Rend. Istit. Mat. Univ. Trieste
Volume 47 (2015), 45–57
DOI: 10.13137/0049-4704/11218

Expressing forms as a sum of pfaffians

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The author is very glad for this opportunity to leave a tribute in honor of Emilia. We met at the beginning of our careers and I always had the feeling that we share the same global view of Mathematics and its applications.

Abstract. *Let $A = (a_{ij})$ be a symmetric non-negative integer $2k \times 2k$ matrix. A is homogeneous if $a_{ij} + a_{kl} = a_{il} + a_{kj}$ for any choice of the four indexes. Let A be a homogeneous matrix and let F be a* general *form in $\mathbb{C}[x_1, \dots x_n]$ with $2\deg(F) = \mathrm{trace}(A)$. We look for the least integer, $s(A)$, so that $F = \mathrm{pfaff}(M_1) + \dots + \mathrm{pfaff}(M_{s(A)})$, where the $M_i = (F^i_{lm})$ are $2k \times 2k$ skew-symmetric matrices of forms with degree matrix A. We consider this problem for $n = 4$ and we prove that $s(A) \le k$ for all A.*

Keywords: Pfaffians.
MS Classification 2010: 14N15.

1. Introduction

Let $F \in \mathbb{C}[x_1, \dots, x_n]$ be a general form and $A = (a_{ij})$ a $2k \times 2k$ integer homogeneous symmetric matrix, whose trace ($\mathrm{tr}(A)$ in the sequel) is equal to twice the degree of F ($\deg F$). In this paper we study representations of F as a sum of pfaffians of skew-symmetric matrices of type $M = (F_{ij})$ where $\deg F_{ij} = a_{ij}$.

In case the number of variables is two then forms F in $\mathbb{C}[x_1, x_2]$ decompose as a product of linear forms. It follows that if A is a matrix as above, with no negative entries and with $\mathrm{tr}(A) = 2\deg(F)$, then F is the pfaffian of a subdiagonal matrix whose degree matrix is A (i.e. a matrix of type

$$\begin{pmatrix} 0 & p_1 & 0 & 0 & 0 & \dots & 0 \\ -p_1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & p_2 & 0 & \dots & 0 \\ 0 & 0 & -p_2 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & p_k \\ 0 & 0 & 0 & 0 & 0 & -p_k & 0 \end{pmatrix}$$

with each p_i equal to a suitable product of linear forms).

For 3 variables, the problem was considered by Beauville who observed, in section 5 of [2], that a general form of degree d is the pfaffian of a $2d \times 2d$ skew-symmetric matrix of linear forms. Indeed Beauville's argument applies to any symmetric integer homogeneous matrix with non-negative entries. We give below a proof of the result, in a more geometric setting (see section 3).

When the number of variables grows, then a similar property fails as soon as k becomes big. In Proposition 7.6 of [2] Beauville noticed that one cannot expect that a general form of degree ≥ 16 in four variables is the pfaffian of a matrix of linear forms, just by a count of parameters. We refer to [9] for a similar result for matrices of quadratic forms, and to [5] for an extension to other constant or almost constant matrices. In any setting, except for particular numerical cases (which become suddenly unbalanced when the size of the matrix grows), one expects that a general form is *not* the pfaffian of a skew-symmetric matrix of forms with fixed degrees. Indeed, even in the case of 4×4 matrices and 4 variables, we do not know a complete description of matrices $A = (a_{ij})$, with trace $2d$, such that the general form of degree d is the pfaffian of a skew-symmetric matrix of forms (F_{ij}) with $\deg(F_{ij}) = a_{ij}$. The problem seems rather laborious, and we refer to [4] for a discussion.

