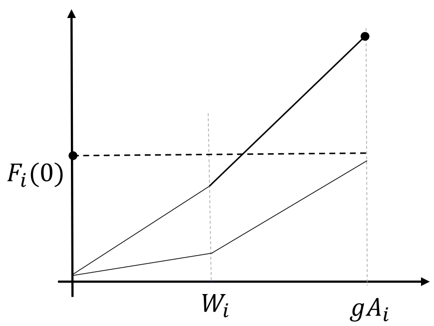

FIG. 4. The maximizer $y_i^* \in \{0, gA_i\}$

Summing up, replacing the maximizers in F_i, we have the following four alternatives:

$$v_i(W_i, A_i, \mu_i) = \begin{cases} [q_i(\mu_i) + G_i(\mu_i)]\,(1-\varphi)W_i + H_i(\mu_i)\mathrm{e}^{-r}(1+b_i)gA_i & \text{if } y_i^* = 0 \\ [q_i(\mu_i) + G_i(\mu_i)]\,(1-\varphi)W_i & \\ + \{1 - [q_i(\mu_i) + G_i(\mu_i)]\,(1-\varphi) + H_i(\mu_i)\mathrm{e}^{-r}\}\, gA_i & \text{if } y_i^* = gA_i < W_i \\ (1-k_i)W_i + k_i gA_i & \text{if } y_i^* = W_i > gA_i \\ [1 + H_i(\mu_i)\mathrm{e}^{-r}]\, gA_i & \text{if } y_i^* = gA_i \geq W_i \end{cases}$$

Therefore $v_i(W_i, A_i, \mu_i) = C_i(\mu_i)W_i + D_i(\mu_i)gA_i$, with $0 \leq C_i(\mu_i) < 1$ and $D_i(\mu_i) > 0$ (almost surely), and this concludes the proof. □

b. Algorithm

Now we outline the algorithm employed to value the GLWB variable annuity. Its execution requires a discretization over the state variables W, A, μ, and interpolation of the value function v over the resulting grid in order to compute the expectation in (23) and (24).

Step 0 For each $i = 1, 2, \ldots, n+1$ discretize the state space $[0, \infty)$ for W_i, $(0, \infty)$ for μ_i, and $(0, A^{\max}]$ for A_i, where $A^{\max} = P\prod_{j=1}^{n-1}(1+b_j)$:[10]

$$\begin{aligned} \mathbb{W} &= \{w_1, \ldots, w_L\}, & 0 &= w_1 < w_2 < \cdots < w_L, \\ \mathbb{A} &= \{a_1, \ldots, a_H\}, & 0 &< a_1 < a_2 < \cdots < a_H = A^{\max}, \\ \mathbb{M} &= \{m_1, \ldots, m_K\}, & 0 &< m_1 < m_2 < \cdots < m_K. \end{aligned}$$

Step 1. Start at $n+1$ by setting $v_{n+1}(w_\ell, a_h, m_k) = 0$ for each $(w_\ell, a_h, m_k) \in \mathbb{W} \times \mathbb{A} \times \mathbb{M}$.

Step 2. Proceed backwards: for $i = n, n-1, \ldots, 1:$

(I) interpolate the $L \cdot H \cdot K$ quadruples $(w_\ell, a_h, m_k, v_{i+1}(w_\ell, a_h, m_k))$, $\ell = 1, 2, \ldots, L$, $h = 1, 2, \ldots, H$ and $k = 1, 2, \ldots, K$, to construct the function $\tilde{v}_{i+1}(w, a, m)$ for $w \geq 0$, $0 < a \leq A^{\max}$, $m > 0$;

