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Twistor Methods in Conformal Almost Symplectic Geometry

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Summary. - *Given a $2n-$dimensional almost symplectic manifold, (M,ω), we consider the conformal class of ω and to each symplectic connection,∇, we associate, in a natural way, a $e^{2\sigma}\omega-$symplectic connection, ∇^σ. We prove that the twistor bundle $Z(M,\omega) := \frac{P(M,Sp(2n))}{U(n)}$, with its canonical almost complex structure induced by ∇, is an invariant of the conformal class of (ω,∇). Then we study the interplay between conformal properties of (M,ω) and complex properties of $Z(M,\omega)$, passing trough the existence of special symplectic connections. Finally we prove that, in the case of a special Kähler manifold, the section of $Z(M,\omega)$ defined by the complex structure of M is an almost complex submanifold with respect to a certain almost complex structure on $Z(M,\omega)$.*

1. Introduction

Let (M,ω) be a $2n-$dimensional almost symplectic manifold, in [10] we studied the *twistor bundle* $Z(M,\omega) := \frac{P(M,Sp(2n))}{U(n)}$ with the natural almost complex structure $\mathbb{J}$ on $Z(M,\omega)$ induced by a symplectic connection. Actually we can introduce two almost complex structures: $\mathbb{J}^+ = \mathbb{J}$ and $\mathbb{J}^-$, (section 2), but $\mathbb{J}^-$ is never integrable.

In this paper we consider the conformal class of ω, $[\omega] := \{e^{2\sigma}\omega \mid$

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$\sigma \in C^\infty(M, \mathbb{R})\}$, and to each $\omega-$symplectic connection, ∇, we associate, in a natural way, a $e^{2\sigma}\omega-$symplectic connection, ∇^σ. We prove that the integrability condition for $\mathbb{J}^+$, as well the horizontal part of the Nijenhuis tensor of $\mathbb{J}^-$, are invariants of the conformal class of (ω, ∇), $[(\omega, \nabla)] := \left\{(e^{2\sigma}\omega, \nabla^\sigma) \mid \sigma \in C^\infty(M, \mathbb{R})\right\}$, and furthermore we prove that, like in the case of the twistor bundle of an oriented, even dimensional, Riemannian manifolds with Levi-Civita connection [4] $(Z(M, e^{2\sigma}\omega), \mathbb{J}^+_{\nabla^\sigma})$ is biholomorphic to $(Z(M, \omega), \mathbb{J}^+_\nabla)$.

Then we try to find an interplay between conformal properties of (M, ω) and complex properties of $Z(M, \omega)$, passing trough the existence of special connections. For example locally conformal symplectic manifolds [12] can be characterized by the existence of connections with a special torsion tensor [8, 9, 11], then, using this result, we can translate this conformal property in the vanishing of the horizontal part of the Nijenhuis tensor of $\mathbb{J}^+$ and $\mathbb{J}^-$.

Section 5 is devoted to the characterization of local sections $J : U \subset M \to Z(M, \omega)$ that define almost complex submanifolds of $(Z(M, \omega), \mathbb{J}^-_\nabla)$.

Finally in last section we apply previous results to *special Kähler manifolds*. Special geometry is of great interest in theoretical physics and in differential geometry [1, 3, 2, 5, 6]. Special Kähler manifolds are Kähler manifolds together with a flat, torsion free, symplectic connection, ∇, such that ∇J is symmetric, then they turn out to be manifolds whose twistor bundle is a Kähler manifold [10]. We get that the complex structure J, of a special Kähler manifold, defines a section of the twistor bundle which is an almost complex submanifold with respect to the almost complex structure $\mathbb{J}^-_\nabla$, moreover this section is almost complex with respect to $\mathbb{J}^+_\nabla$ if and only if ∇ is the Levi Civita connection. In particular this provides examples of integrable complex structures for which the section in the twistor bundle in not almost complex, this in contrast with the Riemannian case [10].

2. Preliminaries

In this section we recall definitions and general facts about *twistor bundles* of almost symplectic manifolds as introduced in [10]; notations will be mainly those used in that paper.

First of all we recall that given a symplectic vector space (V, ω), a complex structure J on V is called $\omega-$*calibrated* if $\omega(v, Jv) > 0$ for any $v \in V \setminus \{0\}$ and $\omega(Jv, Jw) = \omega(v, w)$ for all $v, w \in V$.

