Rend. Istit. Mat. Univ. Trieste
Vol. XXXIV, 121–142 (2002)

Pointwise Versions of Solutions to Cauchy Problems in L^p-spaces

Wolfgang Desch and Krista W. Homan [(*)]

Summary. - *We consider a Cauchy problem*

$$\frac{\partial}{\partial t}\varphi(t,\omega) = (\mathcal{A}\varphi(t,\,\cdot))(\omega), t>0,\ \omega\in\Omega,$$
$$\varphi(0,\omega) = \varphi_0(\omega), \qquad \omega\in\Omega,$$

and assume that it can be solved by a strongly continuous semigroup on a Banach space valued function space $L^p(\Omega;X)$. *For fixed* $t>0$ *the solution* $\varphi(t,\omega)$ *is only defined almost everywhere on* Ω. *Therefore it is not obvious what kind of regularity* $t\mapsto\varphi(t,\omega)$ *has for fixed* $\omega\in\Omega$. *We show that if the semigroup is analytic, then there exists a version of* $\varphi(t,\,\cdot)$ *such that for almost every* $\omega\in\Omega$, $t\mapsto\varphi(t,\omega)$ *is analytic in* $(0,\infty)$.

1. Introduction and notations

Let $(\Omega,\mathcal{F},\mu)$ be a measure space, $(X,\|\cdot\|)$ a Banach space, and $1\le p<\infty$. For a function $\varphi:\Omega\to X$ we denote by $[\varphi]$ the equivalence class of functions $\psi:\Omega\to X$ such that for almost every

---

(*) Authors' addresses: Wolfgang Desch, Institut für Mathematik, Karl-Franzens-Universität Graz, Heinrichstraße 36, A-8010 Graz, Austria.
Krista W. Homan, Department of Applied Mathematical Analysis, Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands.
This work was partially supported by the Spezialforschungsbereich SFB F-003 Optimierung und Kontrolle at the Karl-Franzens-Universität und Technische Universität Graz, and by the Dutch Organization for Scientific Research NWO 613-006-558.

$\omega \in \Omega$, $\psi(\omega) = \varphi(\omega)$. Functions φ and ψ are called versions of the equivalence class $[\varphi]$. Moreover, we denote

$$\begin{aligned} M(\Omega; X) &= \{[\varphi];\ \varphi : \Omega \to X \text{ strongly measurable}\}, \\ \mathcal{L}^p(\Omega; X) &= \Big\{\phi : \Omega \to X;\ \phi \text{ strongly measurable and} \\ &\qquad\qquad \int_\Omega \|\phi(\omega)\|^p \,\mu(\mathrm{d}\omega) < \infty\Big\}, \\ L^p(\Omega; X) &= \{[\phi] \in M(\Omega; X);\ \phi \in \mathcal{L}^p(\Omega; X)\}. \end{aligned}$$

In $\mathcal{L}^p(\Omega; X)$ we consider the Cauchy problem

$$\begin{aligned} \frac{\partial}{\partial t}\varphi(t,\omega) &= (\mathcal{A}\varphi(t,\,\cdot))(\omega), t > 0,\ \omega \in \Omega, \\ \varphi(0,\omega) &= \varphi_0(\omega), \qquad\qquad\qquad \omega \in \Omega, \end{aligned} \tag{1}$$

where $\mathcal{A} : \mathrm{D}(\mathcal{A}) \subseteq \mathcal{L}^p(\Omega; X) \to \mathcal{L}^p(\Omega; X)$ is a linear operator. Frequently, $\mathcal{A}$ is a partial differential operator with respect to ω and Ω a domain in $\mathbb{R}^N$, but it need not be so. We assume that problem (1) can be rewritten in $L^p(\Omega; X)$ as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}[\varphi(t)] &= A[\varphi(t)], t > 0, \\ [\varphi(0)] &= [\varphi_0], \end{aligned} \tag{2}$$

where $A : \mathrm{D}(A) \subseteq L^p(\Omega; X) \to L^p(\Omega; X)$ is the infinitesimal generator of a strongly continuous semigroup $\{e^{tA}\}_{t \geq 0}$ on $L^p(\Omega; X)$. Then problem (2) admits a mild solution $[\varphi(t)] = e^{tA}[\varphi_0]$ and problem (1) seems to be solved.

However, this is not the case, yet. Problem (1) requires for $t > 0$ a version $\varphi(t,\,\cdot)$ of $[\varphi(t)]$. If $\mathcal{A}$ is a partial differential operator and $\{e^{tA}\}_{t \geq 0}$ exhibits some smoothing, then we simply choose $\varphi(t,\,\cdot)$ to be the unique version of $[\varphi(t)]$ such that $\omega \mapsto \varphi(t,\omega)$ is smooth.

If $\mathcal{A}$ is not a partial differential operator, then the choice of a version $\varphi(t,\,\cdot)$ is less evident. To see that this is a nontrivial problem we consider the example where $\{e^{tA}\}_{t \geq 0}$ is a strongly continuous 1-periodic translation semigroup on $L^1(\mathbb{R}; \mathbb{R})$ and where φ_0 is such that every version $\varphi(t,\,\cdot)$ of $[\varphi(t,\,\cdot)] = e^{tA}[\varphi_0]$ has the property that for almost every $\omega \in \mathbb{R}$, $t \mapsto \varphi(t,\omega)$ is discontinuous at every $t > 0$, see Example 3.3.

It helps if $\{e^{tA}\}_{t\geq 0}$ is an analytic semigroup. In this paper we work out the consequences of a result of Stein, see [4, Lemma, page 72], stating that if $t \mapsto [\varphi(t)]$ is analytic from $(0,\infty)$ into $L^p(\Omega; X)$ with $1 < p < \infty$, then for every $t > 0$, $[\varphi(t)]$ has a version $\varphi(t, \cdot)$ such that for almost every $\omega \in \Omega$, $t \mapsto \varphi(t,\omega)$ is analytic from $(0,\infty)$ into X. More precisely, we prove the following theorem:

THEOREM 1.1. *Let X be a complex Banach space. Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space and $\Sigma \subseteq \mathbb{C}$ an open subset. Let for some $1 \leq p < \infty$, $\Phi : \Sigma \to L^p(\Omega; X)$ be an analytic function. Then there exists a function $\varphi : \Sigma \times \Omega \to X$ with the following properties:*

(i) *φ is strongly measurable;*

(ii) *For every $\omega \in \Omega$, $z \mapsto \varphi(z,\omega)$ is analytic in Σ;*

(iii) *For every $z \in \Sigma$ and $j \in \{0, 1, 2, \cdots\}$,*

$$\left[\frac{\partial^j}{\partial z^j}\varphi(z, \cdot)\right] = \frac{d^j}{dz^j}\Phi(z).$$

Section 2 is devoted to the proof of Theorem 1.1. In Section 3 we show how Theorem 1.1 can be applied to the Cauchy problem (1) in $\mathcal{L}^p(\Omega; X)$. Finally, Section 4 gives an application to a semigroup setting for an integral equation where $\mathcal{A}$ is a perturbation of a multiplier rather than a partial differential operator.

Throughout the paper we use the following notations. We write $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and

$$B(z_0, r) = \{z \in \mathbb{C};\ |z - z_0| < r\}, z_0 \in \mathbb{C},\ r > 0.$$

For $F \in \mathcal{F}$ we define the function $\mathbb{1}_F : \Omega \to \mathbb{R}$ by

$$\mathbb{1}_F(\omega) := \begin{cases} 1, & \omega \in F, \\ 0, & \omega \in \Omega \backslash F. \end{cases}$$

2. Versions of analytic functions in $\mathbf{L^p(\Omega; X)}$

The following lemma is essentially a result of Stein, see [4, Lemma, page 72], rewritten for our convenience.