(II) for each $w_\ell, a_h, m_k \in \mathbb{W} \times \mathbb{A} \times \mathbb{M}$ compute

$$v_i(w_\ell, a_h, m_k) = \sup_{y \in \{0,\, g a_h,\, w_\ell\}} \Bigg\{ y - k_i \max\{y - g a_h, 0\} + q_i(m_k) \max\{w_\ell - y, 0\}(1-\varphi) + e^{-r} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \mathrm{e}^{-\xi}\, \tilde{v}_{i+1}(\tilde{w}\mathrm{e}^{\delta}, \tilde{a}, \eta)\, f_1(\delta)\, g_1^{(m_k)}(\xi, \eta)\, \mathrm{d}\delta\, \mathrm{d}\xi\, \mathrm{d}\eta \Bigg\},$$

[10] It would be sufficient do discretize A_i until $P\prod_{j=1}^{i-1}(1+b_j)$, but in this case the grid would depend on i.

where $\tilde{w} = \mathrm{e}^{r+d} \max\{w_\ell - y, 0\}(1-\varphi)$, $\tilde{a} = f^A_{i+1}(w_\ell, a_h, y)$, f_1 is the density of the Lévy process given by Equation (16), d is the drift adjustment (15), and $g_1^{(m_k)}$ is the joint density of $\int_i^{i+1} \mu_u \mathrm{d}u$ and μ_{i+1} conditional on $\mu_i = m_k$ defined in Equation (12) (with $s-t=1$ and $\mu_t = m_k$).

Step 3. The value of the contract at inception is

$$\begin{aligned} v_0(P, P, \mu_0) =& q_0(\mu_0) P (1-\varphi) \\ &+ e^{-r} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \mathrm{e}^{-\xi}\, \tilde{v}_1\left(P\mathrm{e}^{r+d+\delta}(1-\varphi), P, \eta\right)\, f_1(\delta)\, g_1^{(\mu_0)}(\xi,\eta)\, \mathrm{d}\delta\, \mathrm{d}\xi\, \mathrm{d}\eta. \end{aligned}$$

The densities f_1 and $g_1^{(m)}$ can be calculated through inversion of the characteristic function (14) and Laplace transform (11) (see Bailey and Swarztrauber (1994)) and the constant d via numerical integration. Similarly, numerical integration can be used to compute the integrals in Steps 2(II) and 3 of the algorithm.

c. Contract decomposition

It is clear that the valuation algorithm just described, aimed at producing the contract value under the dynamic approach, can be used to obtain, as simplified cases, also the contract values under alternative policyholder behaviours, namely under the static and the mixed approaches. To obtain the value under the static approach, in fact, it is sufficient to fix $y = ga_h$ in Step 2(II) without searching the maximum,[11] while to obtain the value under the mixed approach the search of the maximum must be restricted to the subset $\{ga_h,\, w_\ell\}$. To distinguish between these three different values we denote them, respectively, by $V_0^{dynamic}$, V_0^{static} and V_0^{mixed}.

Then we can see the dynamic contract as the combination of three components: the *basic GLWB contract*, i.e., the static one, the *surrender option* (with value given by $V_0^{surrender} := V_0^{mixed} - V_0^{static}$), and the *roll-up option* (whose value is $V_0^{rollup} := V_0^{dynamic} - V_0^{mixed}$):

$$V_0^{dynamic} = V_0^{static} + V_0^{surrender} + V_0^{rollup}.$$

5. Numerical examples

In this section we perform a sensitivity analysis comparing the numerical results obtained for different parameters and policyholder behaviours.

We focus on a contract with a single premium $P = 100$ and annual withdrawals, so that we use the standard unit of measurement of time, i.e., the year. We consider a policyholder aged $x = 65$ years at inception, which is in line with the retirement age in many countries.

[11] This value could also be obtained directly, without dynamic programming, as the value of a portfolio composed by a standard whole life annuity with constant instalment equal to gP, and a whole life assurance with death benefit given by the residual policy account value.

The parameters of the mortality intensity process defined in Equation (8) are the following: $\mu_0 = 0.009954829$, $\alpha = 0.0001$, $\theta = 0.10068752$, $\sigma_\mu = 0.01$. We have obtained them by fitting the survival probabilities for a 65 years old male implied by the projected life table IPS55. This table is commonly used in the Italian market and is focused on an individual born in 1955, hence with an age close to 65 at present (2022). The expected residual lifetime produced by the calibrated process is equal to 20.38, and we take $\omega = 118$, that is the extreme age in this life table, so that $n = 53$.

