COMPLEXITY OF COMPACTIFICATIONS OF N (*)

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Sommario. - *A causa della sua complessità, la compattificazione di Stone-Čech dei numeri naturali è tra gli spazi topologici più studiati. Altre compattificazioni di* $\mathbf{N}$ *possono essere altrettanto complesse. In questo lavoro si esamina il reticolo delle compattificazioni di* $\mathbf{N}$ *rispetto a due misure di complessità.*

Summary. - *Because of its complexity, the Stone-Čech compactification of the natural numbers is among the most studied of topological spaces. Other compactifications of* $\mathbf{N}$ *share in this complexity. This paper begins an examination of the lattice of compactifications of* $\mathbf{N}$ *with respect to two measures of complexity that compactifications may share with* $\beta\mathbf{N}$.

1. Introduction.

The Stone-Čech compactification of the natural numbers, $\beta\mathbf{N}$, has long been a fascination of topologist. This is almost certainly due to the fact that something of such tantalizing complexity could arise out of such a topologically trivial object as the natural numbers. This complexity has made $\beta\mathbf{N}$ into something of an example machine and one of the principle grounds of interaction between set theory and topology.

Of course the lattice of compactifications of $\mathbf{N}$ is huge and there are very many compactifications of $\mathbf{N}$ which are "near" $\beta\mathbf{N}$ in the lattice. Perhaps these compactification share in the complexity of $\beta\mathbf{N}$. It is this that we begin to address in this paper.

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What do we mean when we say that $\beta\mathbf{N}$ is complex ? We might of course mean any one of an almost inexhaustable list of properties. We will consider two of these:

1. First $\beta\mathbf{N}$ contains no nontrivial convergent sequences.

Vaughan and Dow call spaces with this property "contrasequential" [3]. We will usually use the notation $\omega + 1 \not\hookrightarrow X$. If $\omega + 1 \not\hookrightarrow \alpha\mathbf{N}$, then this alone forces restrictions on how small the remainder can be. In particular $|\alpha\mathbf{N} \setminus \mathbf{N}| \geq 2^{\mathfrak{t}}$, where $\mathfrak{t}$ is the smallest cardinality of a tower in $\mathcal{P}(\mathbf{N})$. In fact it is known that a compact space of cardinality less than $2^{\mathfrak{t}}$ is sequentially compact [14, Thm 6.3], [16, Thm. 5.9]. Of course, within the lattice of compactifications, the presence of nontrivial convergent sequences in some cases yields as much information as their absence.

2. $\beta\mathbf{N}$ contains a wealth of copies of $\beta\mathbf{N}$.

In particular if F is any infinite closed subset of $\beta\mathbf{N}$ then $\beta\mathbf{N} \hookrightarrow F$. Clearly, all compactifications of $\mathbf{N}$ which share this property with $\beta\mathbf{N}$ are contrasequential. However, it is known, at least consistently, that there are compactifications of $\mathbf{N}$ which contain neither $\omega + 1$ or $\beta\mathbf{N}$ [6].

The notation substantially follows [2,7]. Information on cardinal functions can be found in [8], and information on small cardinals in [14,16].

We will denote by $\mathbf{N}$ the set and the discrete space of the natural numbers and by ω the cardinality of $\mathbf{N}$. Also, as is customary, $A^* = Cl_{\beta\mathbf{N}}(A)\setminus A$ for each infinite $A \subseteq \mathbf{N}$. In particular $\mathbf{N}^* = \beta\mathbf{N} \setminus \mathbf{N}$.

If X is a Tychonoff space, the canonical quotient map from a larger compactification γX to a smaller compactification αX will be denoted by $\pi_{\gamma\alpha}$. As usual, $C^*(X)$ denotes the algebra of all bounded real valued continuous functions on X. $C^*(X)$ is a Banach space with the *supremum* norm. In particular, $C^*(\mathbf{N})$ coincides with the space l_∞ of all bounded real valued sequences. As usual, c_0 is the subspace of consisting of those sequences which converge to zero.

The closed subalgebra of $C^*(X)$ generated by a subcollection $\mathcal{F}$ is denoted by $\overline{\langle\mathcal{F}\rangle}$. $C_\alpha(X)$ is the (closed) subalgebra of $C^*(X)$ consisting of those functions which have continuous extensions to the compactification αX. Usually, the unique extension of $f \in C_\alpha(X)$ to αX will be denoted by f^α. The map $f \mapsto f^\alpha$ is an isomorphism from $C_\alpha(\mathbf{N})$ onto $C(\alpha X)$ and one has $\| f^\alpha \| = \| f \|$. The extension of $f \in C^*(\mathbf{N})$ to $\beta\mathbf{N}$ will be denoted, as is customary, by f^*.

If Y is a Banach space and M is a subspace of Y, then we say that M is complemented in Y provided there is another closed subspace N of Y for which $Y = M \oplus N$. A subspace M is complemented if and only if M is the nullspace of a continuous projection defined on Y.

1. First we will consider several nontrivial examples of compactifications of $\mathbf{N}$ which are contrasequential.

If F is a closed subset of $\mathbf{N}^*$, we will denote by $\beta\mathbf{N}/F$ the compactification resulting from identifying F to a single point.

