SELF–SIMILAR SETS AND MEASURES (*)

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A major theme in the theory of fractals is that of self–similarity: the whole fractal set is composed of smaller parts which are geometrically similar to whole set. There are several ways to formulate this concept in a mathematically rigorous way. Here I will deal with Hutchinson's definition of self–similarity. It is the purpose of this lecture to collect some of the basic results concerning the Hausdorff dimension, Hausdorff measure and local structure of self-similar sets and measures. I am not striving for completeness but rather use my own research interests as a guide to the results and problems in the area. Most of the proofs are omitted. Interested readers are refered to the literature.

1. Iterated function systems and their attractors.

In this section I will describe the basic construction for self–similar sets using the terminology of Barnsley [2].

1.1. Definition.

Let (E, d) be a metric space,

a) A map $w : E \to E$ is called a **contraction** if there exists a number $c < 1$ such that $d(w(x), w(y)) \leq cd(x, y)$ for all $x, y \in E$. By $Lip(w)$ we denote the smallest c satisfying the above

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condition and call it the **contraction** or **Lipschitz constant** of w.

b) An **iterated function system** (IFS) on E is an N–tuple $(w_1, \ldots, w_N)$ of contractions from E into itself.

c) A non–empty compact subset A of E is called an **attractor** of the IFS $(w_1, \ldots, w_N)$ if $A = w_1(A) \cup \ldots \cup w_N(A)$.

1.2. THEOREM. (Hutchinson [6]) *If* (E, d) *is complete then every* IFS *on E has a unique attractor.*

Idea of proof. Consider the space $\mathcal{K}(E)$ of all non–empty compact subsets of E with the Hausdorff metric h:

$$h(K, L) = max(max\{d(x, L) : x \in K\}, max\{d(y, K) : y \in L\}).$$

Then $(\mathcal{K}(E), h)$ is complete and $W : \mathcal{K}(E) \to \mathcal{K}(E)$ defined by

$$W(K) = w_1(K) \cup \ldots \cup w_N(K)$$

is a contraction. Hence the theorem follows from Banach's fixed point theorem.

1.3. THEOREM. (Hutchinson [6]) *Let* (E, d) *be complete,* $(w_1, \ldots, w_N)$ *an* IFS *on* E *with attractor* A *and* $x_0 \in E$. *Then, for every* $\eta \in \{1, \ldots, N\}^{\mathbf{N}}$, *the limit*

$$\lim_{n\to\infty} w_{\eta_1} \circ \ldots \circ w_{\eta_n}(x_0)$$

exists and is a point in A. *Moreover, the map* $\pi : \{1, \ldots, N\}^{\mathbf{N}} \to A, \eta \to \lim_{n\to\infty} w_{\eta_1} \circ \ldots \circ w_{\eta_n}(x_0)$ *is continuous and onto.*

2. Natural measures on the attractor of an iterated function system.

Here I will introduce a class of measures related to iterated function systems. In the following (E, d) is always a complete metric space, $(w_1, \ldots, w_N)$ an IFS on E, A its attractor, $p := (p_1, \ldots, p_N)$ is a probability vector, ν_p is the corresponding product measure on $\{1, \ldots, N\}^{\mathbf{N}}$ and $\mu_p = \nu_p \circ \pi^{-1}$ is the image measure of ν_p with respect to π.

2.1. Theorem. (Hutchinson [6]) *The measure μ_p is the unique probability measure μ on E with*

$$\mu = \sum_{i=1}^{N} p_i \mu \circ w_i^{-1}$$

2.2. Remark and definitions.

(i) There is a unique $\alpha \in \mathbf{R}_+$ with $\sum_{i=1}^{N} Lip(w_i)^\alpha = 1$. α is called the **similarity dimension** of $(w_1, \ldots, w_N)$.

(ii) For $p = (Lip(w_1)^\alpha, \ldots, Lip(w_N)^\alpha)$ the measure μ_p is called the **canonical measure** on the attractor A and denoted by μ.

