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Finite Time Blow-up for Solutions of a Hyperbolic System: the Critical Case

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Summary. - *It has already been proved that for the systems forming by m wave equations containing polynomial nonlinearities there exists a manifold that bounds the region of the blow-up in the half-space to which belong the parameters of nonlinearity.*

Here we prove the formation of singularities if the parameters belong to the critical manifold in three space dimensions.

1. Introduction

In the two previous papers we concentrated our attention to the following Cauchy problem:

$$\left\{ \begin{array}{l} \partial^2_{tt}u_1 - \Delta u_1 = |u_2|^{p_1} \\ \partial^2_{tt}u_2 - \Delta u_2 = |u_3|^{p_2} \\ \qquad \cdots \\ \partial^2_{tt}u_m - \Delta u_m = |u_1|^{p_m} \end{array} \right. \qquad \text{in } \mathcal{R}^n \times [0,+\infty[\,, \tag{1}$$

$$u_i(x,0) = f_i(x),\ \partial_t u_i(x,0) = g_i(x), \quad \text{in } \mathcal{R}^n, \tag{2}$$

where $p_i > 1$, and $f_i,\ g_i \in C_0^\infty(\mathcal{R}^n)$ for all $i = 1,...,m$.

If there exists a m-tuple $(u_1, u_2, ..., u_m)$ of C^2 functions defined in $\mathcal{R}^n \times [0,+\infty[$ satisfying (1, 2) we say that the problem admits a global classical solution.

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Blow-up consists in the non-existence of a global solution. In this case there exist local solutions with initial data like in (2) satisfying (1) only in $\mathcal{R}^n \times [0, T[$ where $T < +\infty$ is called the life-span.

For sake of simplicity we started to study the case of three equations. More precisely we proved (see [9]) that if

$$\max\left\{\tfrac{p_2p_3+2p_2+2+p_1^{-1}}{p_1p_2p_3-1}, \tfrac{p_1p_3+2p_3+2+p_2^{-1}}{p_1p_2p_3-1}, \tfrac{p_1p_2+2p_1+2+p_3^{-1}}{p_1p_2p_3-1}\right\} > \frac{n-1}{2}, \tag{3}$$

and the space dimensions $n = 3$ then any non trivial C^2 solution is defined only locally in the time. If (3) is satisfied but $n \neq 3$, then any solution of the system blows-up in finite time if the average of the initial data satisfy a certain positive condition. On the contrary we gave a partial converse result for space dimensions $n = 2, 3$ (see [10]). More exactly we proved that if

$$\max\left\{\tfrac{p_2p_3+2p_2+2+p_1^{-1}}{p_1p_2p_3-1}, \tfrac{p_1p_3+2p_3+2+p_2^{-1}}{p_1p_2p_3-1}, \tfrac{p_1p_2+2p_1+2+p_3^{-1}}{p_1p_2p_3-1}\right\} < \frac{n-1}{2}, \tag{4}$$

and

$$\min\{p_1, p_2, p_3\} > 2,$$

then there exists a tern $(u, v, w) \in C^2(\mathcal{R}^n \times [0, +\infty[\,; \mathcal{R})^3$ that satisfies (1, 2) provided that the initial data are small in a suitable way. Instead if

$$\min\{p, p_2, p_3\} \leq 2,$$

then (4) implies, under the same condition on the initial data, the existence of a global continuous solution of the integral problem corresponding to (1, 2).

After that we dealt with the question of the m equations. We showed in [9] that the condition (3) can be widened for large system. In fact if the parameters of nonlinearity in (1, 2) satisfy

$$\frac{\prod_{i=2}^{m} p_i + 2\left(\prod_{i=2}^{m-1} p_i + \prod_{i=2}^{m-2} p_i + \prod_{i=2}^{m-3} p_i + \ldots + p_2 + 1\right) + p_1^{-1}}{\prod_{i=1}^{m} p_i - 1} > \frac{n-1}{2}, \tag{5}$$

or one of the m inequality obtained by a cyclic permutation, then any solution is defined only locally in the time under the assumption of some positive condition of the initial data.