THEOREM 1.1. *Suppose that $\alpha\mathbf{N} = \beta\mathbf{N}/F$, where F is a retract of $\beta\mathbf{N}$. Then $\omega + 1 \not\hookrightarrow \alpha\mathbf{N}$.*

Proof. Let $r : \beta\mathbf{N} \to F$ be a retraction and let $r_1 = r|_{\mathbf{N}}$. Let I_F be the ideal of functions in $C^*(\mathbf{N}) = l_\infty$ whose extensions to $\beta\mathbf{N}$ vanish on F. Define a continuous linear projection $P : C^*(\mathbf{N}) \to C^*(\mathbf{N})$ by $Pf = f^* \circ r_1$. Clearly the nullspace of P is exactly I_F. Thus I_F is complemented in $C^*(\mathbf{N})$, and since I_F is of codimension one in $C_\alpha(\mathbf{N})$, $C_\alpha(\mathbf{N})$ is complemented as well. Thus $C_\alpha(\mathbf{N})$ is isomorphic to $C^*(\mathbf{N})$ [10]. Now suppose $\omega + 1 \hookrightarrow \alpha\mathbf{N}$. In particular, let $\{p_n \mid n < \omega\} \cup \{p\} \subseteq \alpha\mathbf{N}$, with $p_n \to p$. Define a mapping $Q : C_\alpha(\mathbf{N}) \to c_0$ by $Qf = \{f^\alpha(p_n) - f^\alpha(p)\}_{n=1}^\infty$. Clearly $\| Q \| \leq 2$, so that Q is a continuous linear operator. Since any copy of $\omega + 1$ must be C^*-embedded in $\alpha\mathbf{N}$, Q is surjective. By the open mapping theorem, Q must be open. However, Q must also be weakly compact [8]. This would imply that c_0 is reflexive. Hence $\omega + 1 \not\hookrightarrow \alpha\mathbf{N}$. ♢

The hypothesis that F is a retract of $\beta\mathbf{N}$ is satisfied, for example, if F is of countable π-weight [15, Thm. 1.8.2]. In particular, if $F \subset \mathbf{N}^*$ is homeomorphic to $\beta\mathbf{N}$, then $\omega + 1 \not\hookrightarrow \beta\mathbf{N}/F$.

It is known that there are closed separable subsets of $\beta\mathbf{N}$ which are not retracts of $\beta\mathbf{N}$ [13]. This leads to:

QUESTION. Suppose $\alpha\mathbf{N} = \beta\mathbf{N}/F$, where F is separable. Can $\alpha\mathbf{N}$ contain nontrivial convergent sequences?

In any case it is clear that, if you identify a large enough set, then convergent sequences must arise.

LEMMA 1.2. *Let $\alpha\mathbf{N}$ be a compactification of $\mathbf{N}$. If, for some $p \in \alpha\mathbf{N} \setminus \mathbf{N}$, $\pi_{\beta\alpha}^{-1}(p)$ has nonempty interior in $\mathbf{N}^*$, then $\omega + 1 \hookrightarrow \alpha\mathbf{N}$.*

Proof. Suppose $\pi_{\beta\alpha}^{-1}(p)$ has nonempty interior in $\mathbf{N}^*$. Then $\pi_{\beta\alpha}^{-1}(p)$ contains a set of the form A^* for some $A \subseteq \mathbf{N}$. This set A, considered as a sequence in $\alpha\mathbf{N}$. converges to p. ♢

REMARK 1.3. We may observe that, if there is a sequence in $\mathbf{N}$ which converges to a point p in $\alpha\mathbf{N} \setminus \mathbf{N}$, then $\pi_{\beta\alpha}^{-1}(p)$ has nonempty interior.

Now, how does one typically obtain convergent sequences in a compactification of $\mathbf{N}$ when viewed as a quotient of $\beta\mathbf{N}$? One can proceed as follows. Let D be any countable discrete subset of $\beta\mathbf{N}$. Since D is C^*-embedded in $\beta\mathbf{N}$, $Cl_{\beta\mathbf{N}}(D) \cong \beta\mathbf{N}$. If in this copy of $\beta\mathbf{N}$ we identify the remainder to a point, we obtain a convergent sequence. We might in fact collapse any closed set which contains this remainder and misses the points of D. This is in fact what always happens. Suppose $\{p_n \mid n < \omega\}$ is a nontrivial sequence in $\alpha\mathbf{N}$ converging to a point p. Let $q_n \in \pi_{\beta\alpha}^{-1}(p_n)$ and $D = \{q_n \mid n < \omega\}$. As before D is C^*-embedded in $\beta\mathbf{N}$ so that $Cl_{\beta\mathbf{N}}(D) \setminus D \cong \mathbf{N}^*$. In the quotient $\alpha\mathbf{N}$, this set, and perhaps more, must be collapsed to p. From this it is easy to observe that:

PROPOSITION 1.4. *If $\omega+1 \hookrightarrow \alpha\mathbf{N}$, then there is a compactification $\gamma\mathbf{N}$, strictly larger than $\alpha\mathbf{N}$, for which $\omega + 1 \hookrightarrow \gamma\mathbf{N}$.*

Proof. Let $\{p_n \mid n < \omega\}$ be a nontrivial convergent sequence in $\alpha\mathbf{N}$, and let p be its limit point. Then there is a copy F of $\mathbf{N}^*$ in $\beta\mathbf{N}$ such that $\pi_{\beta\alpha}(F) = \{p\}$. Take a copy of $\beta\mathbf{N}$ in F. The compactification of $\mathbf{N}$ resulting from the collapse of the remainder in this copy of $\beta\mathbf{N}$ to a point is the desired $\gamma\mathbf{N}$. ♢

If X is a locally compact space, K is compact, and $f : X \to K$ is continuous, then, clearly, there exists a minimum compactification αX of X to which f extends. Such a compactification can actually be constructed by endowing the disjoint union $X \cup K$ with a suitable topology, which makes it compact, and putting $\alpha X = Cl_{X\cup K}(X)$ [1,5,11]. The remainder of αX is the *singular set* of f, $\mathcal{S}(f) = \{y \in K \mid$ for each neighborhood V of y, $\overline{f^{-1}(V)}$ is not compact$\}$. The extension $\hat{f}$ of f to αX is the identity on

$\alpha X \setminus X$. Moreover, if γX is a compactification to which f extends, then $\pi_{\gamma\alpha} |_{\gamma X \setminus X} = \tilde{f} |_{\gamma X \setminus X}$, where $\tilde{f}$ is the extension of f to γX.