Let (M, ω) be an almost symplectic manifold of real dimension $2n$ and let $P_\omega := P(M, Sp(2n))$ be the principal bundle of symplectic frames on (M, ω). Let $Z_\omega = Z(M, \omega) := \frac{P(M,Sp(2n))}{U(n)}$ be the *twistor bundle* of (M, ω), let us denote by $\pi_\omega : P_\omega \to M$, $p_\omega : Z_\omega \to M$ and $r_\omega : P_\omega \to Z_\omega$ the bundle projections, the subscript ω will be omitted when there is no ambiguity.

Z_ω is a bundle on M with structure group $Sp(2n)$ and standard fibre the Hermitian symmetric space $\frac{Sp(2n)}{U(n)}$. Z_ω is a manifold of real dimension $n(n+3)$ and the fibre at the point $x \in M$, $p_\omega^{-1}(x)$, parameterizes all complex structures on $T_x M$ which are ω_x-calibrated.

A connection θ on P_ω defines a connection on Z_ω, that is a splitting of the tangent bundle in horizontal and vertical subbundles: $TZ_\omega = H \oplus V$, and, as Z_ω is a bundle of complex structures with complex fibre, this allows us to define almost complex structures, $\mathbb{J}^+$, $\mathbb{J}^-$, on Z_ω in a tautologically way.
In fact let $\{E_1, ..., E_{2n}\}$ be a local symplectic frame on M, let $J \in Z(M, \omega)$ and let $H_J = \left\{\widehat{E_1}, ..., \widehat{E_{2n}}\right\}$ be the horizontal subspace of $T_J Z(M, \omega)$, where, $\forall i = 1, ...2n$, is $\widehat{E_i} = E_i - [\Gamma_i, J] \frac{\partial}{\partial J}$, with Γ_i Cristoffell's symbols.
Let $V_J = \left\{\widehat{Y}(J) = [Y, J] \frac{\partial}{\partial J} \,|Y \in sp(2n)\right\}$ be the vertical subspace, we define:

$$\begin{cases} \mathbb{J}^\pm(J)(\widehat{E_i}) = J_i^k \widehat{E_k} \\ \mathbb{J}^\pm(J)(\widehat{Y}(J)) = \pm \widehat{JY}(J) \end{cases}$$

where we used, as we will do in the following, Einstein's convention on repeated indices.

Let ∇ be the covariant derivative of θ and let $T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$, $R^\nabla(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ be respectively the torsion and the curvature of the connection, a direct computation on the Nijenhuis tensor, $N_{\mathbb{J}^\pm}$, gives conditions on θ under which $\mathbb{J}^\pm$ are integrable, precisely we have:

$$\begin{cases} N_{\mathbb{J}^+}(J)(\widehat{E_i},\widehat{E_j}) = \\ = (-T^\nabla(JE_i,JE_j) + JT^\nabla(JE_i,E_j) + JT^\nabla(E_i,JE_j) + \\ + T^\nabla(E_i,E_j))^r\widehat{E_r} - R^\nabla\widehat{(JE_i,JE_j)} + JR^\nabla\widehat{(JE_i,E_j)} + \\ + JR^\nabla\widehat{(E_i,JE_j)} + R^\nabla\widehat{(E_i,E_j)} \\ N_{\mathbb{J}^+}(J)(\widehat{E_i},\widehat{A}) = 0 \\ N_{\mathbb{J}^+}(J)(\widehat{A},\widehat{B}) = 0 \end{cases}$$

and:

$$\begin{cases} N_{\mathbb{J}^-}(J)(\widehat{E_i},\widehat{E_j}) = \\ = (-T^\nabla(JE_i,JE_j) + JT^\nabla(JE_i,E_j) + JT^\nabla(E_i,JE_j) + \\ + T^\nabla(E_i,E_j))^r\widehat{E_r} - R^\nabla\widehat{(JE_i,JE_j)} - JR^\nabla\widehat{(JE_i,E_j)} + \\ - JR^\nabla\widehat{(E_i,JE_j)} + R^\nabla\widehat{(E_i,E_j)} \\ N_{\mathbb{J}^-}(J)(\widehat{E_i},\widehat{A}) = -2[\widehat{E_i},\widehat{A}] \\ N_{\mathbb{J}^-}(J)(\widehat{A},\widehat{B}) = 0. \end{cases}$$