It is interesting to compare these results with the analogous of the scalar equation and of the Hamiltonian system. It is well known (see [6], [7], [4], [5], [12], [11]) that for the semilinear wave equation

$$\begin{aligned} &\partial_{tt}^2 u - \Delta u = |u|^p \qquad \text{in } \mathcal{R}^n \times [0, +\infty[\,, \\ &u(x,0) = f(x), \quad \partial_t u(x,0) = g(x), \end{aligned} \tag{6}$$

where $f, g \in C_0^\infty(\mathcal{R}^n)$, there exists a critical value p_c, the larger root of the quadratic $(n-1)z^2 - (n+1)z - 2 = 0$, that breaks the behavior of the solution in two: If $1 < p < p_c$ then the solution of (6) is defined only locally in the time, while for $p > p_c$ small Cauchy data imply global existence.

Moreover given the $p - q$ system

$$\begin{cases} \partial_{tt}^2 u - \Delta u = |v|^p \\ \partial_{tt}^2 v - \Delta v = |u|^q \end{cases} \qquad \text{in } \mathcal{R}^n \times [0, +\infty[\,, \tag{7}$$

the equality

$$\max\left\{ \frac{q + 2 + p^{-1}}{pq - 1}, \frac{p + 2 + q^{-1}}{pq - 1} \right\} = \frac{n-1}{2}$$

identifies the critical curve that plays the same role for this Hamiltonian system as the exponent p_c in the scalar case (see [2], [3], [1]).

These issues led us to conjecture that there exists a peculiar phenomenon in dealing with the question of global existence for solution of this kind of systems: given a m equations system, there is always a manifold of $m - 1$ dimensions that divides the region of global existence from that of the formation of the singularities in the half-space to which belong the parameters of the non-linearity.

In the present paper we make one more step in that direction. We show that in three space dimensions if the average of initial speed is positive and

$$\frac{\prod_{i=2}^{m} p_i + 2\left(\prod_{i=2}^{m-1} p_i + \prod_{i=2}^{m-2} p_i + \prod_{i=2}^{m-3} p_i + \ldots + p_2 + 1 \right) + p_1^{-1}}{\prod_{i=1}^{m} p_i - 1} = 1 \tag{8}$$

then any m-tuple $(u_1, u_2, ..., u_m)$ solution of (1, 2) blows-up in finite time.

Hence we prove that the separation manifold belong to the blow-up case as well.

This paper is organized as follows: in the next section we state the main result. Section 3 is devoted to present some important tools in order to prove it. Finally in the last section we give the proof of the theorem.

2. Result

THEOREM 2.1. *Let $m \in \mathcal{N} - \{0,1,2\}$, $T \in]0,+\infty]$ and let $(u_1, u_2, ..., u_m) \in C^2((\mathcal{R}^3 \times [0,T[);\mathcal{R})^m$ be a solution of the following Cauchy problem*

$$\left\{ \begin{array}{c} \partial^2_{tt} u_1 - \Delta u_1 = |u_2|^{p_1} \\ \partial^2_{tt} u_2 - \Delta u_2 = |u_3|^{p_2} \\ \cdots \\ \partial^2_{tt} u_m - \Delta u_m = |u_1|^{p_m} \end{array} \right. \qquad \text{in } \mathcal{R}^3 \times [0,T[\,, \tag{9}$$

$$u_i(x,0) = f_i(x), \ \partial_t u_i(x,0) = g_i(x), \quad \text{in } \mathcal{R}^3, \tag{10}$$

where $f_i, g_i \in C_0^\infty(\mathcal{R}^3)$ for all $i = 1, ..., m$. Suppose that all $p_i > 1$ and

$$\int_{\mathcal{R}^3} g_i(x)\, dx > 0 \quad \text{for all } i = 1, ..., m. \tag{11}$$