If $f \in C^*(X)$, then the minimum compactificaton to which f extends is denoted by $\omega_f X$. Clearly $\omega_f X \setminus X$ is homeomorphic to a compact subspace of $\mathbf{R}$.

If $\mathcal{F} \subseteq C^*(X)$, then we denote by $\omega_{\mathcal{F}} X$ the minimum compactification to which each member of $\mathcal{F}$ extends. Clearly $\omega_{\mathcal{F}} X = \sup\{\omega_f X \mid f \in \mathcal{F}\}$. Therefore $\omega_{\mathcal{F}} X \setminus X$ is homeomorphic to a subspace of the Tychonoff cube $\mathbf{I}^{|\mathcal{F}|}$. Clearly the family of the extensions of the elements of $\mathcal{F}$ to $\omega_{\mathcal{F}} X$ must separate points of $\omega_{\mathcal{F}} X \setminus X$. One has $w(\omega_{\mathcal{F}} X \setminus X) \leq |\mathcal{F}|$ and, if X is second countable, $w(\omega_{\mathcal{F}} X) \leq |\mathcal{F}|$

THEOREM 1.5. *Let g be a map from $\mathbf{N}$ into $\mathbf{N}$ and let $f : \mathbf{N} \to \beta\mathbf{N}$ be the composition of g and the inclusion map. Let $\alpha\mathbf{N}$ be the minimum of the compactifications of $\mathbf{N}$ to which f extends. Then $\omega + 1 \not\hookrightarrow \alpha\mathbf{N}$ if and only if g is is finite-to one.*

Proof. We can put $\alpha\mathbf{N} = \mathbf{N} \cup \mathcal{S}(f) \subseteq \mathbf{N} \cup \beta\mathbf{N}$. Let $\tilde{f}$ denote the extension of f to $\beta\mathbf{N}$. First suppose that there is $n \in \mathbf{N}$ such that $A = g^{-1}(n) = f^{-1}(n)$ is infinite. Then $n \in \mathcal{S}(f) = \alpha\mathbf{N} \setminus \mathbf{N}$ and one has $\tilde{f}(A^*) = \{n\}$. Then $\pi_{\beta\alpha}(A^*) = \{n\}$, hence, by lemma 1.2, $\omega + 1 \hookrightarrow \alpha\mathbf{N}$.

Now, let g be finite to one. Then $g(\mathbf{N})$ is infinite. We can replace $g(\mathbf{N})$ by $\mathbf{N}$ (and $Cl_{\beta\mathbf{N}}(g(\mathbf{N}))$ by $\beta\mathbf{N}$), hence, without loss of generality, we can suppose g surjective. Then, obviously, $\alpha\mathbf{N} \setminus \mathbf{N} = \mathcal{S}(f) = \mathbf{N}^*$, so $\omega + 1 \not\hookrightarrow \alpha\mathbf{N} \setminus \mathbf{N}$. Now, let B be an infinite subset of $\mathbf{N}$. Then $f(B) = g(B)$ is infinite. One has $\tilde{f}(B^*) = \pi_{\beta\alpha}(B^*) \subseteq \alpha\mathbf{N} \setminus \mathbf{N} = \mathbf{N}^*$. Clearly one has $\tilde{f}(B^*) = \tilde{f}(Cl_{\beta\mathbf{N}}(B)) \setminus \mathbf{N}$. Moreover, since $\tilde{f}$ is closed, one has $\tilde{f}(Cl_{\beta\mathbf{N}}(B)) = Cl_{\beta\mathbf{N}}(\tilde{f}(B)) = Cl_{\beta\mathbf{N}}(f(B))$. So we have proved $\pi_{\beta\alpha}(B^*) = Cl_{\beta\mathbf{N}}(f(B)) \setminus \mathbf{N} = (f(B))^*$, which is infinite. Suppose $B = \{b_n\}$ is a sequence in $\mathbf{N}$ converging to $y \in \alpha\mathbf{N} \setminus \mathbf{N}$. Then $\pi_{\beta\alpha}(B^*) = \{y\}$, contradiction. ◊

If $\mathcal{F} \subseteq C^*(\mathbf{N})$ is countable, then $\omega_{\mathcal{F}}\mathbf{N}$ is metrizable, hence it has a wealth of convergent sequences. The following proposition indicates that if $\mathcal{F}$ is "nearly" countable, then the wealth of nontrivial convergent sequences persists. We recall that the cardinal s is the minimum cardinality of a *splitting family* in $\mathcal{P}(\mathbf{N})$.

PROPOSITION 1.6. *Let $\mathcal{F} \subset C^*(\mathbf{N})$ with $|\mathcal{F}| < s$ and let $\alpha\mathbf{N} = \omega_{\mathcal{F}}\mathbf{N}$.*

Then, each infinite closed subset of $\alpha\mathbf{N}$ and every open subset of $\alpha\mathbf{N}$ which intersects $\alpha\mathbf{N}\setminus\mathbf{N}$ contains a copy of $\omega+1$.

Proof. Since $w(\alpha\mathbf{N}) < s$, by [14, Thm. 6.1] every closed subset of $\alpha\mathbf{N}$ is sequentially compact. If U is open in $\alpha\mathbf{N}$ and $p \in U \cap (\alpha\mathbf{N}\setminus\mathbf{N})$, then U contains a closed neighborhood of p which must be infinite. $\diamondsuit$

Note that, in the above theorem, $\mathbf{N}$ can be replaced by any locally compact space of weight $\mu < s$.