The problem is indeed related to the existence of indecomposable rank 2 bundles E without intermediate cohomology (aCM bundles) on the hypersurface defined by $F = 0$ (which we will indicate, by abuse, with the same letter F). In turn, this is equivalent to the existence of some arithmetical Gorenstein subscheme of codimension 2 in F (thus codimension 3 in the projective space), via the algebraic characterization of codimension 3 Gorenstein ideals, given in [3]. For instance, F is a pfaffian of a 4×4 skew-symmetric matrix of forms if and only if there exists a subscheme of F which is complete intersection of 3 forms, whose degrees are related with the degrees of the entries of the matrix. This is the point of view under which the problem is attached in [4], and see also [14] for a similar discussion.

In the present note, we make one step further. Since in most cases one cannot hope to express a general form as the pfaffian of a skew-symmetric matrix of forms with pre-assigned degree matrix A, then we ask for the minimum $s(A)$ such that a general form is a *sum of* $s(A)$ pfaffians of skew-symmetric matrices, with degree matrix A.

We consider the case of forms in four variables and show that the complete answer $s(A) \leq 2$ follows soon for 4×4 matrices A, while for $2k \times 2k$ matrices with $k > 2$, we provide a bound for the number $s(A)$, i.e. $s(A) \leq 2k$. The (weak) sharpness of this bound is discussed in the last section. As showed in [2] and [5], at least for small values of the entries of the integer matrix A (e.g. for matrices of linear forms), the number of pfaffians needed to write a general form can be smaller than our bound. The problem of finding a *sharp* bound for the number $s(A)$ is open.

The procedure is a mixture of algebraic and geometric arguments, involving computations of the dimension of secant varieties and Terracini's Lemma, as well as the description of tangent spaces to the varieties of forms that can be expressed as pfaffians, given in [13], [12] or [1].

We mention that, of course, one could ask a similar question for the *determinant* of a general matrix of forms. In other words, fixing a homogeneous integer matrix A, one could ask for the minimum $s'(A)$ such that the general form of degree $d = tr(A)$ is the sum of the determinants of $s'(A)$ matrices of forms, with degree matrix A. This is indeed the target of a series of papers [8], [6], [7], where it is proved that, in $n \geq 3$ variables, $s'(A) \leq k^{n-3}$ for a $k \times k$ matrix A.

Let us end by noticing that the problem addressed in this note, of clear algebraic and geometric flavor, turns out to also have a connection with some applications in control theory. Indeed, if the algebraic boundary of a region Θ in the plane or in space is described by the pfaffian of a matrix of linear forms, then the study of systems of matrix inequalities, whose domain is Θ, can be considerably simplified. We refer to the papers [15] and [11], for an account of this theory. We believe that expressing Θ as a sum of determinants or pfaffians can have some application for similar problems.

2. The geometric construction

We work in the ring $R = \mathbb{C}[x_0, \dots, x_n]$, i.e. the polynomial ring in $n+1$ variables with coefficients in the complex field. By R_d we indicate the vector space of homogeneous forms of degree d in R.

For any degree d, the space R_d has an associated projective space $\mathbb{P}^N$ with

$$N := N(d) = \binom{n+d}{n} - 1.$$

For any choice of integers a_{ij}, $1 \leq i, j \leq 2k$, consider the numerical $2k \times 2k$ matrix $A = (a_{ij})$.

We will say that a $2k \times 2k$ matrix $M = (F_{ij})$, whose entries are homogeneous forms in R, has *degree matrix* A if for all i, j we have $\deg(F_{ij}) = a_{ij}$. In this case, we will also write that $A = \partial M$.

Notice that when for some i, j we have $F_{ij} = 0$, then there are several possible degree matrices for M, since the degree of the zero polynomial is indeterminate.

We will focus on the case where A is symmetric and M is skew-symmetric.

The set of all skew-symmetric matrices of forms, whose degree matrix is a fixed A, defines a vector space whose dimension is $\sum_{i<j} \dim(R_{a_{ij}})$. From the

geometrical point of view, however, we will consider this set as the *product* of projective spaces

$$\mathcal{V}(A) = \mathbb{P}^{r_{12}} \times \cdots \times \mathbb{P}^{r_{2k-1\,2k}}$$

where $r_{ij} = -1 + \dim(R_{a_{ij}})$.