LEMMA 2.1. *Let* $(X, \|\cdot\|)$ *be a complex Banach space. Let* $(\Omega, \mathcal{F}, \mu)$ *be a finite measure space. Let* $z_0 \in \mathbb{C}$, $r > 0$, *and* $\Sigma := B(z_0, r)$. *Let* $\Phi : \Sigma \to L^1(\Omega; X)$ *be a function with an analytic extension to a neighborhood of* $\overline{\Sigma}$. *Then there exists a function* $\varphi : \Sigma \times \Omega \to X$ *with the following properties:*

(i) φ *is strongly measurable;*

(ii) *For every* $\omega \in \Omega$, $z \mapsto \varphi(z, \omega)$ *is analytic in* Σ;

(iii) *For every* $z \in \Sigma$ *and* $j \in \mathbb{N}_0$,

$$\left[\frac{\partial^j}{\partial z^j}\varphi(z,\,\cdot)\right] = \frac{d^j}{dz^j}\Phi(z).$$

Proof. Let $E := L^1(\Omega; X)$. Without loss of generality we assume that $z_0 = 0$. First we construct a function $\varphi : \Sigma \times \Omega \to X$. Since Φ is analytic in a neighborhood of $\overline{\Sigma}$ there exists a sequence $\{C_k\}_{k=0}^{\infty}$ in E such that

$$\Phi(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} C_k, z \in \overline{\Sigma}. \tag{3}$$

Moreover, the power series in (3) has radius of convergence larger than r and therefore

$$\sum_{k=0}^{\infty} \frac{r^k}{k!} \|C_k\|_E < \infty.$$

For every $k \in \mathbb{N}_0$ we choose a representative $c_k : \Omega \to X$ of the equivalence class C_k. Using Fubini's theorem we then have

$$\int_\Omega \sum_{k=0}^{\infty} \frac{r^k}{k!} \|c_k(\omega)\|\, \mu(\mathrm{d}\omega) = \sum_{k=0}^{\infty} \frac{r^k}{k!} \int_\Omega \|c_k(\omega)\|\, \mu(\mathrm{d}\omega) = \sum_{k=0}^{\infty} \frac{r^k}{k!} \|C_k\|_E < \infty.$$

This implies that there exists a nullset $N \subseteq \Omega$ such that

$$\sum_{k=0}^{\infty} \frac{r^k}{k!} \|c_k(\omega)\| < \infty, \omega \in \Omega \backslash N. \tag{4}$$

Now we define $\varphi : \Sigma \times \Omega \to X$ by

$$\varphi(z,\omega) := \begin{cases} \sum_{k=0}^{\infty} \frac{z^k}{k!} c_k(\omega), & z \in \Sigma,\ \omega \in \Omega \backslash N, \\ 0, & z \in \Sigma,\ \omega \in N. \end{cases}$$

Note that (4) implies that φ is well-defined and hence, for every $\omega \in \Omega$, $z \mapsto \varphi(z,\omega)$ is analytic in Σ, that is, φ has property (ii).

Now we show that φ has property (i). For every $k \in \mathbb{N}_0$, $z \mapsto \frac{z^k}{k!}$ is Borel-measurable on Σ and $\omega \mapsto c_k(\omega)$ is strongly measurable on Ω since $C_k \in E$. Thus for every $k \in \mathbb{N}_0$, $(z,\omega) \mapsto \frac{z^k}{k!} c_k(\omega)$ is strongly measurable on $\Sigma \times \Omega$ and therefore $(z,\omega) \mapsto \sum_{k=0}^{\infty} \frac{z^k}{k!} c_k(\omega)$ is strongly measurable on $\Sigma \times \Omega \backslash N$ as pointwise limit of finite sums. It follows that φ is strongly measurable on $\Sigma \times \Omega$.

To show that φ has property (iii) we fix $j \in \mathbb{N}_0$ and observe that

$$\frac{\partial^j}{\partial z^j} \varphi(z,\omega) = \sum_{k=0}^{\infty} \frac{z^k}{k!} c_{k+j}(\omega), z \in \Sigma,\ \omega \in \Omega \backslash N.$$

For every $n \in \mathbb{N}$ let $\varphi_{j,n} : \Sigma \times \Omega \to X$ be defined by

$$\varphi_{j,n}(z,\omega) := \begin{cases} \sum_{k=0}^{n} \frac{z^k}{k!} c_{k+j}(\omega), & z \in \Sigma,\ \omega \in \Omega \backslash N, \\ 0, & z \in \Sigma,\ \omega \in N. \end{cases}$$

Now we also fix $z \in \Sigma$. On the one hand we have using (3),

$$\lim_{n \to \infty} [\varphi_{j,n}(z,\ \cdot)] = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{z^k}{k!} C_{k+j} = \sum_{k=0}^{\infty} \frac{z^k}{k!} C_{k+j} = \frac{\mathrm{d}^j}{\mathrm{d}z^j} \Phi(z), \quad (5)$$

where the convergence is in E. Note that the power series in (5) has radius of convergence larger than r. On the other hand we have

$$\lim_{n \to \infty} \varphi_{j,n}(z,\omega) = \frac{\partial^j}{\partial z^j} \varphi(z,\omega), \omega \in \Omega,$$

where the convergence is in X. We can apply Lebesgue's dominated convergence theorem since

$$\|\varphi_{j,n}(z,\omega)\| \leq \sum_{k=0}^{n} \frac{|z|^k}{k!} \|c_{k+j}(\omega)\| \leq \sum_{k=0}^{\infty} \frac{r^k}{k!} \|c_{k+j}(\omega)\|, \omega \in \Omega \backslash N,\ n \in \mathbb{N},$$

and, using Fubini's theorem,

$$\int_{\Omega\setminus N}\sum_{k=0}^{\infty}\frac{r^k}{k!}\|c_{k+j}(\omega)\|\,\mu(\mathrm{d}\omega)=\sum_{k=0}^{\infty}\frac{r^k}{k!}\|C_{k+j}\|_E<\infty.$$

Thus we get

$$\lim_{n\to\infty}\int_{\Omega}\left\|\varphi_{j,n}(z,\omega)-\frac{\partial^j}{\partial z^j}\varphi(z,\omega)\right\|\,\mu(\mathrm{d}\omega)=0$$

and hence,

$$\lim_{n\to\infty}[\varphi_{j,n}(z,\,\cdot)]=\left[\frac{\partial^j}{\partial z^j}\varphi(z,\,\cdot)\right],\tag{6}$$

where the latter convergence is in E. By combining (5) and (6) we obtain that φ has property (iii). □

We use Lemma 2.1 to prove that the same result holds for any open set $\Sigma\subseteq\mathbb{C}$.