3. Connection between Hausdorff dimension and similarity dimension for self–similar sets.

3.1. Definition. *A map $S : E \to E$ is called a* **similitude** *if there is a $c \in]0, +\infty[$ with*

$$d(Sx, Sy) = cd(x, y) \qquad \text{for all } x, y \in E.$$

The attractor of an IFS consisting of similitudes is called a **self–similar set**. *For $\beta \in [0, +\infty[, \delta > 0$ and $B \subset E$ define*

$$\mathcal{H}_\delta^\beta(B) = \inf\{\sum_{i \in i} diam(U_i)^\beta \mid B \subset \cup_{i \in I} U_i, U_i \text{ open}, diam(\mathcal{U}_i) \le \delta\}.$$

Then $\mathcal{H}^\beta(B) := \lim_{\delta\to 0} \mathcal{H}^\beta_\delta(B)$ *is the* β*–dimensional* **Hausdorff measure** *of* B.

There exists a unique $\beta_c \in [0, +\infty]$ *with* $\mathcal{H}^\beta(B) = \infty$ *for all* $\beta < \beta_c$ *and* $\mathcal{H}^\beta(B) = 0$ *for all* $\beta > \beta_c$. *The number* β_c *is called the* **Hausdorff dimension** *of* B *and is denoted by* $H\text{–}dim(B)$.

In the following $(S_1, \ldots, S_N)$ *is always an* IFS *consisting of similitudes of* E, A *is its attractor, and* α *its similarity dimension.*

Next I will summarize the main results concerning the connection of α and $H\text{–}dim(A)$.

3.2. Theorem. (Hutchinson [6]) *The* α*–dimensional Hausdorff measure of* A *is finite, in particular the Hausdorff dimension of* A *is less than or equal to* α.

The question under what circumstances the Hausdorff dimension of A actually equals the similarity dimension α led Hutchinson [6] to define the open set condition.

3.3. Definition. The IFS $(S_1, \ldots, S_N)$ satisfies the **open set condition** (OSC) if there is a non–empty open set $U \subset E$ with $S_i(U) \subset U$ and $S_i(U) \cap S_j(U) = \emptyset$ for all $i, j \in \{1, \ldots, N\}$ with $i \neq j$.

If there is such a U with $U \cap A \neq \emptyset$ then $(S_1, \ldots, S_N)$ satisfies the **strong open set condition** (SOSC).

3.4. Theorem. (Schief [14]) *The following implications hold*

(i) $\mathcal{H}^\alpha(A) > 0 \Rightarrow (S_1, \ldots, S_N)$ satisfies the SOSC

(ii) $(S_1, \ldots, S_N)$ *satisfies the SOSC* $\Rightarrow H\text{–}dim(A) = \alpha$.

For general complete metric spaces the converse of both of these implications is false (see Schief [14]). But for euclidean spaces we have the following result:

3.5. THEOREM. (Moran [8], Hutchinson [6], Schief [13]) *If* $E = \mathbf{R}^m$ *then the following statements are equivalent*

(i) $(S_1, \ldots, S_N)$ *satisfies the OSC*

(ii) $(S_1, \ldots, S_N)$ *satisfies the SOSC*

(iii) $0 < \mathcal{H}^\alpha(A)$.

If (i) – (iii) hold then the canonical measure μ *on* A *is the normalization of the* α*–dimensional Hausdorff measure restricted to* A.

The implication (i) $\Rightarrow$ (ii) was proved by Hutchinson [6] who thereby rediscovered an argument used by Moran [8] in a more general context. Hutchinson [6] also proved the statement about the canonical measure. The remaining assertions of the theorem were proved by Schief [13].

As an obvious consequence the preceding theorem has the following corollary.

3.6. COROLLARY. *If* $E = \mathbf{R}^m$ *and* $(S_1, \ldots, S_N)$ *satisfies the OSC then* $\alpha = H\text{–}dim(A)$.

That the converse does not hold even for $E = \mathbf{R}$ is a consequence of the first of the following remarks.

3.7. REMARKS.

(i) Let $S_1, S_2, S_3 : \mathbf{R} \to \mathbf{R}$ be defined by $S_1 x = \frac{1}{3}, S_2 x = \frac{1}{3}x + t, S_3 x = \frac{2}{3}x + \frac{1}{3}$ with $t \in [0, \frac{2}{3}[$. Then (S_1, S_2, S_3) is an IFS consisting of similitudes and there exists a $t \in]0, \frac{2}{3}[$ such that $\alpha = H\text{–}dim(A)$ and $\mathcal{H}^\alpha(A) = 0$. (Bandt–Mattila, oral communication 1989)

(ii) There is a complete metric space (E, d) and an IFS $(S_1, \ldots, S_N)$ on E consisting of similitudes and satisfying the SOSC with $\mathcal{H}^\alpha(A) = 0$. (Schief [14]).