We say that the numerical matrix A is *homogeneous* when, for any choice of the indexes i, j, l, m, we have

$$a_{ij} + a_{lm} = a_{im} + a_{lj}.$$

All submatrices of a homogeneous matrix are homogeneous.

If a skew-symmetric $2k \times 2k$ matrix of forms M has a homogeneous degree matrix, then the pfaffian of M is a homogeneous form. The degree of the pfaffian is one half of the sum of the numbers on the main diagonal of $A = \partial M$, i.e. $tr(A)/2$. It is indeed immediate to see that when A is symmetric and homogeneous of even size, then the trace $tr(A)$ is even.

Let us recall a geometric interpretation of the problem, based on the study of secant varieties, which uses the classical Terracini's Lemma. This a standard construction was already used in [4].

In the *projective* space $\mathbb{P}^N$, which parametrizes all forms of degree d, we have the subset U of all the forms which are the pfaffian of a skew-symmetric matrix of forms whose degree matrix is a given A. This set is a quasi-projective variety, since it corresponds to the image of the (rational) map $\mathcal{V}(A) \to \mathbb{P}^N$, which sends every matrix to its pfaffian (it is undefined when the pfaffian is the zero polynomial). We will denote by V the closure of U. It is clear that V is irreducible, by construction.

Our main question can be rephrased by asking: what is the minimal s such that a general point of $\mathbb{P}^N$ is spanned by s points of V? In classical Algebraic Geometry, (the closure of) the set of points spanned by s points of V is called the *s-th secant variety* $\sigma_s(V)$ of V. Thus, we look for the minimal s such that $\sigma_s(V) = \mathbb{P}^N$. Of course, this is equivalent to ask that the dimension of $\sigma_s(V)$ is N.

Usually, when dealing with similar problems on secant varieties, one can hope to compute the dimension of $\sigma_s(V)$ as the dimension of a general tangent space to $\sigma_s(V)$. Indeed one can invoke the celebrated Terracini's Lemma:

Lemma 2.1. (Terracini) *At a general point $F \in \sigma_s(V)$, expressed as a sum $F = F_1 + \cdots + F_s$, $F_i \in V$ for all i, the tangent space to $\sigma_s(V)$ equals the span of the tangent spaces to V at $F_1, \dots, F_s$.*

Thus, for our purposes, it is crucial to obtain a characterization of the tangent space to V at a general point F. This has been obtained (see e.g. [13]) via the submaximal pfaffians of matrices.

Indeed, for a skew-symmetric matrix of forms M of even size $2k \times 2k$, let us denote as *submaximal pfaffians* the pfaffians of the (skew-symmetric) submatrices of M obtained by erasing two rows and the two columns with the same indexes.

Then we have the following.

PROPOSITION 2.2. *With the previous notation, let F be a general element in V, $F = \mathrm{pfaff}(M)$, where $M = (F_{ij})$ is a $2k \times 2k$ skew-symmetric matrix of forms, whose degree matrix is A.*

Then the tangent space to V at F coincides with the subspace of $R_d/\langle F\rangle$, generated by the classes of the forms of degree d in the ideal $J = \langle F, M_{ij}\rangle$, where the M_{ij}'s are the submaximal pfaffians of the matrix M.

Proof. See [1] or section 2 of [13] or [12]. It can be obtained also by a direct computation over the ring of dual numbers. □

For instance, when the degree matrix A of M has all entries equal to a, then J is generated by $\binom{2k}{2}$ forms of degree $a(k-1)$.

It follows immediately from the previous propositions and Terracini's lemma, that:

REMARK 2.3. *We have the following equivalences:*

- a general form of degree d is the sum of s pfaffians of $2k \times 2k$ matrices, all having degree matrix A

if and only if

- the span of s general tangent spaces to the variety V of pfaffians is the whole space $\mathbb{P}^N$

if and only if

- for a general choice of s matrices of forms $M_1, \dots M_s$, of type $2k \times 2k$, with $\partial M_i = A$ for all i, the ideal generated by the submaximal pfaffians of all the M_i's coincides, in degree d, with the whole space R_d.