LEMMA 2.2. *Let X be a complex Banach space. Let $(\Omega,\mathcal{F},\mu)$ be a finite measure space and let $\Sigma\subseteq\mathbb{C}$ be an open subset. Let $\Phi:\Sigma\to L^1(\Omega;X)$ be an analytic function. Then there exists a function $\varphi:\Sigma\times\Omega\to X$ with the properties (i), (ii), and (iii) stated in Lemma 2.1.*

Proof. As Σ is open in $\mathbb{C}$ it can be covered by countably many open balls. For every $k\in\mathbb{N}$ let $z_k\in\Sigma$ and $r_k>0$ be such that $\overline{B(z_k,r_k)}\subseteq\Sigma$ and $\Sigma=\bigcup_{k=1}^{\infty}B(z_k,r_k)$. It follows from Lemma 2.1 applied to each $\Phi|_{B(z_k,r_k)}$ that for every $k\in\mathbb{N}$ there exists a strongly measurable function $\varphi_k:B(z_k,r_k)\times\Omega\to X$ such that for every $\omega\in\Omega$, $z\mapsto\varphi_k(z,\omega)$ is analytic in $B(z_k,r_k)$ and

$$\left[\frac{\partial^j}{\partial z^j}\varphi_k(z,\,\cdot)\right]=\frac{\mathrm{d}^j}{\mathrm{d}z^j}\Phi(z), z\in B(z_k,r_k),\ j\in\mathbb{N}_0.\tag{7}$$

We shall construct a function $\varphi:\Sigma\times\Omega\to X$. Therefore we define for every $z\in\Sigma$,

$$I_z:=\{k\in\mathbb{N};\ z\in B(z_k,r_k)\}$$

and for every $k,\ l \in I_z$

$$N_{k,l,z} := \{\omega \in \Omega;\ \varphi_k(z,\omega) \neq \varphi_l(z,\omega)\}.$$

It follows from (7) with $j = 0$ that for every $z \in \Sigma$ and $k,\ l \in I_z$, $[\varphi_k(z,\ \cdot)] = \Phi(z) = [\varphi_l(z,\ \cdot)]$ and hence, $N_{k,l,z}$ is a nullset in Ω. For every $z \in \Sigma$ we define the nullset

$$N_z := \bigcup_{k,l \in I_z} N_{k,l,z}.$$

Furthermore, let $\Sigma_0 \subseteq \Sigma$ be a countable dense subset and

$$N := \bigcup_{z \in \Sigma_0} N_z.$$

Note that for every $\omega \in \Omega \backslash N$,

$$\varphi_k(z,\omega) = \varphi_l(z,\omega), z \in \Sigma_0,\ k, l \in I_z. \tag{8}$$

We remark that (8) even holds for every $z \in \Sigma$, since Σ_0 is dense in Σ and for every $k \in \mathbb{N}$, $z \mapsto \varphi_k(z,\omega)$ is continuous. Finally we define $\varphi : \Sigma \times \Omega \to X$ by

$$\varphi(z,\omega) := \begin{cases} \varphi_k(z,\omega), & z \in \Sigma,\ \omega \in \Omega \backslash N,\ k \in I_z, \\ 0, & z \in \Sigma,\ \omega \in N. \end{cases}$$

By construction, φ is independent of the choice of k. Moreover, it follows from (7) that φ satisfies

$$\left[\frac{\partial^j}{\partial z^j}\varphi(z,\omega)\right] = \frac{\mathrm{d}^j}{\mathrm{d}z^j}\Phi(z), z \in \Sigma,\ j \in \mathbb{N}_0.$$

Now we show that for every $\omega \in \Omega$, $z \mapsto \varphi(z,\omega)$ is analytic in every $z_0 \in \Sigma$. As this is obvious when $\omega \in N$, we fix any $\omega \in \Omega \backslash N$. Let $z_0 \in \Sigma$ and $k \in I_{z_0}$, so that $z_0 \in B(z_k, r_k)$ and hence, there exists $\varepsilon > 0$ such that $B(z_0, \varepsilon) \subseteq B(z_k, r_k)$. Since $z \mapsto \varphi(z,\omega) = \varphi_k(z,\omega)$ is analytic in $B(z_k, r_k)$, in particular in $B(z_0, \varepsilon)$, it follows that $z \mapsto \varphi(z,\omega)$ is analytic in z_0.

To show that φ is strongly measurable we construct a disjoint partition $\{B_k\}_{k=1}^{\infty}$ of Σ such that $\bigcup_{k=1}^{\infty} B_k = \Sigma$ via

$$B_k := \begin{cases} B(z_1, r_1), & k = 1, \\ B(z_k, r_k) \setminus \bigcup_{l=1}^{k-1} B(z_l, r_l), & k = 2, 3, \dots . \end{cases}$$

For every $k \in \mathbb{N}$ we define $\tilde{\varphi}_k : \Sigma \times \Omega \to X$ by

$$\tilde{\varphi}_k(z, \omega) := \begin{cases} \varphi_k(z, \omega), & z \in B_k,\, \omega \in \Omega \setminus N, \\ 0, & z \in B_k,\, \omega \in N, \\ 0, & z \in \Sigma \setminus B_k,\, \omega \in \Omega. \end{cases}$$

Note that

$$\varphi(z, \omega) = \sum_{k=1}^{\infty} \tilde{\varphi}_k(z, \omega), z \in \Sigma,\, \omega \in \Omega.$$

We remark that for every $k \in \mathbb{N}$, $\tilde{\varphi}_k$ is strongly measurable on $\Sigma \times \Omega$, since φ_k is strongly measurable on $B(z_k, r_k) \times \Omega$, in particular on $B_k \times \Omega \setminus N$. Therefore, φ is strongly measurable on $\Sigma \times \Omega$ as pointwise limit of finite sums. This proves that φ has the properties (i), (ii), and (iii) in Lemma 2.1. $\square$

The next lemma extends the result of Lemma 2.2 to σ-finite measure spaces. The assumptions in the lemma look rather technical, but they essentially concern locally integrable functions.

LEMMA 2.3. *Let X be a complex Banach space. Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space and $\{\Omega_k\}_{k=1}^{\infty}$ a sequence in $\mathcal{F}$ such that $\bigcup_{k=1}^{\infty} \Omega_k = \Omega$. Let $\Sigma \subseteq \mathbb{C}$ be an open subset. Let $\Phi : \Sigma \to M(\Omega; X)$ be such that for every $k \in \mathbb{N}$ the function $\Phi_k : \Sigma \to M(\Omega_k; X)$ given by $\Phi_k(z) := \Phi(z)|_{\Omega_k}$ for $z \in \Sigma$, satisfies* $\mathrm{Ran}(\Phi_k) \subseteq L^1(\Omega_k; X)$ *and $\Phi_k : \Sigma \to L^1(\Omega_k; X)$ is analytic.*

Then there exists a function $\varphi : \Sigma \times \Omega \to X$ with the following properties:

(i) *φ is strongly measurable;*

(ii) *For every $\omega \in \Omega$, $z \mapsto \varphi(z, \omega)$ is analytic in Σ;*

(iii) *For every* $z \in \Sigma$, $[\varphi(z, \cdot)] = \Phi(z)$;

(iv) *For every* $z \in \Sigma$, $j \in \mathbb{N}_0$, *and* $k \in \mathbb{N}$,

$$\left[\frac{\partial^j}{\partial z^j}\varphi(z, \cdot)|_{\Omega_k}\right] = \frac{d^j}{dz^j}\Phi_k(z).$$

Proof. Without loss of generality we assume that $\{\Omega_k\}_{k=1}^{\infty}$ is pairwise disjoint in $\mathcal{F}$ and for every $k \in \mathbb{N}$, $\mu(\Omega_k) < \infty$. It follows from Lemma 2.2 applied to each Φ_k that for every $k \in \mathbb{N}$ there exists a strongly measurable function $\varphi_k : \Sigma \times \Omega_k \to X$ such that for every $\omega \in \Omega_k$, $z \mapsto \varphi_k(z, \omega)$ is analytic in Σ and

$$\left[\frac{\partial^j}{\partial z^j}\varphi_k(z, \cdot)\right] = \frac{\mathrm{d}^j}{\mathrm{d}z^j}\Phi_k(z), z \in \Sigma,\ j \in \mathbb{N}_0. \tag{9}$$