EXAMPLE 1.7. It is of course possible to lose copies of $\omega+1$ in the supremum of a family of compactifications. In fact $\beta\mathbf{N}$ is the supremum of all 2-point compactifications. However, it is even possible to lose copies of $\omega+1$ in the supremum of two compactifications. Suppose A, B are disjoint infinite subsets of $\mathbf{N}$. Let $\alpha\mathbf{N}$ be the compactification formed from $\beta\mathbf{N}$ by identifying A^* to a point, and let $\gamma\mathbf{N}$ be formed by identifying B^* to a point. Each of these compactifications contains a copy of $\omega+1$. However $\alpha\mathbf{N}\vee\gamma\mathbf{N} = \beta\mathbf{N}$.

Now we consider two compactifications of an arbitrary Tychonoff space X. Suppose that $\alpha X \leq \gamma X$ and that $\omega+1 \hookrightarrow \alpha X$. It should be the case that, if γX is not "too far" above αX, then $\omega+1 \hookrightarrow \gamma X$. In fact:

THEOREM 1.8. *Let $\alpha X \leq \gamma X$ and suppose, for each $p \in \alpha X$, one of the following is true:*

a. $\pi_{\gamma\alpha}^{-1}(p)$ is first-countable;
b. $|\pi_{\gamma\alpha}^{-1}(p)| < 2^{\mathfrak{t}}$;
c. $w(\pi_{\gamma\alpha}^{-1}(p)) < s$.

Then $\omega+1 \hookrightarrow \alpha X$ implies $\omega+1 \hookrightarrow \gamma X$.

Proof. First suppose each fiber of $\pi_{\gamma\alpha}$ is finite. Let $\{p_n \mid n < \omega\}$ a nontrivial sequence in αX converging to a point p. Choose $q_n \in \pi_{\gamma\alpha}^{-1}(p_n)$ and let Q be the set of limit points of $\{q_n\}$. One has $Q \subseteq \pi_{\gamma\alpha}^{-1}(p)$, hence Q is finite. Then some subsequence of $\{q_n\}$ converges. Now suppose $A = \pi_{\gamma\alpha}^{-1}(q)$ is an infinite fiber. Then, by a., b or c., A contains a nontrivial convergent sequence. $\diamondsuit$

Note that, if $\alpha X, \gamma X$ are as in the above theorem, then $\omega+1 \hookrightarrow \alpha X\setminus X$ implies $\omega+1 \hookrightarrow \gamma X \setminus X$.

Let $\alpha X, \gamma X$ be compactifications of X and let $\mathcal{F} \subseteq C^*(X)$. One has

$$C_\gamma(X) = \overline{\langle C_\alpha(X) \cup \mathcal{F} \rangle}$$

if and only if γX is the smallest compactification greater than or equal to αX to which every element of $\mathcal{F}$ extends. In this case the family $\mathcal{F}^\gamma$ of the extensions separates points in every fiber $\pi_{\gamma\alpha}^{-1}(p)$, where $p \in \alpha X \setminus X$. In fact, if two points in $\pi_{\gamma\alpha}^{-1}(p)$ were not separated by $\mathcal{F}^\gamma$, then we could identify them to obtain a compactification still greater than or equal to αX, to which each function in $\mathcal{F}$ extends and at the same time smaller than γX. This contradicts the minimality of γX.

COROLLARY 1.9. *Let* $C_\gamma(X) = \overline{\langle C_\alpha(X) \cup \mathcal{F} \rangle}$, *where* $|\mathcal{F}| < s$. *If* $\omega + 1 \hookrightarrow \alpha X$, *then* $\omega + 1 \hookrightarrow \gamma X$.

Proof. Let $A = \pi_{\gamma\alpha}^{-1}(p)$ be an arbitrary fiber. Then $\mathcal{F}^\gamma$ separates points of A. Since A is compact, this implies that $w(A) < s$ and we can apply the above theorem. ♢

We recall that, if $\gamma X = \alpha X \vee \delta X$, then $\pi_{\gamma\alpha}$ and $\pi_{\gamma\delta}$ separate points of $\gamma X \setminus X$, hence $\pi_{\gamma\delta}$ is injective on the fibers of $\pi_{\gamma\alpha}$.

COROLLARY 1.10. *Let* $\gamma X = \alpha X \vee \delta X$ *and suppose* $|\delta X \setminus X| < 2^{\mathfrak{t}}$. *Then* $\omega + 1 \hookrightarrow \alpha X$ *implies* $\omega + 1 \hookrightarrow \gamma X$.

Proof. Since $\pi_{\gamma\delta}$ is injective on the fibers of $\pi_{\gamma\alpha}$, one has $|\pi_{\gamma\alpha}^{-1}(p)| < 2^{\mathfrak{t}}$ for each $p \in \alpha X$. ♢

EXAMPLE 1.11. Note that $\alpha X \leq \gamma X$ and $\omega + 1 \hookrightarrow \gamma X$ do not imply $\omega + 1 \hookrightarrow \alpha X$. To see this, let K be a copy of $\beta\mathbf{N}$ which is contained in $\mathbf{N}^*$ and let $K_1 \cong \mathbf{N}^*$ be the set of nonisolated points of K. Then $\gamma\mathbf{N} = \beta\mathbf{N}/K_1$ contains a convergent sequence. Clearly $\beta\mathbf{N}/K < \gamma\mathbf{N}$ and $\omega + 1 \not\hookrightarrow \beta\mathbf{N}/K$.