Thus, what we are looking for is the minimal s such that, for general skew-symmetric matrices $G_1, \dots, G_s$ with degree matrix A, the ideal I generated by their submaximal pfaffians coincides with the whole polynomial ring in degree $d = tr(A)/2$.

REMARK 2.4. *If M is a $2k \times 2k$ skew-symmetric matrix of forms with homogeneous degree matrix A, then the pfaffian of M is essentially invariant if we permute rows and the corresponding columns of M. Consequently, we can arrange $A = (a_{ij})$ so that*

$$a_{11} \geq a_{21} \geq \cdots \geq a_{2k\,1}.$$

We will say that A is ordered if it satisfies the previous inequalities.

Notice that A is symmetric, thus if A is ordered then $a_{11} \geq a_{12} \geq \cdots \geq a_{1\,2k}$.

Since A is homogeneous, when A is ordered $a_{ij} \geq a_{i'j}$ *for some j implies that* $a_{ij'} \geq a_{i'j'}$ *for any* j'. *It follows that a homogeneous symmetric ordered matrix A has* a_{11} *as its maximal entry and* $a_{2k\,2k}$ *as its minimal one. Moreover columns are non-increasing going downward, while rows are non-increasing going rightward.*

Notice also that when A is symmetric and homogeneous, then the entries of the diagonal of A are either all odd or all even. Indeed for any i, j *one has*

$$a_{ii} + a_{jj} = a_{ij} + a_{ji} = 2a_{ij}.$$

3. The case of ternary forms

As mentioned in the introduction, the fact that any form of degree d in 2 variables is the pfaffian of a skew-symmetric matrix with prescribed degree matrix A is trivial. Thus the first relevant case concerns forms in three variables.

For three variables, the construction of pfaffian representations of forms via the existence of aCM rank 2 bundles, given by Beauville in section 5 of [2], proves that the following holds:

THEOREM 3.1. *Let* $A = (a_{ij})$ *be a non-negative symmetric homogeneous integer matrix of even size, with trace* $2d$. *Then a general homogeneous form of degree* d *in three variables is the pfaffian of a skew-symmetric matrix of forms* G *with* $\partial G = A$.

Indeed, Beauville states the theorem only for matrices of linear forms. For completeness, we show and inductive method which, starting with Beauville's claim, proves the statement for any non-negative matrix A.

We have the chance, in this way, to introduce our inductive method for the study of pfaffian representations of forms in more variables.

Let us consider a $(2k-1) \times (2k-1)$ integer matrix $A' = (a'_{ij})$, which is moreover symmetric, non-negative, *ordered* and homogeneous. Notice that the trace of A' is equal to

$$tr(A') = a_{12} + a_{23} + \cdots + a_{2k-2\,2k-1} + a_{2k\,1}.$$

Let G' be a skew-symmetric matrix of *general* forms, with degree matrix A'. The *submaximal* pfaffians of G', i.e. the pfaffians of the $(2k-2)\times(2k-2)$ matrices obtained by erasing one row and the corresponding column of G',determine an ideal $I(G')$ whose zero-locus is an arithmetically Gorenstein subscheme of codimension 3, by the celebrated structure theorem of Buchsbaum and Eisenbud ([3]). Moreover, we have a resolution of $I(G')$ of type

$$0 \to R'(-m) \to \oplus R'(-r_j) \to \oplus R'(-s_i) \to I(G') \to 0$$

where R' is the polynomial ring $R' := \mathbb{C}[x, y, z]$, and m is the trace of A'. Since we are working in dimension 2, the ideal $I(G')$ defines the empty set in $\mathbb{P}^2$, and the resolution shows that $I(G')$ coincides with the whole polynomial ring in all degrees $d \geq tr(A') - 2$.