We define $\varphi : \Sigma \times \Omega \to X$ by

$$\varphi(z, \omega) := \sum_{k=1}^{\infty} \varphi_k(z, \omega)\mathbb{1}_{\Omega_k}(\omega), z \in \Sigma,\ \omega \in \Omega.$$

Then φ is well-defined since $\{\Omega_k\}_{k=1}^{\infty}$ is pairwise disjoint, and φ has properties (i) and (ii). Note that for every $z \in \Sigma$ and $k \in \mathbb{N}$, $\varphi(z, \cdot)|_{\Omega_k} = \varphi_k(z, \cdot)$. Hence, it follows from (9) that φ also has property (iv). In particular if $j = 0$, then

$$[\varphi(z, \cdot)|_{\Omega_k}] = \Phi_k(z, \cdot) = \Phi(z)|_{\Omega_k}, z \in \Sigma,\ k \in \mathbb{N}.$$

As $\bigcup_{k=1}^{\infty} \Omega_k = \Omega$, this implies that φ has property (iii). □

Note that if for some $1 \leq p < \infty$ the range of $\Phi : \Sigma \to M(\Omega; X)$ is contained in $L^p(\Omega; X)$ and $\Phi : \Sigma \to L^p(\Omega; X)$ is analytic, then Φ satisfies the assumptions in Lemma 2.3. Therefore Theorem 1.1 is proved.

3. Versions of solutions to Cauchy problems in $\mathbf{L^p}$-spaces

In the next theorem we apply Theorem 1.1 to solve the Cauchy problem (1). We assume that $\mathcal{A}$ satisfies the following hypothesis:

HYPOTHESIS 3.1. *Let X be a complex Banach space, $(\Omega, \mathcal{F}, \mu)$ a σ-finite measure space, and $1 \le p < \infty$. The linear operator $\mathcal{A} : D(\mathcal{A}) \subseteq \mathcal{L}^p(\Omega; X) \to \mathcal{L}^p(\Omega; X)$ has the following properties:*

(i) *If $\varphi_1 \in D(\mathcal{A})$ and $\varphi_2 \in \mathcal{L}^p(\Omega; X)$ such that $\varphi_1 = \varphi_2$ almost everywhere, then $\varphi_2 \in D(\mathcal{A})$ and $\mathcal{A}\varphi_1 = \mathcal{A}\varphi_2$ almost everywhere.*

(ii) *If $\{\varphi_n\}_{n=1}^{\infty}$ is a sequence in $D(\mathcal{A})$ and if there exist φ, $\psi \in \mathcal{L}^p(\Omega; X)$ and a nullset $N \subseteq \Omega$ such that*

$$\begin{aligned} \lim_{n\to\infty} \varphi_n(\omega) &= \varphi(\omega), \omega \in \Omega \backslash N, \\ \lim_{n\to\infty} (\mathcal{A}\varphi_n)(\omega) &= \psi(\omega), \omega \in \Omega \backslash N, \\ \lim_{n\to\infty} [\varphi_n] &= [\varphi], \\ \lim_{n\to\infty} [\mathcal{A}\varphi_n] &= [\psi], \end{aligned}$$

where the convergence in the first two lines is in X and in the last two lines in $L^p(\Omega; X)$, then $\varphi \in D(\mathcal{A})$ and for every $\omega \in \Omega \backslash N$, $\psi(\omega) = (\mathcal{A}\varphi)(\omega)$.

We remark that if $\mathcal{A}$ satisfies Hypothesis 3.1, then we can define a linear operator $A : \mathrm{D}(A) \subseteq L^p(\Omega; X) \to L^p(\Omega; X)$ by

$$\mathrm{D}(A) := \{\Phi \in L^p(\Omega; X);\ \text{there exists } \varphi \in \mathrm{D}(\mathcal{A}) \text{ such that } [\varphi] = \Phi\},$$

$$A\Phi := [\mathcal{A}\varphi], \Phi \in \mathrm{D}(A). \tag{10}$$

This operator is well-defined by Hypothesis 3.1(i).

THEOREM 3.2. *Let $(X, \|\cdot\|)$ be a complex Banach space. Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space. Let for some $1 \le p < \infty$, $\mathcal{A} : D(\mathcal{A}) \subseteq \mathcal{L}^p(\Omega; X) \to \mathcal{L}^p(\Omega; X)$ satisfy Hypothesis 3.1. Let $A : D(A) \subseteq L^p(\Omega; X) \to L^p(\Omega; X)$, defined by (10), be the infinitesimal generator of a semigroup $\{S(t)\}_{t\ge 0}$ on $L^p(\Omega; X)$ that for some $0 < \vartheta \le \pi$ has an analytic extension to*

$$\Sigma := \{z \in \mathbb{C} \backslash \{0\};\ |\mathrm{Arg}(z)| < \vartheta\}.$$

Let $\varphi_0 \in \mathcal{L}^p(\Omega; X)$ and let the analytic function $\Phi : \Sigma \to L^p(\Omega; X)$ be defined by

$$\Phi(z) := S(z)[\varphi_0], z \in \Sigma.$$

Then there exist a function $\varphi : \Sigma \times \Omega \to X$ and a nullset $N \subseteq \Omega$ with the following properties:

(i) *φ is strongly measurable;*

(ii) *For every $z \in \Sigma$, $[\varphi(z, \cdot)] = \Phi(z)$;*

(iii) *For every $\omega \in \Omega$, $z \mapsto \varphi(z, \omega)$ is analytic in Σ;*

(iv) *For every $z \in \Sigma$ and $\omega \in \Omega \backslash N$,*

$$\frac{\partial}{\partial z}\varphi(z,\omega) = (\mathcal{A}\varphi(z, \cdot))(\omega);$$

(v) *If $\varphi_0 \in D(\mathcal{A})$, then for every $\omega \in \Omega \backslash N$, $\lim_{t\downarrow 0} \varphi(t,\omega)$ exists where the convergence is in X, and*

$$\left[\lim_{t\downarrow 0} \varphi(t, \cdot)\right] = [\varphi_0].$$

Proof. We apply Theorem 1.1. We obtain that there exists a strongly measurable function $\varphi : \Sigma \times \Omega \to X$ such that for every $\omega \in \Omega$, $z \mapsto \varphi(z,\omega)$ is analytic in Σ and

$$\left[\frac{\partial^j}{\partial z^j}\varphi(z, \cdot)\right] = \frac{\mathrm{d}^j}{\mathrm{d}z^j}\Phi(z), z \in \Sigma,\ j \in \mathbb{N}_0. \tag{11}$$

Thus φ has properties (i), (ii), and (iii). To show that φ has property (iv) we fix any $z \in \Sigma$. Since Φ is defined by an analytic semigroup generated by A we observe that $\Phi(z) \in \mathrm{D}(A)$ and $\frac{\mathrm{d}}{\mathrm{d}z}\Phi(z) = A\Phi(z)$. By definition of $\mathrm{D}(A)$ there exists $\tilde{\varphi}(z, \cdot) \in \mathrm{D}(\mathcal{A})$ such that $[\tilde{\varphi}(z, \cdot)] = \Phi(z)$ and $A\Phi(z) = [\mathcal{A}\tilde{\varphi}(z, \cdot)]$. However, by (11) we also have $[\varphi(z, \cdot)] = \Phi(z)$ and hence, $\varphi(z, \cdot) = \tilde{\varphi}(z, \cdot)$ almost everywhere. Now Hypothesis 3.1(i) implies that $\varphi(z, \cdot) \in \mathrm{D}(\mathcal{A})$ and $[\mathcal{A}\varphi(z, \cdot)] = [\mathcal{A}\tilde{\varphi}(z, \cdot)]$. Using (11) we therefore have

$$\left[\frac{\partial}{\partial z}\varphi(z, \cdot)\right] = \frac{\mathrm{d}}{\mathrm{d}z}\Phi(z) = A\Phi(z) = [\mathcal{A}\tilde{\varphi}(z, \cdot)] = [\mathcal{A}\varphi(z, \cdot)]. \tag{12}$$