As already mentioned in Section 3, for the reference price process we choose the CGMY model with characteristic function defined in Equation (17). The parameters of this process are the same used in Kirkby and Nguyen (2021), i.e., $C = 0.02$, $G = 5$, $M = 15$, and $\mathcal{Y} = 1.2$. For the interest rate r, unless otherwise mentioned, we fix a base level of 2%, which is in line with the EUR EIOPA term structure of risk-free interest rates for very long maturities (exceeding 50 years).[12]

The contract parameters in the basic case are fixed as follows: penalty rate $k_i \equiv k = 5\%$, withdrawal rate $g = 3\%$, roll-up rate $b_i \equiv b = 8\%$, fee rate $\varphi = 10\%$.

The choice of the withdrawal rate is motivated by the fact that, with a (constant) interest rate of 2% and the survival probabilities implied by our model, the (actuarial) value of a whole-life annuity is 16.37, so that its reciprocal, that would be the conversion rate in a standard lifelong annuity, is 6.11%. However, if one buys such annuity, while being protected against the longevity risk (as in the case of the GLWB rider), completely looses the property of the investment fund. Then, if the policyholder dies during the first years of contract, her heirs do not receive anything while, in our case, the whole account value remains to them. In addition, within the annuity, surrender is never admitted in order to avoid adverse selection. That is the reason why we have fixed a withdrawal rate much lower than the conversion rate of the annuity.

As far as the roll-up rate is concerned, we have first verified, by performing some preliminary numerical calculations of the initial contract value, that lower roll-up levels drive to negligible values of the roll-up option, i.e., to values of the contract under the mixed and the dynamic approaches very close to each other. Then, to disincentive withdrawals, the roll-up rate should be significantly high. On the other hand, this is quite expected. In fact, taking into account that current interest rates are very low, and neglecting them, with e.g. an 8% roll-up rate it takes 12.5 years to recover a 0 withdrawal through an increase in the subsequent ones produced by the roll-up feature, and this recovery can take place only if the policyholder lives long enough.

Finally, also the fee rate that we have fixed may appear exaggeratedly high. However, it should be noted that this level of the fee cannot be compared with that required, e.g., by an accumulation guarantee. In fact, in that case, the fee rate would be applied on a (presumably) high and stable value of the policy account, while in our case this account is destinated to run out, especially if the policyholder lives long, and hence the recovered fees can be very low or even null after an initial period. In particular, our preliminary calculations show that, with the basic parameters here considered, the fair fee rate should be equal to 2.51% under the static approach, 40.12% under the mixed approach and

[12]See `http://www.eiopa.europa.eu/tools-and-data/risk-free-interest-rate-term-structures_en`.

41.40% under the dynamic approach. Of course we have not chosen a basic level for φ of the order of 40% because several empirical studies show that policyholders are not so active in the optimal exercise of their options,[13] specially during the decumulation phase of a life insurance contract, in which their main concern is the longevity risk rather than the investment performance. Moreover, recall that the contract values that we have computed under the dynamic and the mixed approaches cover the worst-case scenario for the insurer, but a more realistic prediction of the policyholder behaviour would lead to a fee rate not too far from that required under the static approach, so that our choice of 10% can be considered fairly conservative.

Coming now to the results of the numerical experiments, in Table 1 we study the initial contract value under the static (V_0^{static}), mixed (V_0^{mixed}) and dynamic ($V_0^{dynamic}$) approaches for different levels of the roll-up rate b. We also report the value in percentage of the static contract ($V_0^{static}(\%)$), the surrender option ($V_0^{surrender}(\%)$) and the roll-up option ($V_0^{rollup}(\%)$) to highlight the three components of the GLWB dynamic contract. Clearly the contract value under static and mixed scenarios is independent of the roll-up rate, while it increases under the dynamic approach. Moreover, as already anticipated, for levels of the roll-up rate not sufficiently high ($b \leq 6\%$), $V_0^{dynamic} = V_0^{mixed}$. In particular, the roll-up option component $V_0^{rollup}(\%)$ is bigger than zero (and increasing) only for $b \geq 7\%$; however, it represents the smallest component of the dynamic contract while the static $V_0^{static}(\%)$ and the surrender option $V_0^{surrender}(\%)$ components prevail. Note that the very high level of the fee rate (10%), compared with that required to make the contract fair under the static approach (2.51%), makes the weights of these last two components close enough, with even a prevalence of the surrender one.