Lemma 3.2. *Let G be a general skew-symmetric matrix of forms in three variables, of odd size $(2k-1) \times (2k-1)$, whose degree-matrix A is non-negative and homogeneous. Call I the ideal generated by the submaximal pfaffians of G, i.e. the pfaffian of the submatrices obtained by erasing one row and the corresponding column of G. Then the multiplication map by a general linear form L defines a surjection $(R'/I)_{d-1} \to (R'/I)_d$ for all $d \geq tr(A)/2 - 1$.*

Proof. Since G is general, by [10] R'/I is artinian and arithmetically Gorenstein and enjoys the weak Lefschetz property. The conclusion follows since the socle degree of R'/I is at most $tr(A) - 2$. □

Proof of Theorem 3.1. We may assume that A is ordered and we will make induction on the trace of A. As explained in Remark 2.4, the entries of the diagonal of A are either all even or all odd. If the entries are even, we use as basis for the induction the null matrix, for which the statement is trivial. If the entries are odd, we use the matrix with all the entries equal to 1, for which the statement holds by [2] Proposition 5.1.

For the inductive step, let B be the matrix obtained by A by subtracting 1 to the first row and the first column (hence subtracting 2 from a_{11}). We have $tr(B) = tr(A) - 2$ and by induction the theorem holds for B. Thus if H is a general matrix of forms with degree matrix B, then by Remark 2.3 the submaximal pfaffians of H generate an ideal $I(H)$ which coincides with R' in degree $\geq tr(A)/2 - 1$.

Consider the symmetric matrix G' obtained from H by erasing the first row and the first column. Then G' is a general skew-symmetric $(2k-1) \times (2k-1)$ matrix of forms, whose degree matrix A' corresponds to A mnus the first row and the first column. Call $I(G')$ the ideal generated by the submaximal pfaffians of the G'. As observed in Lemma 3.2, the multiplication map by a general linear form L determines a surjection $(R'/I(G'))_{d-1} \to (R'/I(G'))_d$ for all $d \geq tr(A')/2 - 1$. In particular, we get that $LI(H) + I(G')$ coincides with R' in all degrees $tr(B)/2 + 1$.

Let G be the matrix obtained from H by multiplying the first row and column by L. We have $\partial G = A$ and moreover the ideal $I(G)$ generated by the submaximal pfaffians of G contains $LI(H) + I(G')$. The claim follows from Remark 2.3. □

We will need in the next section a technical results on submaximal pfaffians of skew-symmetric matrices of odd size. As above, the *submaximal pfaffians* of a skew-symmetric matrix G of type $(2k-1)\times(2k-1)$ are the pfaffians of the submatrices obtained by erasing one row and the corresponding column of G.

PROPOSITION 3.3. *Let Let $A = (a_{ij})$ be a non-negative ordered symmetric homogeneous integer matrix of odd size $(2k-1)\times(2k-1)$, $k > 1$. For a general choice of k matrices of ternary forms $G_1,\dots,G_k$ with $\partial G_i = A$, the submaximal pfaffians of all the G_i's generate an ideal I which coincides with the ring R' in degree $d \geq (a_{11} + tr(A))/2$.*

Proof. Assume that all the entries of A are equal to a. Then start with a general $2k \times 2k$ skew-symmetric matrix G, with all entries of degree a, and consider the matrices G_i obtained from G by erasing the i-th row and column. Observe indeed that for such matrices $G_1,\dots,G_k$ the ideal I coincides with the ideal generated by the submaximal pfaffians of G. Thus the claim follows by Theorem 3.1, since the pfaffian of G has degree equal to $(a_{11} + tr(A))/2$.