While the first remark follows from a projection theorem for the 1–dimensional Hausdorff measure which does not give an explicit value for t the second remark is proved by exhibiting an explicit example.

3.8. Definition.

An *IFS* $(S_1, \ldots, S_N)$ with attractor A satisfies the **relative open set condition** (ROSC) if there exists a non–empty set U, which is open in the relative topology on A, such that

$$\begin{array}{ll} S_i(U) \subset U & \text{and} \\ S_i(U) \cap S_j(U) = \emptyset & \text{for } i \neq j \end{array}$$

and all $i, j \in \{1, \ldots, N\}$.

3.9. Theorem. *If $(S_1, \ldots, S_N)$ satisfies the ROSC then H–dim$(A) = \alpha$.*

Proof. The result is an immediate consequence of Theorem 3.4 (ii) if one takes $E = A$.

3.10. Problem. Does the converse of Theorem 3.9 hold? (The answer is not known even for $E = \mathbf{R}^m$).

4. The dimension of the measures μ_p.

In this section $(S_1, \ldots, S_N)$ is an *IFS* on the euclidean space $\mathbf{R}^m$ consisting of similitudes.

4.1. DEFINITION.

a) Let ν be a probability measure on $\mathbf{R}^m$. Then $H\text{–}dim(\nu) = \inf\{H\text{–}dim(B) \mid B$ Borel set, $\nu(B) = 1\}$ is called the **Hausdorff dimension** of ν.

b) If $p = (p_1, \ldots, p_N)$ is a probability vector and μ_p the corresponding natural probability measure on the attractor A of $(S_1, \ldots, S_N)$. Then

$$\alpha_p = \left(\sum_{i=1}^{N} p_i log p_i\right) / \left(\sum_{i=1}^{N} p_i log Lip(S_i)\right)$$

is called the **similarity dimension** of μ_p.

4.2. REMARK. If $p = (Lip(S_1)^\alpha, \ldots, Lip(S_N)^\alpha)$ then $\alpha_p = \alpha$.

4.3 THEOREM. (Cawley–Mauldin [4])

If $(S_1, \ldots, S_N)$ satisfies the OSC then

$$H\text{–}dim(\mu_p) = \alpha_p.$$

5. The density of self–similar sets.

In this section I will discuss several results related to the classical Lebesgue density theorem.

5.1. DEFINITION.

a) Let β be a non–negative real number and B a Borel subset of $\mathbf{R}^m$. B is called a **β–set** if it has positive and finite β-dimensional Hausdorff measure.

b) For a subset B of $\mathbf{R}^m$ and a point $x \in \mathbf{R}^m$ we call

$$\overline{D}^\beta(B, x) = \limsup_{r \to 0} \frac{\mathcal{H}^\beta(B \cap B(x, r))}{(2r)^\beta}$$

the **upper density** of B at x and

$$\underline{D}^\beta(B,x) = \liminf_{r\to 0} \frac{\mathcal{H}^\beta(B\cap B(x,r))}{(2r)^\beta}$$

the **lower density** of B at x.
Here $B(x,r)$ denotes the open ball of radius r and center x.
If $\overline{D}^\beta(B,x)$ and $\underline{D}^\beta(B,x)$ are both finite and equal then the common value is called the density of B at x and denoted by $D(B,x)$.

5.2. THE LEBESGUE DENSITY THEOREM. Let $B \subset \mathbf{R}^m$ be Lebesgue measurable. Then

$$D^m(B,x) = 1_B(x)$$

for $\mathcal{H}^m$ – a.e. $x \in \mathbf{R}^m$.