There is a class of compactifications of **N** having the property that convergent sequences cannot disappear as you descend in the lattice. A compactification αX of a locally compact space X is said to be *singular* provided $K = \alpha X \setminus X$ is a retract of αX [4]. If $f : X \to K$ is the restriction of a retraction, then the topology on αX may be realized by taking as a base the collection of all open subsets of X together with sets of the form $U \cup (f^{-1}(U) \setminus F)$ where U is open in K and F is an arbitrary compact subset of X.

THEOREM 1.12. *Suppose* $\gamma\mathbf{N}$ *be a singular compactification of* $\mathbf{N}$ *such that* $\omega+1 \hookrightarrow \gamma\mathbf{N}$. *If* $\alpha\mathbf{N} \leq \gamma\mathbf{N}$, *then* $\omega+1 \hookrightarrow \alpha\mathbf{N}$.

Proof. The proof is easy if the sequence in $\gamma\mathbf{N}$ can be chosen in $\mathbf{N}$. Suppose not and let $\{p_n\}$ be a nontrivial sequence in $\gamma\mathbf{N}\setminus\mathbf{N}$ which converges to $p \in \gamma\mathbf{N}\setminus\mathbf{N}$. Let $f: X \to \gamma X \setminus X$ be, as above, the restriction of a retraction of γX to the remainder. Choose $q_n \in f^{-1}(p_n)$, for each n and let $V = U \cup (f^{-1}(U) \setminus F)$ be a basic neighborhood of p. Since all but finitely many of p_n are in U, all but finitely many of q_n are in V. Thus $q_n \to p$. $\diamondsuit$

REMARK 1.13. It is easy to see that, if $\alpha X \leq \gamma X$ and $\pi_{\gamma\alpha}^{-1}(p)$ is finite for each $p \in \alpha X$, then $\omega+1 \hookrightarrow \gamma X$ implies that $\omega+1 \hookrightarrow \alpha X$.

2. We now turn to what is probably a more interesting question: when do compactifications of $\mathbf{N}$ contain copies of $\beta\mathbf{N}$?

First we note the following:

REMARK 2.1. If $\beta\mathbf{N} \hookrightarrow \alpha\mathbf{N}$, then $\beta\mathbf{N} \hookrightarrow \alpha\mathbf{N}\setminus\mathbf{N}$. In fact, if h is the embedding, then $h(\mathbf{N}^*) \cap (\alpha\mathbf{N}\setminus\mathbf{N})$ is an infinite closed subset of $h(\mathbf{N}^*)$, so it contains a copy of $\beta\mathbf{N}$.

It is well known that if the continuous image of a topological space contains a copy of $\beta\mathbf{N}$ then the space must as well. So we have:

PROPOSITION 2.2. *If* $\beta\mathbf{N} \hookrightarrow \alpha X \leq \gamma X$ *then* $\beta\mathbf{N} \hookrightarrow \gamma X$.

In section 1 we gave examples of compactifications of the form $\beta\mathbf{N}/F$ which are contrasequential. For this kind of compactifications one has:

PROPOSITION 2.3. *Let* F *be a closed subset of* $\mathbf{N}^*$ *such that* $\omega+1 \not\hookrightarrow \alpha\mathbf{N} = \beta\mathbf{N}/F$. *Then every infinite closed subset of* $\alpha\mathbf{N}$ *contains a copy of* $\beta\mathbf{N}$.

Proof. Put $\pi_{\beta\alpha}(F) = \{p\}$ and let H be an infinite closed subset of $\alpha\mathbf{N}$. The proof is trivial if $p \notin H$. So suppose $p \in H$ and let $H_1 = \pi_{\beta\alpha}^{-1}(H)$. If there is an open subset U of $\beta\mathbf{N}$ such that $F \subset U$ and $H_1 \setminus U$ is infinite, we are done. In fact, in this case, $H_1 \setminus U$ must contain a copy B of $\beta\mathbf{N}$ and so $\pi_{\beta\alpha}(B) \cong B$ is a copy of $\beta\mathbf{N}$ contained in H. Then let us suppose that $H_1 \setminus U$ is finite for each open U containing F. Since $H_1 \setminus F$ is infinite, it contains a countable discrete subset $\{p_n \mid n < \omega\}$. Let $q_n = \pi_{\beta\alpha}(p_n)$, for each n. Under our assumption, for every open subset U of $\beta\mathbf{N}$ containing F, all but finitely many p_n are in U. This clearly implies $q_n \to p$ in $\alpha\mathbf{N}$, contradiction. ♢

EXAMPLE 2.4. As in Example 1.12, let K be a copy of $\beta\mathbf{N}$ which is contained in $\mathbf{N}^*$ and let $K_1 \cong \mathbf{N}^*$ be the set of nonisolated points of K. Then $\gamma\mathbf{N} = \beta\mathbf{N}/K_1$ contains a convergent sequence and $\gamma\mathbf{N} > \alpha\mathbf{N} = \beta\mathbf{N}/K$, which has the property that every infinite closed subset contains a copy of $\beta\mathbf{N}$. Therefore, that property can be lost "going up" in the lattice.

PROPOSITION 2.5. *Let g be a map from* **N** *into* **N** *and let $f : \mathbf{N} \to \beta\mathbf{N}$ be the composition of g and the inclusion map. Let $\alpha\mathbf{N}$ be the minimum of the compactifications of* **N** *to which f extends. Then $\alpha\mathbf{N}$ has the property that every infinite closed subset contains a copy of $\beta\mathbf{N}$ if and only if g is is finite-to-one.*

Proof. Let f be finite-to-one, so that, by Thm. 1.5, $\omega + 1 \not\hookrightarrow \alpha\mathbf{N}$. The hypothesis implies $\alpha\mathbf{N} \setminus \mathbf{N} \cong \mathbf{N}^*$ (see section 1). Let F be an infinite closed subset of $\alpha\mathbf{N}$. If $F \cap (\alpha\mathbf{N} \setminus \mathbf{N})$ is infinite, then F contains a copy of $\beta\mathbf{N}$. But, if $F \cap (\alpha\mathbf{N} \setminus \mathbf{N})$ were finite, then F would contain a convergent sequence, contradiction. The converse follows direcly from Theorem 1.5. ♢

The next theorem asserts that if $\beta\mathbf{N}$ can be realized as the supremum of two compactification one of which is "simple", then the other must be "complex".