Let $\Sigma_0 \subseteq \Sigma$ be a countable dense subset. For every $z_0 \in \Sigma_0$ we define

$$N_{z_0} := \left\{\omega \in \Omega;\ \frac{\partial}{\partial z}\varphi(z_0,\omega) \neq (\mathcal{A}\varphi(z_0, \cdot))(\omega)\right\}$$

and let

$$N_0 := \bigcup_{z_0 \in \Sigma_0} N_{z_0}.$$

Since $z \in \Sigma$ is fixed but arbitrary in (12) it follows that for every $z_0 \in \Sigma_0$, N_{z_0} is a nullset in Ω. Hence, N_0 is a nullset as well and we have

$$\frac{\partial}{\partial z}\varphi(z_0,\omega) = (\mathcal{A}\varphi(z_0,\,\cdot))(\omega), z_0 \in \Sigma_0,\, \omega \in \Omega\backslash N_0. \tag{13}$$

We shall use (13) and Hypothesis 3.1(ii) to get

$$\frac{\partial}{\partial z}\varphi(z,\omega) = (\mathcal{A}\varphi(z,\,\cdot))(\omega), \omega \in \Omega\backslash N_0. \tag{14}$$

As Σ_{z_0} is dense in Σ there exists a sequence $\{z_n\}_{n=1}^{\infty}$ in Σ_0 such that $\lim_{n\to\infty} z_n = z$. Since for every $\omega \in \Omega$, $z \mapsto \varphi(z,\omega)$ is analytic in Σ we have in particular

$$\lim_{n\to\infty} \varphi(z_n,\omega) = \varphi(z,\omega), \omega \in \Omega\backslash N_0$$

and using (13),

$$\lim_{n\to\infty} (\mathcal{A}\varphi(z_n,\,\cdot))(\omega) = \lim_{n\to\infty} \frac{\partial}{\partial z}\varphi(z_n,\omega) = \frac{\partial}{\partial z}\varphi(z,\omega), \omega \in \Omega\backslash N_0,$$

where the convergence in both lines is in X. Furthermore, it follows from (11), (13), and the analyticity of Φ that

$$\lim_{n\to\infty} [\varphi(z_n,\,\cdot)] = \lim_{n\to\infty} \Phi(z_n) = \Phi(z) = [\varphi(z,\,\cdot)]$$

and

$$\lim_{n\to\infty} [\mathcal{A}\varphi(z_n,\,\cdot)] = $$
$$= \lim_{n\to\infty} \left[\frac{\partial}{\partial z}\varphi(z_n,\,\cdot)\right] = \lim_{n\to\infty} \frac{\mathrm{d}}{\mathrm{d}z}\Phi(z_n) = \frac{\mathrm{d}}{\mathrm{d}z}\Phi(z) = \left[\frac{\partial}{\partial z}\varphi(z,\,\cdot)\right],$$

where the convergence is in $L^p(\Omega;X)$. It is a result of Hypothesis 3.1(ii) with φ_n, φ, ψ, and N replaced by respectively $\varphi(z_n,\,\cdot)$, $\varphi(z,\,\cdot)$,

$\frac{\partial}{\partial z}\varphi(z,\cdot)$, and N_0, that $\varphi(z,\cdot) \in \mathrm{D}(\mathcal{A})$ and (14) holds. This shows that φ has property (iv) with N replaced by N_0.

To show that φ has property (v) let $\{\Omega_k\}_{k=1}^{\infty}$ be a pairwise disjoint sequence in Ω such that $\bigcup_{k=1}^{\infty}\Omega_k = \Omega$ and for every $k \in \mathbb{N}$, $\mu(\Omega_k) < \infty$. We fix any $k \in \mathbb{N}$. Let $\Phi_k : \Sigma \to L^1(\Omega_k; X)$ be defined by

$$\Phi_k(z) := \Phi(z)|_{\Omega_k}, z \in \Sigma. \tag{15}$$

Using the analyticity of Φ, the fact that $L^p(\Omega_k; X) \hookrightarrow L^1(\Omega_k; X)$, (11), and Fubini's theorem, we have for any $T > 0$

$$\int_{\Omega_k}\left(\int_0^T \left\|\frac{\partial}{\partial t}\varphi(t,\omega)\right\| \mathrm{d}t\right)\mu(\mathrm{d}\omega) =$$
$$= \int_0^T \left(\int_{\Omega_k}\left\|\frac{\partial}{\partial t}\varphi(t,\omega)\right\| \mu(\mathrm{d}\omega)\right)\mathrm{d}t = \int_0^T \left\|\frac{\mathrm{d}}{\mathrm{d}t}\Phi_k(t)\right\|_{L^1(\Omega_k;X)} \mathrm{d}t < \infty.$$

This implies that there exists a nullset $N_k \subseteq \Omega_k$ such that

$$\int_0^T \left\|\frac{\partial}{\partial t}\varphi(t,\omega)\right\| \mathrm{d}t < \infty, \omega \in \Omega_k \backslash N_k.$$

Hence, for every $\omega \in \Omega_k\backslash N_k$, $\lim_{t\downarrow 0}\varphi(t,\omega)$ exists in X. Since $k \in \mathbb{N}$ is fixed but arbitrary we can define

$$N_\infty := \bigcup_{k=1}^{\infty} N_k$$

so that for every $\omega \in \Omega\backslash N_\infty$, $\lim_{t\downarrow 0}\varphi(t,\omega)$ exists. Now we show that

$$\left[\lim_{t\downarrow 0}\varphi(t,\cdot)\right] = [\varphi_0]. \tag{16}$$

By definition of Φ and the analyticity of the semigroup we have

$$\lim_{\substack{z\to 0\\ z\in\Sigma}}\Phi(z) = \lim_{\substack{z\to 0\\ z\in\Sigma}} S(z)[\varphi_0] = [\varphi_0], \tag{17}$$

where the convergence is in $L^p(\Omega; X)$. Let $\{t_n\}_{n=1}^{\infty}$ be a non-increasing sequence in $[0,\infty)$ such that $\lim_{n\to\infty} t_n = 0$. Then (17) and the definition of Φ_k in (15) imply that

$$\lim_{n\to\infty}\Phi_k(t_n) = [\varphi_0]|_{\Omega_k},$$

where the convergence is in $L^1(\Omega_k; X)$. Furthermore, it follows from (11) that

$$\Phi_k(t_n) = [\varphi(t_n, \cdot)|_{\Omega_k}], n \in \mathbb{N}.$$

Therefore there exists a subsequence $\{t_{n_j}\}_{j=1}^{\infty}$ such that for almost every $\omega \in \Omega_k$,

$$\lim_{j\to\infty} \varphi(t_{n_j}, \omega) = \varphi_0(\omega), \tag{18}$$

where the convergence is in X. Since $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$, (18) even holds for almost every $\omega \in \Omega$ and thus (16) holds. This shows that φ has property (v) with N replaced by N_∞ and the theorem is proved with $N := N_0 \bigcup N_\infty$. □

To finish this section we consider the example mentioned in the introduction.