TABLE 1. Contract values and contract components (in %) for different roll-up rates b

$b(\%)$	V_0^{static}	V_0^{mixed}	$V_0^{dynamic}$	$V_0^{static}(\%)$	$V_0^{surrender}(\%)$	$V_0^{rollup}(\%)$
6	61,44	125,87	125,87	48,81	51,19	0,00
7	61,44	125,87	127,27	48,28	50,62	1,10
8	61,44	125,87	129,26	47,53	49,84	2,63
9	61,44	125,87	132,39	46,41	48,67	4,92
10	61,44	125,87	136,74	44,94	47,12	7,95

In Table 2, the initial contract value in all scenarios is displayed for different levels of the penalty rate k. Clearly, V_0^{static} remains constant not depending on k, while V_0^{mixed} and $V_0^{dynamic}$ decrease. The significant difference between V_0^{static} and V_0^{mixed} is, once again, justified by the use of $\varphi = 10\%$, which is heavily penalizing for the static contract. Hence, even by taking a very high penalty, V_0^{mixed} results substantially higher than V_0^{static}. From preliminary numerical calculations we have verified that, to get $V_0^{static} = V_0^{mixed}$ and discourage surrender under the chosen set of basic parameters, the penalty rate should be $\geq 90\%$! Then, as in Table 1, the roll-up option remains the smallest component of the dynamic contract. It should be noticed that the decrease in the weight of the surrender component is not always offset by an increase in both the remaining components: in particular, the static contract weight always increases with k, but the roll-up option

[13]See, e.g., Bauer et al. (2017) and the references therein.

weight initially increases, reaches a maximum when $k = 5\%$, and after decreases. This shows that the effect on the contract composition of a change in a contractual parameter does not always go in the same direction depending on how all the parameters interact with each other.

TABLE 2. Contract values and contract components (in %) for different penalty rates k

$k(\%)$	V_0^{static}	V_0^{mixed}	$V_0^{dynamic}$	$V_0^{static}(\%)$	$V_0^{surrender}(\%)$	$V_0^{rollup}(\%)$
0	61,44	132,50	133,69	45,96	53,15	0,89
3	61,44	129,85	131,03	46,89	52,20	0,91
5	61,44	125,87	129,26	47,53	49,84	2,63
7	61,44	125,30	127,49	48,19	50,09	1,72
9	61,44	124,53	125,72	48,87	50,18	0,94

In Table 3 we move the withdrawal rate g up to 5%. Recall that, being the conversion rate associated to a standard whole life annuity equal to 6.11%, higher levels for g would result not at all conservative and would increase the value of the contract too much, even in the static case; that is why we do not consider them. Of course the value of the contract in all scenarios strongly increases with g. As far as the contract components are concerned, we can make similar comments as those related to Table 2: now the weight of the static component is obviously increasing and that of the surrender option is always decreasing since with a higher g the contract is more attractive and hence surrender is discouraged; the roll-up option weight, instead, initially increases, reaches a maximum when $g = 3\%$, and after becomes swinging.

