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Analogue of Gidas-Ni-Nirenberg Result in Hyperbolic Space and Sphere

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Summary. - *Let $u \in C^2(\overline{\Omega})$ be a positive solution of the differential equation $\Delta u + f(u) = 0$ in Ω with boundary condition $u = 0$ on $\partial\Omega$ where f is a C^1 function and Ω is a geodesic ball in the hyperbolic space $\mathbf{H^n}$ (respectively sphere $\mathbf{S^n}$). Further in case of sphere we assume that $\overline{\Omega}$ is contained in a hemisphere. Then we prove that u is radially symmetric.*

1. Introduction

In our paper, "Analogue of Serrin's result for domains in hyperbolic space and sphere" [4] we had used the moving plane method to prove the symmetry of solution and symmetry of the domains in hyperbolic space and sphere. Here we use the same technique to prove the analogue of a theorem of Gidas-Ni-Nirenberg [2] for domains in hyperbolic space $\mathbf{H^n}$ and sphere $\mathbf{S^n}$. More precisely, we prove

Theorem 1.1. *Let Ω be a geodesic ball in $\mathbf{H^n}$ and $u \in C^2(\overline{\Omega})$ be a positive solution of the differential equation*

$$\Delta u + f(u) = 0 \quad \text{in } \Omega \tag{1}$$

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$$u = 0 \quad \text{on } \partial\Omega \tag{2}$$

where f is a C^1 function. Then u is radially symmetric.

Theorem 1.2. *Let Ω be a geodesic ball in $\mathbf{S^n}$ such that $\overline{\Omega}$ is contained in a hemisphere. Let $u \in C^2(\overline{\Omega})$ be a positive solution of the differential equation*

$$\Delta u + f(u) = 0 \quad \text{in } \Omega \tag{3}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{4}$$

where f is a C^1 function. Then u is radially symmetric.

Remark. We learnt later that Pablo Padilla has proved a version of Theorem 1.2 in his thesis [5]. However, we would like to mention that we have given an intrinsic geometric interpretation of "moving plane method" for the Sphere which allows us to derive results like [4]. Further, to our knowledge the result of Theorem 1.1 is new.

Before giving the proof of theorems, we shall first recall briefly the necessary prerequisites and notation. The details can be found in [4].

2. Prerequisites

We shall consider the upper half-space model of the n-dimensional hyperbolic space, i.e., $\mathbf{H^n}$ denotes the open upper half space $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$ with the Póincare metric $ds^2 := {x_n}^{-2}\sum_i {dx_i}^2$. Also, $\mathbf{S^n}$ denotes the unit sphere $\{x \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\}$. It is known that for $\mathbf{H^n}$ and $\mathbf{S^n}$, the isometries are generated by "reflections with respect to closed, totally geodesic hypersurfaces" of the respective spaces (see [3]). By a totally geodesic hypersurface , of a Riemannian manifold **(M, g)**, we mean a hypersurface with the property that any geodesic of , (with induced metric) is also a geodesic in $(\mathbf{M}, \mathbf{g})$. Given a closed, totally geodesic hypersurface , (of $\mathbf{H^n}$ or $\mathbf{S^n}$) we define the reflection R_Γ with respect to , as follows: for $x \in \mathbf{H^n}$ (respectively $\mathbf{S^n}$), let γ denote the distance minimising geodesic from x to , such that $\gamma(0) \in$, and $\gamma(t) = x$. Since $\mathbf{H^n}$ (respectively $\mathbf{S^n}$) is complete, $\gamma(s)$ is defined for all $s \in \mathbb{R}$. We define $R_\Gamma(x) = \gamma(-t)$. Moreover, from [4] we have

Theorem 2.1. *Let* , $\subset \mathbf{H^n}$ *be a totally geodesic hypersurface. Let* φ *be an isometry of* $\mathbf{H^n}$ *which maps* , *onto* Σ *which is necessarily a totally geodesic hypersurface. If* R_Γ *(respectively* R_Σ*) denotes the reflection with respect to* , *(respectively* Σ*) then*

$$R_\Sigma \circ \varphi = \varphi \circ R_\Gamma.$$

Note that Δ denotes the Laplace-Beltrami operator on the respective spaces and we shall use the fact the Laplace-Beltrami operator is invariant under isometries.