5.3. REMARK. It should be noted that $\mathcal{H}^m$ is a multiple of the m–dimensional Lebesgue measure λ^m. One has $\mathcal{H}^m = \frac{1}{\lambda^m(B(0,\frac{1}{2}))}\lambda^m$ and it is well–known that $\lambda^m(B(0,\frac{1}{2})) = \pi^{\frac{1}{2}n}/2^n(\frac{1}{2})!$ This relation between m–dimensional Hausdorff and Lebesgue measure implies that the above result is indeed a version of the classical Lebesgue density theorem because

$$\frac{\mathcal{H}^m(B\cap B(x,r))}{(2r)^m} = \frac{\lambda^m(B\cap B(x,r))}{\lambda^m(B(x,r))}.$$

In the following I will investigate how the Lebesgue density theorem can be generalized to β–sets. The next theorem shows that a direct generalization is not possible.

5.4. THEOREM. (Marstrand [7]) *If β is a non–negative number which is not an integer and if B is a β–subset of $\mathbf{R}^m$, then*

$$\underline{D}^\beta(B,x) < \overline{D}^\beta(B,x)$$

for $\mathcal{H}^\beta$*– a.e.* $x \in B$ *and, moreover,*

$$D^\beta(B, x) = 0$$

for $\mathcal{H}^\beta$*– a.e.* $x \in \mathbf{R}^m \setminus B$*.*

Using the concept of density of order two developed and studied by Bedford–Fisher [3] I obtained the following result concerning the average density which was independently proved by Patzschke–Zähle [10] in a more general context and applying different methods.

5.5. Theorem. *If* $(S_1, \ldots, S_N)$ *is an* IFS *on* $\mathbf{R}^m$ *consisting of similitudes and satisfying the OSC and if* A *is its attractor and* α *its similarity dimension then there exists a* $c \in]0, +\infty[$ *such that*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{\mathcal{H}^\alpha(B(x, e^{-t}) \cap A)}{2^\alpha e^{-\alpha t}} dt = c$$

for $\mathcal{H}^\alpha$ *– a.e.* $x \in A$*.*

5.6. Remarks.

a) For the Cantor set in the line Patzschke–Zähle [10] calculated the number c. In the general case of Theorem 5.5 there is a formula for c (see Graf [5], Patzschke–Zähle [11]) but its numerical value is still hard to compute.

b) It seems to be an open problem whether in the situation of Theorem 5.5, for every Borel set $B \subseteq A$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{\mathcal{H}^\alpha(B \cap B(x, e^t))}{2^\alpha e^{-\alpha t}} dt = c 1_B(x)$$

for $\mathcal{H}^\alpha$–a.e. $x \in A$.

6. The tangential structure of self-similar sets and measures.

It is the purpose of this section to review some of the results concerning the local structure of self–similar measures. First I will recall the definition of tangent measure due to Preiss [12] and state a fundamental result of his concerning rectifiability which illustrates the meaning of tangent measures.

6.1. DEFINTION.

(i) For $x \in \mathbf{R}^m$ and $r > 0$ define $T_{x,r} : \mathbf{R}^m \to \mathbf{R}^m$ by

$$T_{x,r}(z) = \frac{1}{r}(z - x).$$

(ii) Let Φ and Ψ be locally finite Borel measures on $\mathbf{R}^m$ and $x \in \mathbf{R}^m$. Ψ is called a **tangent measure** of Φ at x if $\Psi \neq 0$ and there are sequences $r_k \downarrow 0$ and $c_k > 0$ such that Ψ is the vague limit of the sequence $(c_k \Phi \circ T_{x,r_k}^{-1})_{k \in \mathbf{N}}$. Let $Tan(\Phi, x)$ denote the set of all tangent measures of Φ at x.

(iii) For $k \in \{1, \dots, m\}$ a locally finite Borel measure Φ on $\mathbf{R}^m$ is called k**–rectifiable** if there exists a sequence (C_l) of $\mathcal{C}^1$–manifolds of dimension k such that $\Phi(\mathbf{R}^m \setminus \bigcup_l C_l) = 0$.