TABLE 3. Contract values and contract components (in %) for different withdrawal rates g

$g(\%)$	V_0^{static}	V_0^{mixed}	$V_0^{dynamic}$	$V_0^{static}(\%)$	$V_0^{surrender}(\%)$	$V_0^{rollup}(\%)$
1	32,10	99,47	99,87	32,15	67,46	0,40
2	46,77	113,77	114,56	40,83	58,48	0,69
3	61,44	125,87	129,26	47,53	49,84	2,63
4	76,11	142,38	143,96	52,87	46,03	1,10
5	90,78	156,68	158,66	57,22	41,53	1,25

In Table 4 we vary the interest rate r (as usual keeping fixed the remaining basic parameters). As expected, we observe a general decreasing of the contract value in all scenarios. Moreover, for $r \geq 3.5\%$, $V_0^{mixed} = V_0^{dynamic}$: this means that when r is sufficiently high the exercise of the roll-up option does not return any benefit. Recall, in fact, that with $r = 0\%$ and $b = 8\%$ one year of non-withdrawal can be compensated by the increase of the guaranteed amount in 12.5 years. However, if r is higher, more time is needed because the following withdrawals are also subject to (financial) discount,[14] and hence count less. As expected, the effect of the interest rate is more pronounced under the static scenario, so that the weight of the static component is always decreasing and is

[14]In addition, of course, to the "demographic discount".

completely offset (at least for $r \geq 3.5\%$) by an increase in that of the surrender component. Finally, we have reported also the results obtained for exaggeratedly high levels of r in order to glimpse when a 10% fee rate would result fair even under the dynamic and mixed approaches, and this happens when $r \approx 15\%$.

TABLE 4. Contract values and contract components (in %) for different interest rates r

$r(\%)$	V_0^{static}	V_0^{mixed}	$V_0^{dynamic}$	$V_0^{static}(\%)$	$V_0^{surrender}(\%)$	$V_0^{rollup}(\%)$
1	66,68	133,18	136,52	48,84	48,71	2,45
2	61,44	125,87	129,26	47,53	49,84	2,63
3,5	55,03	121,83	121,83	45,17	54,83	0,00
5	49,95	116,88	116,88	42,74	57,26	0,00
7,5	43,59	110,68	110,68	39,38	60,62	0,00
11	37,54	104,78	104,78	35,83	64,17	0,00
13	35,06	102,35	102,35	34,26	65,74	0,00
15	33,05	100,39	100,39	32,92	67,08	0,00
17	31,4	98,78	98,78	31,79	68,21	0,00

Now, to highlight the influence of the fee rate on the initial contract value under the different policyholder behaviours we plot, in Figure 5, V_0^{static}, V_0^{mixed} and $V_0^{dynamic}$ against φ maintaining fixed all the remaining basic parameters. In particular, one can capture at first glance the decrease in V_0 with respect to φ as well as the differences among the contract values under the various approaches. Moreover, note that the fair fee rate φ^* is given by the abscissa of the intersection between the contract graph and the horizontal line at level $P = 100$: $\varphi^* = 2.51\%$ for the static value, $\varphi^* = 40.12\%$ for the mixed value and $\varphi^* = 41.40\%$ for the dynamic value.