THEOREM 2.6. *Suppose $\beta\mathbf{N} = \alpha\mathbf{N} \vee \gamma\mathbf{N}$ and $\omega + 1 \hookrightarrow \alpha\mathbf{N}$. Then $\beta\mathbf{N} \hookrightarrow \gamma\mathbf{N}$.*

Proof. If $\omega + 1 \hookrightarrow \alpha\mathbf{N}$, then, by Thm. 1.8, some fiber $A = \pi_{\beta\alpha}^{-1}(p)$ is infinite. Since A is closed, it must contain a copy of $\beta\mathbf{N}$. But $\pi_{\beta\gamma}$ is injective on A, hence its restriction to A is an embedding. Thus $\beta\mathbf{N} \hookrightarrow \gamma\mathbf{N}$. ♢

Now, as the results of a theorem of Shapirovskii's, we see that it is impossible to create copies of $\beta\mathbf{N}$ from small collections of simple compactifications.

THEOREM 2.7. [12, Cor. 3]. *If* $\beta\mathbf{N} \hookrightarrow \prod_{\lambda<\kappa} Y_\lambda$ *and* $\kappa < cf(c)$ *then there is* $\lambda_0 < \kappa$ *such that* $\beta\mathbf{N} \hookrightarrow Y_{\lambda_0}$.

The pertinence of this to our considerations follows:

COROLLARY 2.8. *Let* X *be any Tychonoff space and let* $\alpha X = \sup\{\alpha_\lambda X \mid \lambda < \kappa\}$ *with* $\kappa < cf(c)$. *If* $\beta X \not\hookrightarrow \alpha_\lambda X$ *for each* λ, *then* $\beta X \not\hookrightarrow \alpha X$.

Proof. We know that αX is (homeomorphic to) a subset of $\prod_{\lambda<\kappa} \alpha_\lambda X$. $\diamondsuit$

THEOREM 2.9. *If* $\beta\mathbf{N} = \alpha\mathbf{N} \vee \gamma\mathbf{N}$ *and* $\beta\mathbf{N} \not\hookrightarrow \gamma\mathbf{N}$, *then every infinite closed subset of* $\alpha\mathbf{N}$ *contains a copy of* $\beta\mathbf{N}$.

Proof. One has $\beta\mathbf{N} \subseteq \alpha\mathbf{N} \times \gamma\mathbf{N}$ and $\pi_{\beta\alpha}$ is the restriction of the first projection. Put $G = \pi_{\beta\alpha}^{-1}(F)$. Then $G \subseteq F \times \gamma\mathbf{N}$. Since $\beta\mathbf{N} \hookrightarrow G$, by theorem 2.7 $\beta\mathbf{N} \hookrightarrow F$. $\diamondsuit$

COROLLARY 2.10. *Let* $\beta\mathbf{N} = \alpha\mathbf{N} \vee (\sup\{\gamma_\lambda\mathbf{N} \mid \lambda < \kappa\})$ *with* $\kappa < cf(c)$. *If* $\beta\mathbf{N} \not\hookrightarrow \gamma_\lambda\mathbf{N}$ *for each* λ, *then every infinite closed subset of* $\alpha\mathbf{N}$ *contains a copy of* $\beta\mathbf{N}$.

THEOREM 2.11. *Suppose* $\mathcal{F}$ *is a subset of* $C^*(\mathbf{N})$ *such that* $|\mathcal{F}| < c$. *Suppose also that* $\alpha\mathbf{N}$ *satisfies*

$$C^*(\mathbf{N}) = \overline{\langle C_\alpha(\mathbf{N}) \cup \mathcal{F}\rangle}.$$

Then each infinite closed subset of $\alpha\mathbf{N}$ *contains a copy of* $\beta\mathbf{N}$.

Proof. Clearly, one has $\beta\mathbf{N} = \alpha\mathbf{N} \vee \omega_{\mathcal{F}}\mathbf{N}$. We know that $w(\omega_{\mathcal{F}}\mathbf{N}) < c$ (see Section 1) and this implies $\beta\mathbf{N} \not\hookrightarrow \omega_{\mathcal{F}}\mathbf{N}$. Then we can apply Theorem 2.9. $\diamondsuit$

REMARK 2.12. In Theorem 2.6 we could replace the hypothesis $\beta\mathbf{N} = \alpha\mathbf{N} \vee \gamma\mathbf{N}$ by $\delta\mathbf{N} = \alpha\mathbf{N} \vee \gamma\mathbf{N}$, where $\delta\mathbf{N}$ is a compactification such that every infinite closed subset contains a copy of $\beta\mathbf{N}$. We can do the same with Theorem 2.9 and Cor. 2.10. Also, in Theorem 2.11, we can replace $C^*(\mathbf{N})$ by $C_\delta(\mathbf{N})$, where $\delta\mathbf{N}$ is a compactification which satisfies the same condition.