3. Proof of Theorem 1.1

Recall that the Laplace-Beltrami operator on $\mathbf{H^n}$ is given by

$$\Delta = {x_n}^2 \left(\sum_{i=1}^{n} \frac{\partial^2}{\partial^2 {x_i}^2} \right) + (2-n) x_n \frac{\partial}{\partial x_n}, \tag{5}$$

where $x_1, \ldots, x_n$ denotes the usual coordinate system on $\mathbb{R}^n$. Since Ω is a geodesic ball, given a direction $\overrightarrow{\nu}$ in $\mathbb{R}^n$ there exists a totally geodesic hypersurface orthogonal to $\overrightarrow{\nu}$ such that $x \in \Omega$ is symmetric about , i.e., $R_\Gamma \Omega = \Omega$, where R_Γ denotes the reflection with respect to , as defined above. We shall prove that $u(x) = u(R_\Gamma x)$ for every such , ; which proves Theorem 1.1. As in [4], we prove the symmetry of solution with respect to the particular closed, totally geodesic submanifold $T_\lambda = \{(x_1, \ldots, x_n) \in \mathbf{H^n} : x_1 = \lambda\}$ of $\mathbf{H^n}$; which is orthogonal to x_1-direction. We shift the hyperplane T_λ i.e., decrease λ until it begins to intersect Ω. Let λ_0 be the first λ such that T_{λ_0} is tangential to $\partial\Omega$. For $\lambda < \lambda_0$, let Σ_λ denote that portion of Ω which lies on the same side of T_λ as the x_1-direction. Let $\Sigma'_\lambda := R_\lambda \Sigma_\lambda$, where R_λ denotes the reflection with respect to T_λ. Further, let $\lambda_1 < \lambda_0$ be such that Ω is symmetric about T_{λ_1}. We claim that

$$u(x) = u(R_{\lambda_1} x) \quad \text{for } x \in \Sigma_{\lambda_1}. \tag{6}$$

For $\lambda < \lambda_0$, define $v_\lambda(x) = u(R_\lambda x)$, $x \in \Sigma_\lambda$. It follows that v_λ satisfies the differential equation

$$\Delta v_\lambda + f(v_\lambda) = 0$$

on Σ_λ and the boundary conditions

$$v_\lambda = u \quad \text{on} \quad \partial\Sigma_\lambda \cap T_\lambda,$$
$$v_\lambda > 0 \quad \text{on} \quad \partial\Sigma_\lambda \setminus T_\lambda.$$

Consider the function $w_\lambda(x) = v_\lambda(x) - u(x)$, $x \in \Sigma_\lambda$ which satisfies the differential equation

$$\Delta w_\lambda + h(x) w_\lambda = 0 \quad \text{on} \quad \Sigma_\lambda \tag{7}$$

for L^∞ function h and boundary conditions

$$w_\lambda = 0 \quad \text{on} \quad \partial\Sigma_\lambda \cap T_\lambda,$$
$$w_\lambda \geq 0 \quad \text{on} \quad \partial\Sigma_\lambda \setminus T_\lambda.$$

i.e.,

$$w_\lambda \geq 0 \quad \text{on} \quad \partial\Sigma_\lambda. \tag{8}$$

Note that w_λ satisfies the equations (7) and (8) for all $\lambda < \lambda_0$. We shall show that $w_{\lambda_1} \equiv 0$, which proves (6).

Claim: for λ near λ_0, $w_\lambda > 0$ on Σ_λ.

For the proof of claim, we require the following version of maximum principle [1, Proposition 1.1]:

Proposition 3.1. *Let Ω be a domain in $\mathbb{R}^n$ with $\operatorname{diam}(\Omega) \leq d$. Consider a second order elliptic operator L on Ω given by*

$$L = a_{ij}(x)\partial_{ij} + b_i(x)\partial_i + c(x),$$

with L^∞ coefficients and which is uniformly elliptic

$$c_0|\xi|^2 \leq a_{ij}(\xi) \leq C_0|\xi|^2, \quad c_0, C_0 > 0 \quad \text{for all } \xi \in \mathbb{R}^n,$$

and satisfying

$$\left(\sum {b_i}^2\right)^{\frac{1}{2}}, \quad |c| \leq b.$$

Let $z \in W^{2,n}_{loc}(\Omega)$ be such that

$$Lz \geq 0 \quad \text{in } \Omega$$

and

$$\overline{\lim_{x \to \partial\Omega}} z(x) \leq 0.$$

Then there exists $\delta > 0$ depending only on n, d, c_0 and b such that if $\operatorname{meas}(\Omega) = |\Omega| < \delta$ then $z(x) \leq 0$ in Ω.