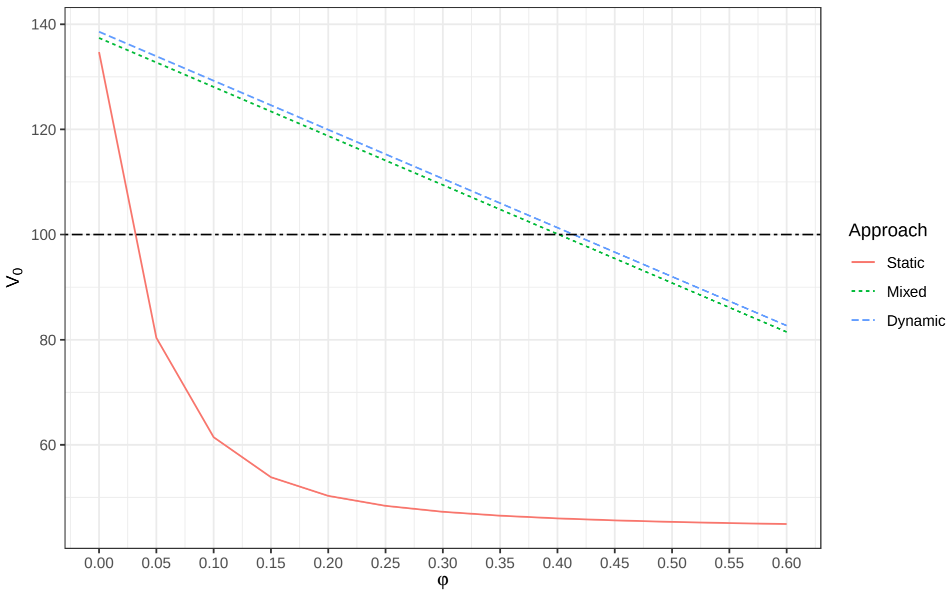

FIG. 5. Contract value against the fee rate φ under the different policyholder behaviours.

Finally, focusing on the dynamic approach, in Figure 6 we show $V_0^{dynamic}$ against φ for different roll-up rates b, while in Figure 7 we make the same for different penalty rates k. Again, for a given roll-up rate, or a given penalty rate, the contract value is decreasing with respect to φ. If instead we fix the fee rate, the contract value increases with b and decreases with k. As a consequence, also φ^* increases with b and decreases with k.

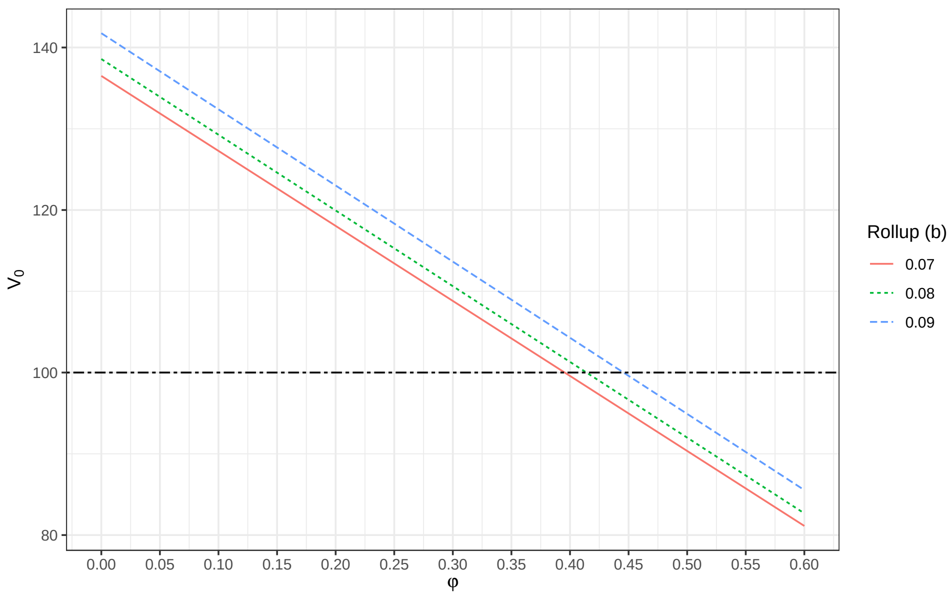

FIG. 6. Contract value in the dynamic approach for different fee and roll-up rates φ and b