In the general case, assume that A is ordered and let B be the matrix obtained from A by decreasing the first row and column by 1. Assume the claim holds from B. Notice that $(b_{11} + tr(B))/2 = d - 2$. Take k general skew-symmetric matrices of forms $H_1,\dots,H_k$, with $\partial H_i = B$. Then the ideal I' generated by the submaximal minors of the H_i's coincides with R' in degree $(b_{11} + tr(B))/2 = (a_{11} + tr(A))/2 - 2$. Let I_1 be the ideal generated by the submaximal pfaffians of G_1. By Lemma 3.2 the multiplication map $(R'/I_1)_{d-2} \to (R'/I_1)_{d-1}$ surjects. Let $G_2,\dots,G_k$ be the matrices obtained from the H_i's by multiplying the first row and column by a general linear form L. Then $\partial G_i = A$ and the ideal I'' generated by the submaximal pfaffians of $H_1, G_2,\dots,G_k$ contains $LI' + I_1$, thus it coincides with R' in degree $d-1$. Let now I_2 be the ideal generated by the submaximal pfaffians of G_2. By Lemma 3.2 the multiplication map $(R'/I_2)_{d-1} \to (R'/I_2)_d$ surjects. Let G_1 the matrix obtained from H_1 by multiplying the first row and column by a general linear form L. Then $\partial G_1 = A$ and the ideal generated by the submaximal minors of $G_1,\dots,G_k$ contains $I_2 + LI''$, thus it coincides with R in degree d. Hence the claim holds for A.

The proof is concluded by observing that any symmetric, homogeneous matrix A reduces to a matrix with constant entries by steps consisting in subtracting 1 from one row and one column. □

4. The four by four case

We move now to the case of quaternary forms (and surfaces in $\mathbb{P}^3$). For 4×4 matrices a complete answer to the problem of the pfaffian representation of forms is given by the following.

THEOREM 4.1. *Let $A = (a_{ij})$ be a 4×4 symmetric homogeneous matrix of non-negative integers. Let $d = tr(A)/2$. Then a general form of degree d in*

$\mathbb{C}[x_1, \dots, x_4]$ *is the sum of two pfaffians of skew-symmetric matrices, whose degree matrix is* A.

Proof. Let $M = (M_{ij})$ be a general 4×4 skew-symmetric matrix of forms, with $\partial M = A$. If F is the pfaffian of M, then Proposition 2.2 tells us that the tangent space at F of the variety V (of forms which are pfaffians of matrices with degree matrix A) is generated by the 2×2 submaximal pfaffians of M. These pfaffians correspond to the six entries $M_{12}, M_{13}, M_{14}, M_{23}, M_{24}, M_{34}$, thus they are six general forms, of degrees (respectively) $a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}$, The homogeneity of A implies that

$$a_{12} + a_{34} = a_{13} + a_{24} = a_{14} + a_{23}$$

Thus, after Remark 2.3, the claim follows if we prove that 12 general forms, of degrees respectively

$$a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}, \ a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}$$

generate the polynomial ring $\mathbb{C}[x_1, \dots, x_4]$ in degree $a_{12} + a_{34} = tr(A)/2$.

On the other hand, it is a consequence of Theorem 2.9 of [6] that already 8 general forms of degrees respectively a, b, c, e, a, b, c, e, with $a + e = b + c$, generate $\mathbb{C}[x_1, \dots, x_4]$ in degree $a + e$. The claim thus follows by taking $a = a_{12}, b = a_{13}, c = a_{24}, e = a_{34}$. □

After Beauville's work (see e.g. Theorem 2.1 of [5]), a form F is the pfaffian of a matrix with degree matrix A if and only if the surface F contains a complete intersection set of points, of type a_{12}, a_{13}, a_{14}.

Thus we just proved that:

Corollary 4.2. *For any choice of numbers* d, a, b, c *with* $d > a, b, c$, *a general form of degree* d *is the sum of two forms* F_1, F_2 *corresponding to two surfaces, both containing a complete intersection set of points of type* a, b, c.

Compare this statement with the results of [4].