Define

$$\Psi(p_1, ..., p_m) = \frac{\prod_{i=2}^{m} p_i + 2\left(\prod_{i=2}^{m-1} p_i + \prod_{i=2}^{m-2} p_i + \prod_{i=2}^{m-3} p_i + ... + p_2 + 1\right) + p_1^{-1}}{\prod_{i=1}^{m} p_i - 1},$$

and for all the cyclic permutations π_j, $j = 1, ..., m$, of $(1, 2, ..., m)$, call

$$\Psi_j = \Psi(p_{\pi_j(1)}, ..., p_{\pi_j(m)}).$$

If

$$\max_{j=1,\dots,m} \Psi_j = 1, \tag{12}$$

then

$$T < +\infty.$$

3. Preliminary results

To prove the main theorem of this paper we use the following two lemmas.

LEMMA 3.1. *Let* $T \in \left]0,+\infty\right]$, $R > 0$, $G \in C^1(\mathcal{R}^3 \times [0,T[\,;\mathcal{R})$, *and* $f, g \in C_0^\infty(\mathcal{R}^3)$. *Let* V *be the solution of*

$$\Box V(x,t) = G(x,t) \quad \text{in } \mathcal{R}^3 \times [0,T[\,,$$

$$V(x,0) = f(x),\ \partial_t V(x,0) = g(x) \quad \text{in } \mathcal{R}^3.$$

Suppose that G *is a non-negative function with*

$$\operatorname{supp} G \subset \left\{(x,t) \in \mathcal{R}^3 \times [0,T[\, : |x| \le R + t\right\},$$

and furthermore that

$$\operatorname{supp} f, g \subset \left\{x \in \mathcal{R}^3 : |x| \le R\right\}, \int_{\mathcal{R}^3} g(x)\, dx > 0.$$

Then

$$\int_{\frac{t}{2} \le |x| \le t+R} V(x,t)\, dx \ge \int_{|x| \le \frac{t}{2}+R} V(x, \frac{t}{2})\, dx, \tag{13}$$

for all $t \in [2R, T[\,$.

For the proof of this first lemma we refer the interested reader to [3, lemma 2.1].

Lemma 3.2. *Let $a,\ b \in \,]e,+\infty]$ with $a < b$, and let $C_0,\ R > 0$. Given m functions $F_i \in C^2([a,b[\,,\mathcal{R})$, $i = 1,...,m$, suppose that there exist $l \geq 1$, and m real numbers $p_i \in \,]1,+\infty[$, $i = 1,...,m$, such that, letting $\alpha_i = 3(p_i - 1)$, the following conditions hold:*

$$\begin{aligned} F_1(t) &\geq C_0(R+t)^l(\log(R+t)), \\ F_i(t) &\geq C_0(R+t) \quad \text{for } i = 1,...,m, \\ F_i''(t) &\geq C_0(R+t)^{-\alpha_i}\,|F_{i+1}(t)|^{p_i}, \quad \text{for } i = 1,...,m-1, \\ F_m''(t) &\geq C_0(R+t)^{-\alpha_m}\,|F_1(t)|^{p_m}, \end{aligned} \tag{14}$$

for all $t \in [a,b[$. Suppose finally that

$$l\left(\prod_{i=1}^{m} p_i - 1\right) = 3\left(\prod_{i=1}^{m} p_i - 1\right) - 2\left(1 + p_1 + p_1p_2 + ... + \prod_{i=1}^{m-1} p_i\right). \tag{15}$$

Then

$$b \;<\; +\infty.$$

Proof of lemma 3.2. We prove this Lemma by assuming that $b = +\infty$ and deducing a contradiction. Positive constants arising in the estimates will be denoted by C, and will change from line to line.