EXAMPLE 3.3. *We use the notation*

$$\mathcal{L}_1^1(\mathbb{R}) = \Big\{\varphi : \mathbb{R} \to \mathbb{R};\ \varphi \text{ Borel measurable, for every } \omega \in \mathbb{R}, \\ \varphi(\omega + 1) = \varphi(\omega), \text{ and } \int_0^1 |\varphi(\omega)|\, d\omega < \infty\Big\},$$

$$L_1^1(\mathbb{R}) = \{[\varphi];\ \varphi \in \mathcal{L}_1^1(\mathbb{R})\}, \|[\varphi]\|_{L_1^1(\mathbb{R})} = \int_0^1 |\varphi(\omega)|\, d\omega.$$

For $t \geq 0$ *let* $\mathcal{S}(t) : \mathcal{L}_1^1(\mathbb{R}) \to \mathcal{L}_1^1(\mathbb{R})$ *and* $S(t) : L_1^1(\mathbb{R}) \to L_1^1(\mathbb{R})$ *be given by respectively*

$$(\mathcal{S}(t)\varphi_0)(\omega) := \varphi_0(t + \omega), \omega \in \Omega,\ \varphi_0 \in \mathcal{L}_1^1(\mathbb{R}),$$
$$S(t)[\varphi_0] := [\mathcal{S}(t)\varphi_0], \qquad \varphi_0 \in \mathcal{L}_1^1(\mathbb{R}).$$

Then there exists $\varphi_0 \in \mathcal{L}_1^1(\mathbb{R})$ *with the following property: if* $\varphi : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ *is a Borel measurable function such that for every* $t > 0$, $[\varphi(t, \cdot)] = S(t)[\varphi_0]$, *then for almost every* $\omega \in \mathbb{R}$, $\lim_{t\downarrow 0} \varphi(t, \omega)$ *does not exist.*

Proof. To construct φ_0 let$\{\varphi_k\}_{k=1}^{\infty}$ be a sequence of functions in $\mathcal{L}_1^1(\mathbb{R})$ given by

$$\varphi_k(\omega) = \begin{cases} 1, & \omega \in \bigcup_{j=0}^{2^k-1} [j \cdot 2^{-k}, j \cdot 2^{-k} + 2^{-2k}], \\ 0, & \omega \text{ elsewhere in } [0,1). \end{cases}$$

Note that $\|[\varphi_k]\|_{L_1^1(\mathbb{R})} = 2^k \cdot 2^{-2k} = 2^{-k}$ so that the series $\sum_{k=1}^{\infty}[\varphi_k]$ converges in $L_1^1(\mathbb{R})$. It follows from Lebesgue's monotone convergence theorem that $\int_0^1 \sum_{k=1}^{\infty} \varphi_k(\omega)\, \mathrm{d}\omega < \infty$. Hence, for almost every $\omega \in \mathbb{R}$, $\sum_{k=1}^{\infty} \varphi_k(\omega)$ is a convergent series. Now we define $\varphi_0 : \mathbb{R} \to \mathbb{R}$ by

$$\varphi_0(\omega) = \begin{cases} \sum_{k=1}^{\infty} \varphi_k(\omega), & \omega \text{ such that the series converges}, \\ 0, & \text{elsewhere}. \end{cases}$$

Fatou's lemma implies that $\varphi_0 \in \mathcal{L}_1^1(\mathbb{R})$. Moreover, it is a result of Lebesgue's dominated convergence theorem that $[\varphi_0] = \sum_{k=1}^{\infty}[\varphi_k]$.

We remark that for m, $n \in \mathbb{N}$ such that $1 \leq m < n$ and $\omega \in [j \cdot 2^{-m}, j \cdot 2^{-m} + 2^{-2n}]$ we have $\sum_{k=m}^{n-1} \varphi_k(\omega) = n - m$. This implies that for any $M > 0$ and any interval $I \subseteq \mathbb{R}$ there exists an open interval $J \subseteq I$ such that

$$\varphi_0(\omega) > M, \omega \in J. \tag{19}$$

To show that φ_0 has the requested property let $\varphi : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a Borel measurable function such that for every $t > 0$, $[\varphi(t, \cdot)] = S(t)[\varphi_0]$. Then there exist a countable dense subset $K \subseteq (0, \infty)$ and a Borel nullset $N \subseteq \mathbb{R}$ such that

$$\varphi(t, \omega) = \varphi_0(t + \omega), t \in K,\ \omega \in \mathbb{R} \backslash N. \tag{20}$$

Seeking a contradiction we assume that there exists $\omega \in \mathbb{R} \backslash N$ such that $L := \lim_{t \downarrow 0} \varphi(t, \omega)$ exists. If we can construct a nonincreasing sequence $\{t_n\}_{n=1}^{\infty}$ in K such that $\lim_{n \to \infty} t_n = 0$ and for every $n \in \mathbb{N}$, $\varphi_0(t_n + \omega) > L + 1$, then using (20) we obtain the following contradiction:

$$L = \lim_{n \to \infty} \varphi(t_n, \omega) = \lim_{n \to \infty} \varphi_0(t_n + \omega) \geq L + 1,$$

which would finish the proof. To construct $\{t_n\}_{n=1}^{\infty}$ we use (19) with M replaced by $L+1$. If $I_1 = [\omega, 1)$, then there exist $J_1 \subseteq (\omega, 1)$ and, by density of K, $t_1 \in K$ such that $t_1+\omega \in J_1$ and $\varphi_0(t_1+\omega) > L+1$. If $I_2 = [\omega, t_1+\omega)$, then there exist $J_2 \subseteq (\omega, t_1+\omega)$ and $t_2 \in K$ such that $t_2+\omega \in J_2$ and $\varphi_0(t_2+\omega) > L+1$. Proceeding like this will give the sequence $\{t_n\}_{n=1}^{\infty}$. □

4. An application

For an application of Theorem 3.2 we consider a homogeneous abstract Cauchy problem in a Hilbert space of equivalence classes and show that its solution has an analytic version. For the setting of the problem we refer to [1] and [2]. In these papers we consider the scalar Volterra integrodifferential equation of convolution type

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{t} a(t-s)u(s)\,\mathrm{d}s = f(t),\ t>0, \\ u(t) = u_0(t), t \le 0, \end{aligned} \tag{21}$$

with a completely monotonic kernel $a : (0,\infty) \to \mathbb{R}$. In the homogeneous case, that is, f is identically zero, we can rewrite problem (21) to the homogeneous abstract Cauchy problem

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\psi(t) &= A\psi(t), t>0, \\ \psi(0) &= \psi_0, \end{aligned} \tag{22}$$

with a suitable A that we shall define later. Using an analytic semigroup we find a solution ψ to (22) in a Hilbert space of equivalence classes. From ψ we obtain a solution u to (21) by means of a linear functional. To show that u is indeed a solution to (21) we need pointwise interpretation of ψ. It is at this point that we need a version of ψ that is at least differentiable.