In order to give a nontrivial example of a compatifications $\alpha\mathbf{N}$ and of a family $\mathcal{F}$ of functions which, compatibly, satisfy the hypotheses of Theorem 2.11, we need the following Lemma:

LEMMA 2.13. *Let X be a normal space. Suppose F_1, F_2 are closed subsets of X and let $h : F_1 \to F_2$ be a homeomorphism such that, $\forall x \in F_1 \cap F_2$ one has $h(x) = x$. Let Y be the quotient space obtained by identifying x and $h(x)$ $\forall x \in F_1$. Then Y is Hausdorff.*

Proof. Let q be the quotient map and $F = F_1 \cup F_2$. Let y, z be distinct point of Y. The only nontrivial case if when $y, z \in q(F)$. Let $y_1, z_1 \in F_1$ be such that $q(y_1) = y$, $q(z_1) = z$. Let U, V be open subsets of F_1 such that $y_1 \in U, z_1 \in V$ and $\overline{U} \cap \overline{V} = \emptyset$. The hypotheses imply that $U \cup h(U)$, $V \cup h(V)$ are open in F, and that $\overline{U} \cup \overline{h(U)}$ and $\overline{V} \cup \overline{h(V)}$ are disjoint closed subsets of X. Let W, T be open subsets of X such that $W \cap F = U \cup h(U)$, $T \cap F = V \cup h(V)$. We can choose W, T so that $W \cap T = \emptyset$. In fact, let W' and T' be disjoint open subsets of X which contain $\overline{U} \cup \overline{h(U)}$ and $\overline{V} \cup \overline{h(V)}$ respectively. If necessary, we can replace W, T by $W \cap W'$ and $T \cap T'$, respectively. Clearly, $q^{-1}(q(W)) = W$ and $q^{-1}(q(T)) = T$, so that $q(W)$ and $q(T)$ are disjoint open subsets of Y which contain y and z respectively. $\diamondsuit$

EXAMPLE 2.14. Let Y be a P-space of the form $Z \cup \{x\}$, where $Z = \{x_\lambda : \lambda < \omega_1\}$, every $x_\lambda \in Z$ is isolated in Y and the neighborhoods of x are sets of the form $\{x\} \cup F$, where $F \subseteq Z$ has countable complement. Now Y can be embedded in $\mathbf{N}^*$ [15, Thm. 4.4.4]. Clearly $Y \cup \mathbf{N}$ is regular and Lindelöf and hence normal. Since Y is closed in $Y \cup \mathbf{N}$, Y is C^*-embedded in $\mathbf{N}^*$, hence $Cl_{\mathbf{N}^*}(Y) \cong \beta Y$. Clearly, $Cl_{\mathbf{N}^*}(Y)$ consists of $\{x\}$ together with all points of $\mathbf{N}^*$ which are in the closure of some countable subset of Z.

Let $S, T \subset Y$, with $Z = S \cup T$, $S \cap T = \emptyset$ and $|S| = |T| = \omega_1$. Put $S' = S \cup \{x\}$, $T' = T \cup \{x\}$. Clearly both S' and T' are copies of Y, then $\beta S', \beta T' \subseteq \mathbf{N}^*$ and there is a homeomorphism $h : \beta S' \to \beta T'$ with $h(x) = x$. Since $\beta S'$ and $\beta T'$ have only x in common, by the above Lemma we can create a compactification $\alpha\mathbf{N}$ of $\mathbf{N}$ by identifying each point p

in $\beta S'$ with the point $h(p)$ in $\beta T'$. Notice that no countable collection of continuous functions can separate every p in $\beta S'$ from the associated points $h(p)$ in $\beta T'$.

We are now in a position to define a collection of functions $\mathcal{F}$ of cardinality ω_1, such that $C^*(\mathbf{N}) = \overline{\langle C_\alpha(\mathbf{N}) \cup \mathcal{F} \rangle}$. We can put $S = \{p_\mu : \mu < \omega_1\}$ and $T = \{q_\mu : \mu < \omega_1\}$, with $h(p_\mu) = q_\mu$. For each $\kappa < \omega_1$ let $F_\kappa = Cl_{\mathbf{N}^*}(\{p_\mu : \mu < \kappa\})$ and $G_\kappa = Cl_{\mathbf{N}^*}(\{q_\mu : \mu < \kappa\})$. Now, for each $\kappa < \omega_1$, let $f_\kappa \in C^*(\mathbf{N})$ be such that $f^*_\kappa(F_\kappa) = \{1\}$ and $f^*_\kappa(\beta T') = \{0\}$. Likewise let g_κ be such that $g^*_\kappa(G_\kappa) = \{1\}$ and $g^*_\kappa(\beta S') = \{0\}$. Then $\mathcal{F} = \{f_\kappa : \kappa < \omega_1\} \cup \{g_\kappa : \kappa < \omega_1\}$ is the desired collection of functions.

Note that, if $\alpha\mathbf{N}$ and $\mathcal{F}$ satisfy the hypotheses of Theorem 2.11, then $\pi^{-1}_{\beta\alpha}(p)$ is finite for each $p \in \alpha\mathbf{N}$. In fact, one has $w(\pi^{-1}_{\beta\alpha}(p)) \leq |\mathcal{F}| < c$, whereas infinite closed subsets of $\beta\mathbf{N}$ must be of weight c. Then Theorem 2.11 could be deduced from Theorem 2.16 below. To prove it, we need the following lemma:

LEMMA 2.15. *If $\alpha\mathbf{N}$ is a compactification of $\mathbf{N}$ such that $\pi^{-1}_{\beta\alpha}(p)$ is finite $\forall p \in \alpha\mathbf{N}$, then there exists $m \in \mathbf{N}$ such that $|\pi^{-1}_{\beta\alpha}(p)| < m, \forall p$.*