First of all we note that all the F_i diverge as $t \to \infty$. Moreover by the convexity we can assume that there exists $T_1 \geq a$ such that for $t \geq T_1$, $F_i'(t) > 0$ for all $i = 1,...m$. Hence for $t \geq T_1$, we have

$$F_1'(t)F_m''(t) \geq C_0(R+t)^{-\alpha_m}(F_1(t))^{p_m}F_1'(t).$$

Integrating by parts on $[T_1, t]$ we obtain

$$\begin{aligned} &\left|F_1'(t)F_m'(t)\right|_{T_m}^{t} - \int_{T_m}^{t} F_1''(t)F_m'(t)d\tau \\ &\qquad\qquad \geq C_0\int_{T_m}^{t}(R+\tau)^{-\alpha_m}(F_1(t))^{p_m}F_1''(\tau)d\tau, \end{aligned}$$

and since $\alpha_m > 0$, it follows that there exist $T_2 \geq T_1,\ C > 0$, such that

$$F_1'(t) F_m'(t) \geq C(R+t)^{-\alpha_m} (F_1(t))^{p_m+1}$$

for all $t \geq T_2$. Multipling again by $F_1''(t)$, and then integrating on $[T_2, t]$, we deduce that there exists $T_3 \geq T_2$ such that

$$F_m(t) \geq C \frac{(R+t)^{-\alpha_m} |F_1(t)|^{p_m+2}}{(F_1'(t))^2}$$

for all $t \geq T_3$. Using now $F''_{m-1}(t) \geq C_0 (R+t)^{-\alpha_{m-1}} |F_m(t)|^{p_{m-1}}$, we can repeat the previous argument to get the following inequality

$$F_{m-1}(t) \geq C \frac{(R+t)^{-(\alpha_{m-1}+p_{m-1}\alpha_m)} |F_1(t)|^{p_{m-1}(p_m+2)+2}}{(F_1'(t))^{2p_m+2}}$$

for all $t \geq T_4$, where $T_4 \geq T_3$ is large enough.

More generally we derive that for t large, say $t \geq T_5$, one has

$$F_{m-j}(t) \geq C \frac{(R+t)^{-\beta_{m-j}} |F_1(t)|^{\gamma_{m-j}}}{(F_1'(t))^{\delta_{m-j}}} \tag{16}$$

for $0 \leq j \leq m-2$, where

$$\beta_m = \alpha_m, \ \ \gamma_m = p_m + 2, \tag{17}$$

and for $1 \leq j \leq m-2$

$$\beta_{m-j} = \alpha_{m-j} + p_{m-j}(\beta_{m-j+1}), \qquad \gamma_{m-j} = p_{m-j}(\gamma_{m-j+1}) + 2,$$

$$\delta_m = 2, \qquad \delta_{m-j} = p_{m-j}(\delta_{m-j+1}) + 2. \tag{18}$$

Thus

$$F_1''(t) \geq C \frac{(R+t)^{-\alpha_1 - p_1\beta_2} |F_1(t)|^{p_1\gamma_2}}{(F_1'(t))^{p_1\delta_2}},$$

whereby after an integration we get

$$F_1'(t) \geq C(R+t)^{-\frac{\alpha_1+p_1\beta_2}{p_1\delta_2+2}} |F_1(t)|^{\frac{p_1\gamma_2+1}{p_1\delta_2+2}} \tag{19}$$

for $t \geq T_6 \geq T_5$. Observe now that

$\frac{p_1\gamma_2+1}{p_1\delta_2+2} = \frac{p_1(\gamma_2-\delta_2)-1}{p_1\delta_2+2} + 1 = \frac{p_1(p_2(\gamma_3-\delta_3))-1}{p_1\delta_2+2} + 1 = \ldots = \frac{\prod_{i=1}^m p_i - 1}{p_1\delta_2+2} + 1,$

so that the (19) can be rewritten as

$$F_1'(t) \geq C(R+t)^{-\frac{\alpha_1+p_1\beta_2}{p_1\delta_2+2}} |F_1(t)|^{\left(\frac{\prod_{i=1}^m p_i-1}{p_1\delta_2+2}+1\right)}. \tag{20}$$