Let $a : (0,\infty) \to \mathbb{R}$ be a completely monotonic kernel such that $\int_0^1 a(t)\,\mathrm{d}t < \infty$ and $a(0+) = +\infty$. Let ν be the unique nonnegative Borel measure on $[0,\infty)$ such that

$$a(t) = \int_{[0,\infty)} \mathrm{e}^{-\omega t}\,\nu(\mathrm{d}\omega), t>0,$$

see Bernstein's theorem [5, Theorem 12b, page 161]. Let μ be the nonnegative Borel measure on $[0,\infty)$ given by

$$\mu(\mathrm{d}\omega) := (\omega+1)\,\nu(\mathrm{d}\omega).$$

Note that if $N \subseteq [0,\infty)$ is a μ-nullset, then N is a ν-nullset. Let $\mathcal{H}$ and H denote respectively

$$\mathcal{H} = \mathcal{L}^2(([0,\infty), \mathcal{B}[0,\infty), \mu); \mathbb{C}),$$

$$H = L^2(([0,\infty), \mathcal{B}[0,\infty), \mu); \mathbb{C}).$$

We define the linear functional $J : \mathrm{D}(J) \subseteq H \to \mathbb{C}$ by

$$\mathrm{D}(J) := \{\Phi \in H;\ \text{there exist } u \in \mathbb{C} \text{ and } \varphi \in \mathcal{H} \text{ such that } [\varphi] = \Phi \text{ and } \omega \mapsto u - \omega\varphi(\omega) \in \mathcal{H}\},$$

$$J(\Phi) := u, \Phi \in \mathrm{D}(J).$$

Then J is well-defined, see [2, Lemma 4.4]. Note that if $\Phi \in \mathrm{D}(J)$ with $u \in \mathbb{C}$ and $\varphi \in \mathcal{H}$ such that $[\varphi] = \Phi$ and $\omega \mapsto u - \omega\varphi(\omega) \in \mathcal{H}$, then for every $\tilde{\varphi} \in \mathcal{H}$ such that $[\tilde{\varphi}] = \Phi$ we have $\omega \mapsto u - \omega\tilde{\varphi}(\omega) \in \mathcal{H}$. We define the linear operator $A : \mathrm{D}(A) \subseteq H \to H$ by

$$\mathrm{D}(A) := \left\{\Phi \in \mathrm{D}(J);\ \text{if } \varphi \in \mathcal{H} \text{ is such that } [\varphi] = \Phi, \text{ then } \int_{[0,\infty)} \big(J(\Phi) - \omega\varphi(\omega)\big)\,\nu(\mathrm{d}\omega) = 0\right\},$$

$$A\Phi := [\omega \mapsto J(\Phi) - \omega\varphi(\omega)], \Phi \in \mathrm{D}(A).$$

Then A is well-defined since $H \hookrightarrow L^1(([0,\infty), \mathcal{B}[0,\infty), \nu); \mathbb{C})$, see [2, Lemma 4.2]. Moreover, A is the infinitesimal generator of a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ on H and there exists $0 < \vartheta \leq \pi$ such that $\{S(t)\}_{\geq 0}$ has an analytic extension to $\Sigma := \{z \in \mathbb{C}\backslash\{0\};\ |\mathrm{Arg}(z)| < \vartheta\}$, see [2, Theorem 3.4]. We define the linear operator $\mathcal{A} : \mathrm{D}(\mathcal{A}) \subseteq \mathcal{H} \to \mathcal{H}$ by

$$\mathrm{D}(\mathcal{A}) := \left\{\varphi \in \mathcal{H};\ [\varphi] \in \mathrm{D}(J) \text{ and } \int_{[0,\infty)} \big(J([\varphi]) - \omega\varphi(\omega)\big)\,\nu(\mathrm{d}\omega) = 0\right\},$$

$$(\mathcal{A}\varphi)(\omega) := J([\varphi]) - \omega\varphi(\omega), \varphi \in \mathrm{D}(\mathcal{A}),\ \omega \geq 0.$$

From the definition of A and $\mathcal{A}$ it follows that

$$\mathrm{D}(A) = \{\Phi \in H;\ \text{ there exists } \varphi \in \mathrm{D}(\mathcal{A}) \text{ such that } [\varphi] = \Phi\} \quad (23)$$

and

$$A[\varphi] = [\mathcal{A}\varphi], \varphi \in \mathcal{A}. \quad (24)$$

Note that this implies that A could have been defined by (10).

Lemma 4.1. *Operator $\mathcal{A}$ satisfies Hypothesis 3.1.*

Proof. First we show that $\mathcal{A}$ satisfies Hypothesis 3.1(i). Let $\varphi_1 \in \mathrm{D}(\mathcal{A})$ and $\varphi_2 \in \mathcal{H}$ be such that $\varphi_1 = \varphi_2$ μ-almost everywhere. Then $[\varphi_2] = [\varphi_1] \in \mathrm{D}(J)$ and in particular $\varphi_1 = \varphi_2$ ν-almost everywhere. Therefore we have

$$\int_{[0,\infty)} \big(J([\varphi_2]) - \omega\varphi_2(\omega)\big)\,\nu(\mathrm{d}\omega) = \int_{[0,\infty)} \big(J([\varphi_1]) - \omega\varphi_1(\omega)\big)\,\nu(\mathrm{d}\omega) = 0.$$

Thus $\varphi_2 \in \mathrm{D}(\mathcal{A})$ and for μ-almost every $\omega \in [0,\infty)$,

$$(\mathcal{A}\varphi_2)(\omega) = J([\varphi_2]) - \omega\varphi_2(\omega) = J([\varphi_1]) - \omega\varphi_1(\omega) = (\mathcal{A}\varphi_1)(\omega).$$

Now we show that $\mathcal{A}$ satisfies Hypothesis 3.1(ii). Let $\{\varphi_n\}_{n=1}^{\infty}$ be a sequence in $\mathrm{D}(\mathcal{A})$, let φ, $\psi \in \mathcal{H}$, and let $N \subseteq [0,\infty)$ be a μ-nullset such that

$$\lim_{n\to\infty} \varphi_n(\omega) = \varphi(\omega), \quad \omega \in [0,\infty)\backslash N, \quad (25)$$

$$\lim_{n\to\infty} (\mathcal{A}\varphi_n)(\omega) = \psi(\omega), \quad \omega \in [0,\infty)\backslash N, \quad (26)$$

$$\lim_{n\to\infty} [\varphi_n] = [\varphi], \quad (27)$$

$$\lim_{n\to\infty} [\mathcal{A}\varphi_n] = [\psi], \quad (28)$$

where the convergence in the last two lines is in H. From (24) and (28) it follows that

$$\lim_{n\to\infty} A[\varphi_n] = \lim_{n\to\infty} [\mathcal{A}\varphi_n] = [\psi]. \quad (29)$$

Since A is a closed operator, (27) and (29) imply that $[\varphi] \in \mathrm{D}(A)$ and $A[\varphi] = [\psi]$. This has two consequences. Firstly, by (23) there exists $\tilde{\varphi} \in \mathrm{D}(\mathcal{A})$ such that $[\tilde{\varphi}] = [\varphi]$ and hence, by Hypothesis 3.1(i) we have $\varphi \in \mathrm{D}(\mathcal{A})$. Secondly, using (29) we have $\lim_{n\to\infty} A[\varphi_n] = A[\varphi]$. Combined with (27) we therefore have $\lim_{n\to\infty}[\varphi_n] = [\varphi]$ where the convergence is in the Banach space $\mathrm{D}(A)$ endowed with the graph norm $\|\cdot\|_{\mathrm{D}(A)}$. As $J|_{\mathrm{D}(A)} : (\mathrm{D}(A), \|\cdot\|_{\mathrm{D}(A)}) \to \mathbb{C}$ is continuous, see [2, Lemma 4.8], it follows that $\lim_{n\to\infty} J([\varphi_n]) = J([\varphi])$. Together with (25) this implies that for every $\omega \in [0,\infty)\backslash N$,

$$\lim_{n\to\infty} (\mathcal{A}\varphi_n)(\omega) = \lim_{n\to\infty} \big(J([\varphi_n]) - \omega\varphi_n(\omega)\big) = J([\varphi]) - \omega\varphi(\omega) = (\mathcal{A}\varphi)(\omega).$$