For the application of the above proposition to $\mathbf{H^n}$, we use the fact that the measure on $\mathbf{H^n}$ is absolutely continuous with respect to the usual Lebesgue measure on $\mathbb{R}^\mathbf{n}$.

For λ near λ_0, the measure of Σ_λ is less than the δ given by the above proposition, where L is now the Laplace-Beltrami operator on $\mathbf{H^n}$ given by (5). Since w_λ satisfies the differential equation

$$\Delta w_\lambda + h(x) w_\lambda = 0 \quad \text{on } \Sigma_\lambda,$$

for L^∞ function h and boundary condition

$$w_\lambda \geq 0 \quad \text{on } \partial\Sigma_\lambda,$$

it follows from the proposition that $w_\lambda \geq 0$ on Σ_λ. Now from the restricted version of maximum principle [5], either $w_\lambda \equiv 0$ or $w_\lambda > 0$. For $\lambda_0 - \lambda$ small, $w_\lambda \not\equiv 0$ for otherwise we get a contradiction to $u > 0$ in Ω. Hence $w_\lambda > 0$ on Σ_λ for λ near λ_0.

Define $\mu = \sup\{\lambda : w_s > 0 \text{ for all } s \in (\lambda, \lambda_0)\}$.

Claim: $\mu = \lambda_1$.

Proof of the claim: suppose $\mu > \lambda_1$. By continuity, we have $w_\mu \geq 0$. Further since $\mu > \lambda_1$, w_μ satisfies the equations (7) and (8). Hence by restricted version of maximum principle, either $w_\mu \equiv 0$ or $w_\mu > 0$ in Σ_μ. Now, $w_\mu \equiv 0$ gives a contradiction to fact that $u > 0$ in Ω. Hence $w_\mu > 0$ in Σ_μ.

Choose a compact set $K \subset \Sigma_\mu$ such that

$$\text{meas}(\Sigma_\mu \setminus K) < \frac{\delta}{2},$$

where δ is the constant chosen in the proposition above. Then $w_\mu > 0$ on K. Since K is compact, there exists $\overline{\lambda}$ near μ and $\lambda_1 < \overline{\lambda} < \mu$ such that

$$w_{\overline{\lambda}} > 0 \quad \text{on } K. \tag{9}$$

Further we may choose $\overline{\lambda}$ such that

$$\text{meas}(\Sigma_{\overline{\lambda}} \setminus K) < \delta.$$

On $\Sigma_{\overline{\lambda}} \setminus K$, $w_{\overline{\lambda}}$ satisfies the differential equation (7) with boundary condition $w_{\overline{\lambda}} \geq 0$ on $\partial(\Sigma_{\overline{\lambda}} \setminus K)$. Since $\text{meas}(\Sigma_{\overline{\lambda}} \setminus K) < \delta$, by proposition it follows that $w_{\overline{\lambda}} \geq 0$ on $\Sigma_{\overline{\lambda}} \setminus K$. Therefore, $w_{\overline{\lambda}} \geq 0$ on $\Sigma_{\overline{\lambda}}$.

Since $\overline{\lambda} > \lambda_1$, $w_{\overline{\lambda}} \not\equiv 0$. Hence $w_{\overline{\lambda}} > 0$ on $\Sigma_{\overline{\lambda}}$; a contradiction to the definition of μ. Therefore, the assumption is wrong. Hence $\mu = \lambda_1$.

By continuity, it follows that $w_{\lambda_1} \geq 0$. If we shift the plane from $-x_1$-direction, then by symmetry of the domain we get the inequality $w_{\lambda_1} \leq 0$. Hence $w_{\lambda_1} \equiv 0$ in Ω, i.e., $u(x) = u(R_{\lambda_1} x)$ for all $x \in \Omega$. Using the Theorem 2.1, we further conclude that u is radially symmetric (see [4]). □

REMARK. It is clear that one go through the steps mentioned above just as well to conclude Theorem 1.2.

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