Next from (14) we have

$$F_1(t)^{\frac{\prod_{i=1}^m p_i-1}{p_1\delta_2+2}} \geq C(R+t)^{l\frac{\prod_{i=1}^m p_i-1}{p_1\delta_2+2}} (\log(R+t))^{\frac{\prod_{i=1}^m p_i-1}{p_1\delta_2+2}},$$

and equality (15) implies that

$$\begin{aligned} &F_1(t)^{\frac{\prod_{i=1}^m p_i-1}{p_1\delta_2+2}} \\ &\geq C(R+t)^{\frac{3(\prod_{i=1}^m p_i-1)-2\left(1+p_{1+\ldots.+}\prod_{i=1}^{m-1} p_i\right)}{p_1\delta_2+2}} (\log(R+t))^{\frac{\prod_{i=1}^m p_i-1}{p_1\delta_2+2}}. \end{aligned} \tag{21}$$

Observe now that

$$\frac{-\alpha_1 - p_1\beta_2 + 3\left(\prod_{i=1}^m p_i - 1\right) - 2\left(1 + p_{1+\ldots.+}\prod_{i=1}^{m-1} p_i\right)}{p_1\delta_2+2}$$

$$= \frac{-3(p_1-1)-p_1(3(p_2-1)+p_2\beta_3)+3\left(\prod_{i=1}^m p_i-1\right)-2\left(1+p_{1+\ldots+}\prod_{i=1}^{m-1} p_i\right)}{p_1\delta_2+2}$$

$$= \frac{-3p_1p_2 - p_1p_2\beta_3 + 3\prod_{i=1}^m p_i - 2\left(1+p_{1+\ldots+}\prod_{i=1}^{m-1} p_i\right)}{p_1(p_2\delta_3+2)+2}$$

$$= \ldots = \frac{-3\prod_{i=1}^m p_i + 3\prod_{i=1}^m p_i - 2\left(1+p_{1+\ldots+}\prod_{i=1}^{m-1} p_i\right)}{2\left(1+p_{1+\ldots+}\prod_{i=1}^{m-1} p_i\right)} = -1,$$

hence combining (20) and (3), we derive that

$$F_1'(t) \geq C(R+t)^{-1}F_1(t)(\log(R+t))^{\frac{\prod_{i=1}^m p_i - 1}{p_1\delta_2+2}}.$$

Integrating this inequality we deduce that there exists $T_7 \geq T_6$, such that

$$F_1(t) \geq C(R+t)^{[\log(R+t)]^{\frac{\prod_{i=1}^m p_i - 1}{p_1\delta_2+2}}}$$

for all $t \geq T_7$, but that means that

$$[F_1(t)]^{\frac{\prod_{i=1}^m p_i - 1}{2(p_1\delta_2+2)}} \geq C(R+t)^{\frac{\alpha_1+p_1\beta_2}{p_1\delta_2+2}} \qquad (22)$$

for all $t \geq T$, where $T \geq T_7$ is large enough.

At last by (20) and (22), we arrive at the following ordinary differential inequality

$$F_1'(t) \geq C\left(F_1(t)\right)^{1+\frac{\prod_{i=1}^m p_i - 1}{2(p_1\delta_2+2)}}, \qquad (23)$$

that holds for all $t \geq T$.

But it is easy to see that a positive function defined in $[T, +\infty[$ cannot satisfy condition (23). This concludes the proof of lemma 3.2. □

4. Proof of Theorem

In view of the symmetry of the problem we suppose the following relation among the parameters of the nonlinearity.

$$p_1 \leq p_m \leq p_{m-1} \leq \ldots \leq p_2.$$

Thus the (12) becomes

$$\frac{\prod_{i=2}^m p_i + 2\left(\prod_{i=2}^{m-1} p_i + \prod_{i=2}^{m-2} p_i + \prod_{i=2}^{m-3} p_i + \ldots + p_2 + 1\right) + p_1^{-1}}{\prod_{i=1}^m p_i - 1} = 1. \qquad (24)$$

We shall argue by contradiction. Hence suppose that $(u_1, u_2, ..., u_m)$ is a smooth solution of (9), (10) with $T = +\infty$.