6.2. THEOREM. (Preiss [12]) *For a locally finite Borel measure Φ on $\mathbf{R}^m$ the following statements are equivalent*

(i) *Φ is k–rectifiable*

(ii) $\lim\limits_{r \to 0} \frac{\Phi(B(x,r))}{r^k}$ *exists and is finite and positive for Φ–a.e. $x \in \mathbf{R}^m$*

(iii) *For Φ–a.e. $x \in \mathbf{R}^m$ one has lim $\inf_{r \to 0} \frac{\Phi(B(x,r))}{r^k} < \infty$ and there is a k–dimensional subspace V of $\mathbf{R}^m$ with $Tan(\Phi, x) = \{c\mathcal{H}^k_{|V} \mid c > 0\}$.*

Considering this result of Preiss it seems to be of interest to study the set of tangent measures for more general measures on $\mathbf{R}^m$. In particular one has the following

6.3. PROBLEMS. Let $(S_1, \ldots, S_N)$ be an IFS consisting of similitudes on $\mathbf{R}^m$ and let $p = (p_1, \ldots, p_N)$ be a probability vector and μ_p the corresponding natural measure on the attractor A (see Section 2).
a) Determine $T(\mu_p, x)$ (at least for μ_p–a.e. x).
b) Is $Tan(\mu_p, x) = Tan(\mu_p, y)$ for $\mu_p \otimes \mu_p$–a.e. (x, y)?

For fractal measures Φ the sets $Tan(\Phi, x)$ are usually rather complicated. Inspired by an idea of U. Zähle [15] Bandt [1], therefore, introduced the concept of random tangent measures or, equivalenty, probability distributions on the set of tangent measures, the so–called tangent measure distributions.

6.4. DEFINITION. Let $\mathcal{M}_m$ be the space of all locally finite measures on $\mathbf{R}^m$ with the topology of vague convergence. For a locally finite Borel measure on $\mathbf{R}^m$ a Borel probability P on $\mathcal{M}_m \setminus \{0\}$ is called a **tangent measure distribution** of Φ at $x \in \mathbf{R}^m$ if there exists a non–decreasing function $h : \mathbf{R}_+ \to \mathbf{R}_+ \setminus \{0\}$ and a sequence $(\nu_k)_{k \in \mathbf{N}}$ of Borel probabilities on $\mathbf{R}_+ \setminus \{0\}$ with $\lim\limits_{k \to \infty} \nu_k = \varepsilon_0$ (where ε_0 is the Dirac measure at 0 and the convergence is weak convergence) such that the image probabilities P_k of ν_k with respect to the map $\mathbf{R}_+ \setminus \{0\} \to \mathcal{M}_m, r \to (h(r))^{-1} \Phi \circ T_{x,r}^{-1}$ converge to P (weakly).

6.5. REMARK. To my knowledge no statements about general tangent measure distributions have been proved so far. For the known results the class of probabilities on $\mathbf{R}_+ \setminus \{0\}$ from which the ν_k are chosen and the function h are specialized. In this context Mörters [9] has investigated the basic properties of uniquely determined tangent measure distributions.

6.6. Definition. For $1 > R > 0$ let κ_R be the Borel probability on $[R, 1]$ defined by

$$\begin{aligned}\kappa_R(B) &= -\frac{1}{lnR}\int_R^1 1_B(r)\frac{dr}{r}\\ &= -\frac{1}{lnR}\int_0^{-lnR} 1_B(e^{-t})dt\ .\end{aligned}$$

6.7. Remark. κ_R is Haar measure on the group $(R_+ \setminus \{0\}, \cdot)$ restricted to the interval $[R, 1]$ and normalized. Moreover one has

$$\lim_{R\to 0}\kappa_R = \varepsilon_0 \qquad \text{(weak convergence)}.$$

The following theorem was conjectured and proved in special cases by Bandt [1].

6.8 Theorem. (Graf [5]). *Let $(S_1, \ldots, S_N)$ be an IFS consisting of similitudes on $\mathbf{R}^m$ and A its attractor. Let $p = (p_1, \ldots, p_N)$ be a probability vector and μ_p the corresponding natural measure on A. For $x \in A$ let P_R^x be the image of κ_R with respect to the map $\mathbf{R}_+ \setminus \{0\} \to \mathcal{M}_m, r \to \mu_p(B(x,r))^{-1}\mu_p \circ T_{x,r}^{-1}$. Then there exists a Borel probability P on $\mathcal{M}_m$ such that*

$$\lim_{R\to 0} P_R^x = P \qquad \textit{(weak convergence)}$$

for μ_p–a.e. $x \in A$.

6.9. Corollary. The P in Theorem 6.8 is a tangent measure distribution of μ_p at x for μ_p–a.e. $x \in A$.

6.10. Problem. Given two different tangent measure distributions of μ_p at x. What is their relationship (μ_p–a.e.)?

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