Thus we get:

PROPOSITION 2.1. $\mathbb{J}^+$ *is integrable if and only if it results:*

$$\begin{cases} T^\nabla(JX,JY) - JT^\nabla(JX,Y) - JT^\nabla(X,JY) - T^\nabla(X,Y) = 0 \\ R^\nabla\widehat{(JX,JY)} - JR^\nabla\widehat{(JX,Y)} - JR^\nabla\widehat{(X,JY)} - R^\nabla\widehat{(X,Y)} = 0 \end{cases} \tag{1}$$

for all $X, Y \in T_xM$, for all $x \in M$, for all $J \in p^{-1}(x)$.

And:

PROPOSITION 2.2. $\mathbb{J}^-$ *is never integrable.*

Almost complex structures depend on the given connection, we can give conditions under which two connections induce same almost complex structures.

Let θ and θ' be two symplectic connections, let ∇ and ∇' be the associated covariant derivatives and let $\nabla' - \nabla = \Theta$, let H_J^∇ and $H_J^{\nabla'}$ be the horizontal tangent spaces at the point J with respect to given connections, we have:

$$H_J^{\nabla'} = \{X + L(X) \mid X \in H_J^\nabla,\ L \in L(H_J^\nabla, V_J)\}$$

where L is the linear map defined by: $L(\widehat{E_i}) = -[\Theta_{E_i}, J]\dfrac{\partial}{\partial J} = -\widehat{\Theta_i}$.

We get:

$$\mathbb{J}^{\pm}_{\nabla'}(\widehat{E_i} + L(\widehat{E_i})) = J_i^k(\widehat{E_k} + L(\widehat{E_k}))$$

and:

$$\begin{aligned} \mathbb{J}^{\pm}_{\nabla}(\widehat{E_i} + L(\widehat{E_i})) &= J_i^k\widehat{E_k} \pm J(-\widehat{\Theta}_i) = \\ &= J_i^k(\widehat{E_k} + L(\widehat{E_k})) - J_i^k L(\widehat{E_k}) \pm J(-\widehat{\Theta}_i). \end{aligned}$$

Thus we have:

PROPOSITION 2.3. $\mathbb{J}^{\pm}_{\nabla'} = \mathbb{J}^{\pm}_{\nabla}$ *if and only if for all* $J \in Z(M,\omega)$ *and for all* $i = 1, ..., 2n$, *results:*

$$J_i^k\Theta_k J - J_i^k J\Theta_k = \pm(J\Theta_i J + \Theta_i).$$

Before closing this section we define almost symplectic structures $\Omega^{\pm}$ on $Z(M,\omega)$ in the following natural way:

$$\begin{cases} \Omega^{\pm}(\widehat{E_i}, \widehat{E_j}) = \omega(E_i, E_j) \\ \Omega^{\pm}(\widehat{E_i}, \widehat{A}) = 0 \\ \Omega^{\pm}(\widehat{A}, \widehat{B}) = \mp\frac{1}{2}Trace(\widehat{AJB}). \end{cases}$$

We have immediately that $\mathbb{J}^+$ (respectively $\mathbb{J}^-$) is Ω^+ (respectively Ω^-) $-$ calibrated.

3. Conformal invariance

In this section we study the behavior of the twistor bundle under a conformal change of the almost symplectic structure.

DEFINITION 3.1. *Let* ω *and* ω' *be almost symplectic structures on* M, (M,ω') *is said to be* ***conformally equivalent*** *to* (M,ω) *if there exists* $\sigma \in C^{\infty}(M,\mathbb{R})$ *such that* $\omega' = e^{2\sigma}\omega$.

Let (M, ω') conformally equivalent to (M, ω), let $P_{\omega'} = P(M, Sp(2n))$ and $Z_{\omega'} = Z(M, \omega')$ be the principal bundle of almost symplectic frames and the twistor bundle of (M, ω') respectively, we have the following:

PROPOSITION 3.2. *There exists a diffeomorphism $\Phi : P_\omega \to P_{\omega'}$ such that the induced map $\widehat{\Phi} : Z(M, \omega) \to Z(M, \omega')$ is the identity.*