Since the functions have their support in $\{x : |x| \leq R + t\}$ (see [8, theorems 4, 4a]), we can define for $t \geq 0$ the following C^2 functions

$$F_i(t) = \int_{\{x:|x|\leq R+t\}} u_i(x,t)\ dx, \quad \text{for } i = 1, 2, ..., m \tag{25}$$

By Hölder's inequality it follows that

$$|F_2(t)|^{p_1} = \left| \int_{\{x:|x|\leq R+t\}} u_2(x,t)\ dx \right|^{p_1}$$

$$\leq C\, \|u_2(\cdot,t)\|^{p_1}_{L^{p_1}(\mathcal{R}^3)}\, (R+t)^{3(p_1-1)}, \tag{26}$$

and, in the same way, letting $u_{m+1} = u_1,\ F_{m+1} = F_1$, we have

$$|F_{i+1}(t)|^{p_i} \leq C\, \|u_{i+1}(\cdot,t)\|^{p_i}_{L^{p_i}(\mathcal{R}^3)}\, (R+t)^{3(p_i-1)} \quad \text{for } i = 2, ..., m. \tag{27}$$

On the other hand differentiating one obtains

$$F_i''(t) = \frac{d^2}{dt^2} \int_{\{x:|x|\leq R+t\}} u_i(x,t)\ dx = \int_{\{x:|x|\leq R+t\}} \frac{\partial^2}{\partial t^2} u_i(x,t)\ dx$$

$$= \int_{\{x:|x|\leq R+t\}} (\Box u_i(x,t) + \Delta u_i(x,t))dx,$$

for all $i = 1, ..., m$. Finally the divergence theorem implies that

$$F_i''(t) = \int_{\{x:|x|\leq R+t\}} \Box u_i(x,t)dx$$

$$= \int_{\{x:|x|\leq R+t\}} |u_{i+1}|^{p_i}\ dx = \|u_{i+1}(\cdot,t)\|^{p_i}_{L^{p_i}(\mathcal{R}^3)}\,. \tag{28}$$

Combining (26, 27, 28), it follows that

$$F_i''(t) \geq C(R+t)^{-3(p_i-1)}\, |F_{i+1}(t)|^{p_i} \quad \text{for all } i = 1, ..., m. \tag{29}$$

Furthermore we note that the functions are convex and, by (11), one has

$$F_i'(0) = \left| \frac{d}{dt} \int_{\mathcal{R}^3} u_i(x,t)dx \right|_{t=0} = \int_{\mathcal{R}^3} \frac{\partial}{\partial t} u_i(x,0)dx = \int_{\mathcal{R}^3} g_i(x)dx > 0.$$

As a consequence there exist $T_0 \geq 0$, such that

$$F_i(t) \geq C(R+t), \tag{30}$$

for all $t \geq T_0$, for all $i = 1, ..., m$.

Our aim is to show that the m function F_i satisfy the hypothesis of lemma 3.2. For this purpose we have only to find a further estimate on the first function F_1. We start recalling that the positivity of the Riemann function in low space dimensions and the Huygens'principle imply that

$$\begin{aligned} \int_{\{x:|x| \leq R+t\}} |u_2|^{p_1} \, dx &\geq \int_{\{x:|x| \leq R+t\}} \left|u_2^0\right|^{p_1} dx \\ &\geq \int_{\{x:R-t \leq |x| \leq R+t\}} \left|u_2^0\right|^{p_1} dx \geq C(R+t)^{2-p_1}, \end{aligned}$$

where u_2^0 is the solution of the homogenous wave equation with the same initial data of u_2. This argument yields that there exists T_1 such that

$$F_1(t) \geq C(R+t)^{4-p_1} \tag{31}$$

for all $t \geq T_1$. Consequently from (29) it follows that

$$F_m''(t) \geq C(R+t)^{-3(p_m-1)}(R+t)^{p_m(4-p_1)}$$

for $t \geq T_1$, whereby, integrating twice, we get

$$F_m(t) \geq C(R+t)^{5+p_m-p_1p_m} \tag{32}$$

for all $t \geq T_2$, where $T_2 \geq T_1$ is large enough. Repeating this argument several times, we obtain that