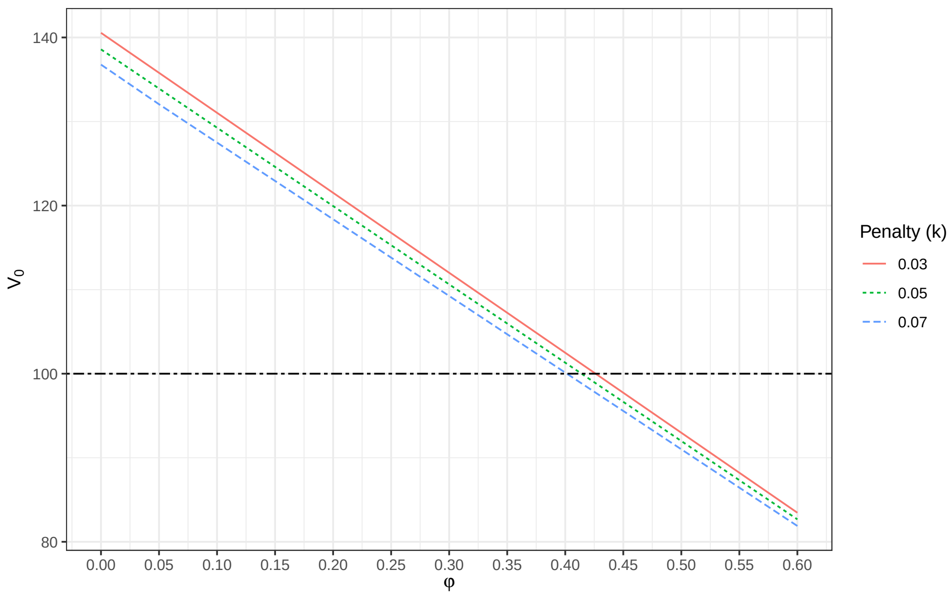

FIG. 7. Contract value in the dynamic approach for different fee and penalty rates φ and k

6. Conclusions

In this paper we have proposed a discrete time model, based on dynamic programming, to price GLWB variable annuities under the dynamic approach within a stochastic mortality framework. We have verified, by backward induction, the bang-bang condition for the set of discrete withdrawal strategies of the model, and offered an interesting contract decomposition. We have considered a quite general set-up, only requiring the Markovian property for the mortality intensity and the asset price processes. However, to keep the curse of dimensionality of our valuation algorithm manageable, we have assumed constant interest rates. Then, we have numerically implemented the model by focusing on a non mean-reverting square root process for the mortality intensity and an exponential Lévy process for the asset price. In particular, we have considered the Carr, Geman, Madan, Yor (CGMY) process and computed the initial value of the contract for different parameters and policyholder behaviours. In such sensitivity analysis we have also highlighted the components into which the contract can be splitted and the fair fees under the various scenarios.

The results obtained under the mixed and the dynamic approaches show that the cost of the guarantee, in terms of fair fee rate, seems excessively high, especially in a low interest rate environment, and therefore makes the product unmarketable. Recall, however, that this is apparently an excessive cost, because the fee rate is applied on a decreasing account value and it is expected that, after an initial period, the periodical fees subtracted from this account become very low or even null.

Of course the fee rate proposed to the customers could be lowered, with respect to the fair one resulting from our dynamic model, by taking into account that their actual behaviour can lead to a cost not too far from that required under the static approach (as we have made when fixing the basic parameters in our numerical experiments). Nevertheless, to make the product more attractive, some action needs to be taken.

A first action could be to apply the fee rate on the benefit base instead of on the account value: the problem that fees are subtracted only if the account value is sufficient would remain, but at least the fair fee rate would result lower because the base on which fees are applied would be high and stable. Although this action does not really involve a cost reduction, it can help in terms of impact on potential customers.

An action that instead may imply a drastic reduction in the cost is to disallow withdrawals exceeding the guaranteed amount, i.e., taking into account the bang-bang condition, to disallow surrender. In this case the roll-up option can be maintained, and also the remaining account value would be paid as a death benefit. Hence, in some way, the product becomes very close to an annuity with capital protection (or money-back annuity), along with some elements of flexibility given by the possibility to withdraw less than the guaranteed amount (once again, taking into account the bang-bang condition, this reduces to the roll-up option). Moreover, the withdrawal rate g can be higher than what we have fixed in our numerical examples, without excessively raising the cost of the guarantee, and the choice of suitable combinations of (g, φ, b) making contracts fair can be let to the policyholder.

Conversely, a way to leave more flexibility to the policyholder in order to permit her to cope with possible liquidity needs could be to allow withdrawals exceeding the guaranteed amount only up to a maximum limit, e.g. 1.5/2 times this amount (provided the account value is large enough and subject to penalties), but not complete surrender. Of course this last solution may result expensive, but anyway less than what we have found in the paper under a (complete) dynamic approach.

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