Remark 4.3. *We derived our statement from Theorem 2.9 of [6], which geometrically proves that for any* d, a, b *with* $a, b < d$ *a general form of degree* d *is the sum of two forms* F_1, F_2 *corresponding to two surfaces, both containing a complete intersection curve of type* a, b.

From this point of view, a geometric reading of the proof of Theorem 4.1 seems straightforward.

5. General quaternary forms as sum of pfaffians

In this section, we want to extend the results for quaternary forms and general degree matrices.

Through the section, let we denote with $R = \mathbb{C}[x, y, z, t]$ the polynomial ring in four variables and with R' the quotient of R by a general linear form (i.e. R' is isomorphic to a polynomial ring in three variables).

We will need two results, derived directly from the previous sections.

LEMMA 5.1. *Let G be a general skew-symmetric $2k \times 2k$ matrix of linear quaternary forms. Call I the ideal generated by the submaximal $(2k-2) \times (2k-2)$ pfaffians of G.*

Then the multiplication by a general linear form L determines a surjection $(R/I)_{d-1} \to (R/I)_d$ for all $d \geq k$.

Proof. Let I be the ideal generated by the submaximal pfaffians of G and let L be a general linear form. Consider the exact sequence

$$(R/I)_{d-1} \xrightarrow{L} (R/I)_d \to (R/(I, L))_d \to 0.$$

By Theorem 3.1 and Remark 2.3, the module on the right side is 0, when $d \geq k - 1$. The claim follows. □

Just with the same procedure, but using Proposition 3.3 instead of Theorem 3.1, we get the following.

LEMMA 5.2. *Let Let $A = (a_{ij})$ be a non-negative ordered symmetric homogeneous integer matrix of odd size $(2k-1) \times (2k-1)$, $k > 1$. For a general choice of k matrices of quaternary forms $G_1, \ldots, G_k$ with $\partial G_i = A$, call I the ideal generated by the submaximal pfaffians of all the G_i's. Then the multiplication by a general linear form L determines a surjection $(R/I)_{d-1} \to (R/I)_d$ for all $d \geq (a_{11} + tr(A))/2$.*

We consider first the case of matrices of linear forms.

THEOREM 5.3. *For a general choice of k matrices of linear forms $H_1, \ldots, H_k$ of size $2k \times 2k$, the submaximal pfaffians of the H_i's generate an ideal which coincides with the polynomial ring R in all degrees $d \geq k$.*

Proof. Use induction on k. The case $k = 2$ holds trivially since the submaximal pfaffians correspond to the choice of six general linear forms.

In the general case, by induction, all forms of degree $k - 1$ in four variables sit in the ideal I' generated by the submaximal pfaffians of $k - 1$ general skew-symmetric matrices of linear forms $G_1, \ldots, G_{k-1}$, of size $(2k - 2) \times (2k - 2)$. Choose one general $2k \times 2k$ skew-symmetric matrix of linear forms M. By Lemma 5.1, if I is the ideal generated by the submaximal pfaffians of M, then the multiplication by a general linear form L determines a surjection $(R/I)_{d-1} \to (R/I)_d$ for all $d \geq k$.

Let $H_1, \ldots, H_{k-1}$ be the matrices obtained by the G_i's by adding the two rows $(0\ L\ 0\ \ldots\ 0)$ and $(-L\ 0\ 0\ \ldots\ 0)$ and the corresponding columns. Then the non-zero submaximal pfaffians of the H_i's are the submaximal pfaffians of the G_i's multiplied by L. Thus the submaximal pfaffians of the matrices $H_1, \dots, H_{k-1}, M$ generate an ideal which contains $I + LI'$, hence it coincides with R in degree $d \geq k$. The claim follows. □

Theorem 5.4. *Fix a $2k \times 2k$ symmetric homogeneous matrix A of non-negative integers. Then for a general choice of k matrices of forms $G_1, \dots, G_k$ with $\partial G_i = A$ for all i, the submaximal pfaffians of the G_i's generate an ideal which coincides with the polynomial ring R in degree $d \geq tr(A)/2$.*

Proof. We may assume $k \geq 2$, the case $k = 1$ being trivial. We make induction on the trace of A.