$$F_{m-j}(t) \geq C(R+t)^{\eta_{m-j}} \tag{33}$$

for $t \geq T_3 \geq T_2$, where $j = 0, ..., m-3$, and

$$\eta_m = 5 + p_m - p_1 p_m,$$
$$\eta_{m-j} = 5 - 3p_{m-j} + p_{m-j}\eta_{m-j+1} \quad \text{for } 1 \leq j \leq m-3, \tag{34}$$

so that finally

$$F_3(t) \geq C(R+t)^{\eta_3}, \tag{35}$$

where

$$\eta_3 = 5 + 2(p_3 + p_3 p_4 + ... + \prod_{i=3}^{m-1} p_i) + (1-p_1)(\prod_{i=3}^{m} p_i). \tag{36}$$

Observe now that u_3 obeys the hypothesis of lemma 14, thus

$$\int_{\frac{t}{2} \leq |x| \leq R+t} u_3(x,t)dx \geq \int_{|x| \leq \frac{t}{2}+R} u_3(x, \frac{t}{2})dx \geq C(R + \frac{t}{2})^{\eta_3}$$

for $t \geq 2R$. Next using Hölder's inequality we infer that

$$\int_{\frac{t}{2} \leq |x| \leq R+t} |u_3(x,t)|^{p_2}\, dx \geq C(R+t)^{3+p_2(\eta_3-3)} \tag{37}$$

for t sufficiently large, say $t \geq T_4$, and from (24) we derive that

$$3 + p_2(\eta_3 - 3) = -p_1^{-1}. \tag{38}$$

Let now $t \geq T_5 = (15/2)T_4 + 9R$, and let $k \in \mathcal{N}$, such that

$$\frac{3T_4}{2R} + 1 \leq k \leq \frac{1}{5} + \frac{t}{2R}. \tag{39}$$

We set

$$H_k(t) = \left\{x \in \mathcal{R}^3 : t - (2k+1)R \leq |x| \leq t - (2k-1)R\right\}. \tag{40}$$

Since $k \geq 2$, by the D'Alembert's formula (see [3] for more details) we derive

$$\int_{H_k(t)} u_2(x,t)dx \geq 2\pi \int_{t-(2k+1)R}^{t-(2k-1)R} r dr \int_0^t d\tau \int_{|r-t+\tau|}^{r+t-\tau} \rho\overline{|u_3|^{p_2}}(\rho,\tau)d\rho. \tag{41}$$

Next define

$$A=\left\{(\rho,\tau)\in\mathcal{R}^2:0\le\tau\le t,\ |r-t+\tau|\le\rho\le r+t-\tau\right\},$$

$$B=\left\{(\rho,\tau)\in\mathcal{R}^2:\tfrac{2}{3}(2k+1)R\le\tau\le 2(2k-1)R,\tfrac{\tau}{2}\le\rho\le R+\tau\right\}. \tag{42}$$

If $t-(2k+1)R\le r\le t-(2k-1)R$, then

$$B\subset A.$$

Hence from (41) we obtain

$$\int\limits_{H_k(t)} u_2(x,t)dx \ge 2\pi \int\limits_{t-(2k+1)R}^{t-(2k-1)R} rdr \int\limits_{2/3(2k+1)R}^{2(2k-1)R} d\tau \int\limits_{\tau/2}^{R+\tau} \rho\overline{|u_3|^{p_2}}(\rho,\tau)d\rho$$

$$\ge \frac{1}{2}\int\limits_{t-(2k+1)R}^{t-(2k-1)R} rdr \int\limits_{2/3(2k+1)R}^{2(2k-1)R} (R+\tau)^{-1}d\tau \int\limits_{\tau/2}^{R+\tau} 4\pi\rho^2\overline{|u_3|^{p_2}}(\rho,\tau)d\rho, \tag{43}$$

and since the left hand side of (39) implies that $\tau\ge T_4$, conditions (37), (38) yield

$$\int\limits_{H_k(t)} u_2(x,t)dx \ge \frac{1}{2}\int\limits_{t-(2k+1)R}^{t-(2k-1)R} rdr \int\limits_{2/3(2k+1)R}^{2(2k-1)R} C(R+\tau)^{-p_1^{-1}-1}d\tau. \tag{44}$$