Proof. Suppose that the cardinality of the fibers is unbounded, so that, $\forall n \in \mathbf{N}$ there exists $y_n \in \alpha\mathbf{N}\backslash\mathbf{N}$ such that $|\pi^{-1}_{\beta\alpha}(y_n)| > n$. We can choose a discrete $A = \{a_n\} \subset \{y_n\}$ which still has the property $|\pi^{-1}_{\beta\alpha}(a_n)| > n$. For each a_n, choose n distinct points $b^{(n)}_1, \ldots, b^{(n)}_n \in \pi^{-1}_{\beta\alpha}(a_n)$. The sets $B_i = \{b^{(n)}_i | n \geq i\}$ are discrete and pairwise disjoint, then their closures in $\beta\mathbf{N}$ are pairwise disjoint and homeomorphic to $\beta\mathbf{N}$. Let $x_1 \in Cl_{\beta\mathbf{N}}(B_1)\backslash B_1$ and let $k \in \mathbf{N}$. Put $B' = \{B_1 \setminus \{b^{(1)}_1, \ldots, b^{(k-1)}_1\}$. Then $Cl_{\beta\mathbf{N}}(B') \setminus B' = Cl_{\beta\mathbf{N}}(B_1) \setminus B_1$ and there is the bijection $h : b^{(n)}_1 \mapsto b^{(n)}_k$ from B' to B_k with the property $\pi_{\beta\alpha}(b^{(n)}_1) = \pi_{\beta\alpha}(h(b^{(n)}_1)), \forall n \geq k$. Then there is a point $x_k \in Cl_{\beta\mathbf{N}}(B_k) \setminus B_k$ such that $\pi_{\beta\alpha}(x_k) = \pi_{\beta\alpha}(x_1)$. Since this is true for every $k \in \mathbf{N}$, $\pi_{\beta\alpha}$ has an infinite fiber. ♢

THEOREM 2.16. *If $\alpha\mathbf{N}$ is a compactification of $\mathbf{N}$ such that $\pi^{-1}_{\beta\alpha}(p)$ is finite $\forall p \in \alpha\mathbf{N}$, then every infinite closed subset of $\alpha\mathbf{N}$ contains a copy of $\beta\mathbf{N}$.*

Proof. Let F be an infinite closed subset of $\alpha\mathbf{N}$. First suppose that the set $\{p \in F | \, |\pi^{-1}_{\beta\alpha}(p)| > 1\}$ is finite. Then, clearly, $\pi^{-1}_{\beta\alpha}(F)$ contains a copy

B of $\beta\mathbf{N}$ on which $\pi_{\beta\alpha}$ is injective. Hence $\pi_{\beta\alpha}|_B$ an embedding.

Now suppose that $\pi_{\beta\alpha}^{-1}(F)$ contains infinitely many nontrivial fibers. Put $k = \max\{h \in \mathbf{N} |$ there exist infinitely many y in F such that $|\pi_{\beta\alpha}^{-1}(y)| = h\}$. The existence of k is ensured by Lemma 2.15. Let $A = \{a_n | n < \omega\}$ be a discrete subset of F such that $|\pi_{\beta\alpha}^{-1}(a_n)| = k$ for each n. Put, for every n, $\pi_{\beta\alpha}^{-1}(a_n) = \{y_1^{(n)}, \ldots, y_k^{(n)}\}$ and $B_i = Cl_{\beta\mathbf{N}}(\{y_i^{(n)} | n < \omega\})$. Then $B_1, \ldots, B_k$ are pairwise disjoint copies of $\beta\mathbf{N}$ contained in $\pi_{\beta\alpha}^{-1}(F)$. Clearly, for each $z_1 \in B_1$ there is $z_i \in B_i$, such that $\pi_{\beta\alpha}(z_i) = \pi_{\beta\alpha}(z_1)$, $i = 2, \ldots, k$. Then, by the definition of k, there are only finitely many $p \in \pi_{\beta\alpha}(B_1)$ such that $|\pi_{\beta\alpha}^{-1}(p) \cap B_1| > 1$. This implies that B_1 contains a closed subset $B \cong \beta\mathbf{N}$ such that $\pi_{\beta\alpha}|_B$ is injective. ♢

Let $\alpha\mathbf{N}$ be the compactification constructed in Example 2.14. We have already remarked that, if $\mathcal{G} \subseteq C^*(\mathbf{N})$ satisfies $C^*(\mathbf{N}) = \overline{\langle C_\alpha(\mathbf{N}) \cup \mathcal{G}\rangle}$, then $|\mathcal{G}| \geq \omega_1$. Then, under CH, no family of functions satisfies, with respect to $\alpha\mathbf{N}$, the hypotheses of Theorem 2.11. However, each fiber of $\pi_{\beta\alpha}$ is finite.

We can generalize the above theorem as follows:

THEOREM 2.17. *Suppose that* $\alpha\mathbf{N} < \gamma\mathbf{N}$ *and* $\pi_{\gamma\alpha}$ *is finite-to-one. Then* $\gamma\mathbf{N}$ *has the property that every infinite closed set contains a copy of* $\beta\mathbf{N}$ *if and only if* $\alpha\mathbf{N}$ *does.*

Proof. First suppose that $\alpha\mathbf{N}$ satifies the requested property. Let G be an infinite closed subset of $\gamma\mathbf{N}$ and put $F = \pi_{\gamma\alpha}(G)$. Then F is closed and infinite, hence it contains a copy of $\beta\mathbf{N}$. But this implies that G contains a copy of $\beta\mathbf{N}$.

Conversely, first observe that, by Theorem 2.16, every image of $\mathbf{N}^*$ with respect to a continuous finite-to-one map contains some copies of $\beta\mathbf{N}$. Now, let F be an infinite closed subset of $\alpha\mathbf{N}$ and let $G = \pi_{\gamma\alpha}^{-1}(F)$. Let B be a copy of $\mathbf{N}^*$ contained in G. Since $\pi_{\gamma\alpha}|B$ is finite-to-one, one has $\beta\mathbf{N} \hookrightarrow \pi_{\gamma\alpha}(B) \subset F$. ♢

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