After Remark 2.4, we know that the entries in the diagonal of the matrix A are either all even or all odd. In the first case, we use as basis for the induction the null matrix A (for which the claim is obvious). In the latter case we use a matrix with all entries equal to 1 (for which the claim follows by Theorem 5.3).

In the inductive step, let A be ordered and call B the matrix obtained by A by subtracting 1 from the first row and the first column (thus subtracting 2 from the upper-left element, so that $tr(B) = tr(A) - 2$). As the theorem holds for B, for a general choice of skew-symmetric matrices $G_1, \dots, G_k$ with $\partial G_i = B$, the ideal I generated by the submaximal pfaffians of the G_i's coincides with the ring R in degree $d \geq tr(A)/2 - 1$.

Let G'_i be the matrix obtained from G_i by erasing the first row and column and call I_0 the ideal generated by the $(2k-2) \times (2k-2)$ pfaffians of all the G'_i's. The degree matrix $A' = (a'_{ij})$ of the G'_i's is the matrix obtained from A by cancelling the first row and column. Since A is ordered, we have $tr(A) \geq a'_{11} + tr(A')$. Thus, by Lemma 5.2 the multiplication by a general linear form L determines a surjection $(R/I_0)_{d-1} \to (R/I_0)_d$, for $d \geq tr(A') + a'_{11})$, hence also for $d \geq tr(A)$.

Let H_i be the matrix obtained from G_i by multiplying the first row and column by a general linear form L. Then $\partial H_i = A$ and the ideal I' generated by the submaximal pfaffians of the H_i's contains both I_0 and LI.

The claim follows. □

From Remark 2.3 it follows soon our main result.

Theorem 5.5. *Fix a $2k \times 2k$ symmetric homogeneous matrix A of non-negative integers, with trace $2d$. Then a general form F of degree d in four variables is the sum of the pfaffians of k skew-symmetric matrices of forms, with degree matrix A.*

In other words, we obtain $s(A) \leq k$.

6. Sharpness

It is very reasonable to ask how far is the bound for $s(A)$ given in Theorem 5.5 to be sharp.

This can be answered by computing the dimension of the (projective) variety V of forms which are the pfaffian of a single skew-symmetric matrix G.

REMARK 6.1. *When A is 4×4, then the bound $s(A) = 2$ is sharp for most values of the entries of A, as explained in [4].*

As the size $2k$ of A grows, however, the given bound is probably no longer sharp.

For instance, when all the entries of A are 1's (so we deal with skew-symmetric matrices of linear forms), then formula 3.6 and the exact sequence 3.1 of [5] show that $\dim V = 4k^2 + o(k)$. So one expects, at least for $k \gg 0$, that the s-secant variety of V fills the space of all forms of degree k as soon as $s \geq k/24 + o(k)$. In other words, we can state the following.

CONJECTURE 6.2. *A general form of degree $k \gg 0$ can be expressed as a sum of s pfaffians of skew-symmetric $2k \times 2k$ matrices of linear forms, for*

$$s = \frac{k}{24} + o(k).$$

Notice that our bound $s = k$ is already linear in k, but with a larger coefficient.

The same phenomenon is expected to occur for other types of homogeneous symmetric matrices A of large size.

For instance, if all the entries of A are equal to a constant $b \gg k$, then formula 3.6 and the exact sequence 3.1 of [5] tell us that $\dim(V) = k^2b^3/3 + o(b)$. Thus the expected value $s(A)$ such that the $s(A)$-secant variety of V fills the space of forms of degree kb is $s(A) = k/2 + o(k)$, which is (asymptotically) $1/2$ of our bound.

We hope that a refinement of our method will provide, in a future, advances towards sharper bounds for $s(A)$.

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Received September 30, 2014
Revised December 18, 2014