Furthermore

$$\frac{1}{2}\int\limits_{t-(2k+1)R}^{t-(2k-1)R} rdr=(t-2kR)R\ge C(R+t), \tag{45}$$

and

$$\int_{2/3(2k+1)R}^{2(2k-1)R} (R+\tau)^{-p_1^{-1}-1} d\tau \geq Ck^{-p_1^{-1}}, \tag{46}$$

so that we arrive at the following estimate

$$\int_{H_k(t)} u_2(x,t)dx \geq C(R+t)k^{-p_1^{-1}}. \tag{47}$$

Finally from (47) using Hölder's inequality with exponents p_1 and $p_1/(p_1-1)$, we infer

$$\int_{H_k(t)} |u_2(x,t)|^{p_1}\, dx \geq C(R+t)^{(2-p_1)}k^{-1}, \tag{48}$$

and noticing that

$$\sum_{\frac{3T_4}{2R}+1 \leq k \leq \frac{1}{5}+\frac{t}{2R}} k^{-1} \geq C\log(R+t) \tag{49}$$

for all $t \geq T_5$, we conclude that

$$F_1''(t) \geq C(R+t)^{2-p_1}\log(R+t) \tag{50}$$

for all $t \geq T_5$. Integrating twice this last inequality, we deduce that there exists $T_6 \geq T_5$ such that

$$F_1(t) \geq C(R+t)^{4-p_1}\log(R+t) \tag{51}$$

for all $t \geq T_6$.

But condition (51) together with (29), (30), and (12) imply that the m functions F_i satisfy the hypothesis of lemma 3.2. Thus we have reached the desired contradiction. The proof is complete.

References

[1] Daniele Del Santo, *Global existence and blow-up for a hyperbolic system in three space dimensions*, Rend. Ist. Mat. Univ. Trieste **29** (1997), 115–140.

[2] Daniele Del Santo, Vladimir Georgiev, and Enzo Mitidieri, *Global existence of the solutions and formation of singularities for a classof hyperbolic systems*, Geometrical Optics and Related Topics (F. Colombini and N. Lerner, eds.), vol. PNLDE 32, Birkhäuser, Boston, 1997, pp. 117–140.

[3] Daniele Del Santo and Enzo Mitidieri, *Blow-up for a hyperbolic system: the critical case*, Differential Equations (Differential 'Nye Uravnenija) **34** (1998), 1–7.

[4] Robert T. Glassey, *Existence in the large for $\Box u = F(u)$ in two space dimensions*, Mathematische Zeitschrift **178** (1981), 233–261.

[5] Robert T. Glassey, *Finite time blow-up for solutions of nonlinear wave equations*, Mathematische Zeitschrift **177** (1981), 323–340.

[6] Fritz John, *Partial Differential Equations*, Springer-Verlag, 1971.

[7] Fritz John, *Blow-up of non linear wave equations in three space dimensions*, Manuscripta Mathematica **28** (1979), 233–261.

[8] Fritz John, *Blow-up for quasi-linear wave equations in three space dimensions*, Comm. Pure Appl. Math. **34** (1981), 29–51.

[9] Atanasio Pantarrotas, *Finite time blow-up for solutions of a hyperbolic system.*

[10] Atanasio Pantarrotas, *Global existence of the solutions of a hyperbolic system in low space dimensions.*

[11] Jack Schaeffer, *The equation $u_{tt} - \Delta u = |u|^p$ for the critical value of p*, Proceedings of the Royal Society of Edinburgh **101A** (1985), 31–44.

[12] Thomas C. Sideris, *Nonexistence of global solutions to semilinear wave equations in high dimensions*, Journal of Differential Equations **52** (1984), 378–406.

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