In combination with (26) this shows that

$$\psi(\omega) = (\mathcal{A}\varphi)(\omega), \omega \in [0,\infty)\backslash N.$$

Thus $\mathcal{A}$ satisfies Hypothesis 3.1(ii) and the lemma is proved. □

Let the function $u_0 : (-\infty, 0] \to \mathbb{R}$ have the following properties:

(i) u_0 is Borel measurable;

(ii) There exist $M_1 > 0$ and $\alpha > 0$ such that

$$|u_0(t)| \le M_1 \mathrm{e}^{\alpha t}, t \le 0;$$

(iii) There exist $M_2 > 0$ and $\delta > 0$ such that

$$|u_0(0) - u_0(t)| \le M_2|t|, -\delta \le t \le 0;$$

(iv) $\frac{\mathrm{d}^-}{\mathrm{d}t}\Big|_{t=0} \int_{-\infty}^{t} a(t-s)u(s)\,\mathrm{d}s = 0.$

We define the function $\varphi_0 : [0,\infty) \to \mathbb{C}$ by

$$\varphi_0(\omega) := \int_0^\infty \mathrm{e}^{-\omega t} u_0(-t)\,\mathrm{d}t, \omega \ge 0.$$

Then $\varphi_0 \in \mathrm{D}(\mathcal{A})$, see [2, Lemma 5.1], and with $\Phi_0 := [\varphi_0]$ we have $\Phi_0 \in \mathrm{D}(A)$. We consider the following homogeneous abstract Cauchy problem in H:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) &= A\Phi(t), t > 0, \\ \Phi(0) &= \Phi_0. \end{aligned} \tag{30}$$

DEFINITION 4.2. *A strict solution to (30) in* $[0,\infty)$ *is a function* $\Phi : [0,\infty) \to H$ *such that for every* $T > 0$, $\Phi \in C([0,\infty); H_1) \bigcap C^1([0,\infty); H)$ *and*

$$\frac{d}{dt}\Phi(t) = A\Phi(t), t \geq 0,$$
$$\Phi(0) = \Phi_0.$$

We define the function $\Phi : [0,\infty) \to H$ by $\Phi(t) := S(t)\Phi_0$ for $t \geq 0$. Then Φ is the unique strict solution to (30), see [3, Theorem 4.3.1(ii), page 134]. Furthermore, Φ has an analytic extension to Σ.

THEOREM 4.3. *There exist a function* $\varphi : [0,\infty) \times [0,\infty) \to \mathbb{C}$ *and a* μ*-nullset* $N \subseteq [0,\infty)$ *with the following properties:*

(i) φ *is Borel measurable;*

(ii) *For every* $t \geq 0$, $[\varphi(t,\cdot)] = \Phi(t)$*;*

(iii) *For every* $\omega \in [0,\infty)\backslash N$, $t \mapsto \varphi(t,\omega)$ *is continuous in* $[0,\infty)$ *and has an analytic extension to* Σ*;*

(iv) *For every* $t > 0$ *and* $\omega \in [0,\infty)\backslash N$,

$$\frac{\partial}{\partial t}\varphi(t,\omega) = J(\Phi(t)) - \omega\varphi(t,\omega);$$

(v) *For every* $\omega \in [0,\infty)\backslash N$, $\varphi(0,\omega) = \varphi_0(\omega)$.

Proof. We are in position to apply Theorem 3.2 with $X = \mathbb{C}$, $\Omega = [0,\infty)$, $\mathcal{F} = \mathcal{B}[0,\infty)$, and $p = 2$. Thus there exist a Borel measurable function $\tilde{\varphi} : \Sigma \times [0,\infty) \to \mathbb{C}$ and a μ-nullset $\tilde{N} \subseteq [0,\infty)$ such that for every $\omega \in [0,\infty)$, $z \mapsto \tilde{\varphi}(z,\omega)$ is analytic in Σ and for every $\omega \in [0,\infty)\backslash\tilde{N}$, $\lim_{t\downarrow 0}\tilde{\varphi}(t,\omega)$ exists with $[\lim_{t\downarrow 0}\tilde{\varphi}(t,\cdot)] = \Phi_0$, where the latter convergence is in $\mathcal{H}$. Moreover, we have

$$[\tilde{\varphi}(z,\cdot)] = \Phi(z), z \in \Sigma \tag{31}$$

and

$$\frac{\partial}{\partial z}\tilde{\varphi}(z,\omega) = (\mathcal{A}\tilde{\varphi}(z,\cdot))(\omega), z \in \Sigma,\ \omega \in [0,\infty)\backslash\tilde{N}. \tag{32}$$

Now we define $\varphi : [0,\infty) \times [0,\infty) \to \mathbb{C}$ by

$$\varphi(t,\omega) := \begin{cases} \tilde{\varphi}(t,\omega), & t > 0,\ \omega \in [0,\infty), \\ \lim_{s\downarrow 0} \tilde{\varphi}(s,\omega), & t = 0,\ \omega \in [0,\infty)\backslash \tilde{N}, \\ 0, & t = 0,\ \omega \in \tilde{N}. \end{cases}$$

Then φ is well-defined and has properties (i), (ii), and (iii) with N replaced by $\tilde{N}$. Using (31) and (32) we have for every $t > 0$ and $\omega \in [0,\infty)\backslash \tilde{N}$,

$$\begin{aligned} \frac{\partial}{\partial t}\varphi(t,\omega) &= \frac{\partial}{\partial t}\tilde{\varphi}(t,\omega) = (\mathcal{A}\tilde{\varphi}(t,\,\cdot))(\omega) \\ &= J([\tilde{\varphi}(t,\,\cdot)]) - \omega\tilde{\varphi}(t,\omega) = J(\Phi(t)) - \omega\varphi(t,\omega). \end{aligned}$$

Hence, φ has property (iv) with N replaced by $\tilde{N}$. To show that φ has property (v) we observe that

$$[\varphi(0,\,\cdot)] = \left[\lim_{s\downarrow 0} \tilde{\varphi}(s,\,\cdot)\right] = \Phi_0.$$

Thus there exists a μ-nullset $N_0 \subseteq [0,\infty)$ such that

$$\varphi(0,\omega) = \varphi_0(\omega), \omega \in [0,\infty)\backslash N_0.$$

Finally we define the μ-nullset $N := \tilde{N} \bigcup N_0$ and the theorem is proved. □

References

[1] Ph. Clément, W. Desch, and K.W. Homan, *An analytic semigroup setting for a class of Volterra equations*, Submitted to J. Integral Equations Appl.

[2] Ph. Clément, W. Desch, and K.W. Homan, *An analytic semigroup setting for a class of Volterra equations*, Optimierung und Kontrolle, Karl-Franzens-Universität und Technische Universität Graz **SFB F003 Bericht** (2002), no. 242.

[3] A. Lunardi, *Analytic semigroups and optimal regularity in parabolic problems*, Birkhäuser Verlag, Basel, 1995.

[4] E.M. Stein, *Topics in harmonic analysis related to the Littlewood-Paley theory*, Princeton University Press, Princeton, N.J., 1970.

[5] D.V. Widder, *The Laplace transform*, Princeton University Press, Princeton, N. J., 1941.

Received September 26, 2002.