Proof. Let $a \in P_\omega$, let $\pi_\omega(a) = x \in M$ and let $\{E_1, ..., E_{2n}\}$ be a symplectic frame around x, define: $\Phi(a) = (x; e^{-\sigma}E_1, ..., e^{-\sigma}E_{2n})$. Φ is a diffeomorphism between P_ω and $P_{\omega'}$, in fact: let $A \in Sp(2n)$ and let $aA = (x; A_1^l E_l, ..., A_{2n}^l E_l)$, then we have $\Phi(aA) = \Phi(a)A$.
Now Φ induces a map $\widehat{\Phi} : Z(M, \omega) \to Z(M, \omega')$ by: $\widehat{\Phi} := (r_{\omega'}\Phi(r_\omega)^{-1})$ which turns out to be the identity, in fact:
let $J \in Z(M, \omega)$, then $J \in End(T_{p_\omega(J)}M)$ and $J^2 = -I$; let $a \in (r_\omega)^{-1}(J)$, is $a = (\pi_\omega(a); E_1, ..., E_{2n})$, define $w_a : \mathbb{R}^{2n} \to T_{\pi_\omega(a)}M$ by $w_a(e_i) := E_i$, then $J(E_i) = w_a \circ J_n \circ (w_a)^{-1}(E_i)$, where $\{e_1, ..., e_{2n}\}$ is the canonical basis of $\mathbb{R}^{2n}$ and J_n is the standard complex structure of $\mathbb{R}^{2n}$.
Let $a' \in (r_\omega)^{-1}(J)$, is $a' = (\pi_\omega(a); A_1^l E_l, ..., A_{2n}^l E_l)$, where $A \in U(n)$, then $w_{a'}(e_i) = A_i^l E_l$; now we have:

$$\Phi(a) = (x; e^{-\sigma}E_1, ..., e^{-\sigma}E_{2n}),$$

$$\Phi(a') = (x; e^{-\sigma}A_1^l E_l, ..., e^{-\sigma}A_{2n}^l E_l)$$

and:

$$r_{\omega'}(\Phi(a)) = w_{\Phi(a)} \circ J_n \circ (w_{\Phi(a)})^{-1} = e^{-\sigma}w_a \circ J_n \circ e^{-\sigma}(w_a)^{-1} = J,$$

$$r_{\omega'}(\Phi(a')) = w_{\Phi(a')} \circ J_n \circ (w_{\Phi(a')})^{-1} = e^{-\sigma}w_{a'} \circ J_n \circ e^{-\sigma}(w_{a'})^{-1} =$$

$$= w_a \circ AJ_nA^{-1} \circ (w_a)^{-1} = w_a \circ J_n \circ (w_a)^{-1} = J.$$

□

Thus the twistor bundle $Z(M, \omega)$ is an **invariant of the conformal class of the almost symplectic structure**.

Let $A = grad_\omega$ be the **symplectic gradient** on (M, ω) defined on a map $f \in C^\infty(M, \mathbb{R})$ by:

$$\omega(grad_\omega f, X) := \omega(A^f, X) := X(f).$$

Let $\omega^\sigma = e^{2\sigma}\omega$ and let $A^\sigma = grad_\omega \sigma$, given a symplectic connection ∇ on (M, ω) we define a new connection by:

$$\begin{aligned} \nabla^\sigma_X Y \quad : &= \nabla_X Y + \omega(A^\sigma, X)Y + \omega(A^\sigma, Y)X + \omega(X,Y)A^\sigma \\ &= \nabla_X Y + X(\sigma)Y + Y(\sigma)X + \omega(X,Y)A^\sigma . \end{aligned}$$

We remark that often we use the term connection for the covariant derivative of the connection.

A direct computation gives immediately the following:

LEMMA 3.3. *∇^σ is a $e^{2\sigma}\omega$−symplectic connection, that is:*

$$\nabla^\sigma (e^{2\sigma}\omega) = 0.$$

We set:

DEFINITION 3.4. *The **conformal class of an almost symplectic manifold with a fixed symplectic connection**, (M, ω, ∇) is the class $[(\omega, \nabla)]$ defined by:*

$$[(\omega, \nabla)] := \left\{ (e^{2\sigma}\omega, \nabla^\sigma) \;\mid \sigma \in C^\infty(M, \mathbb{R}) \right\}.$$

Let T^σ be the torsion of ∇^σ, let $J \in Z_{\omega^\sigma}$, let $X, Y \in T_{p_{\omega^\sigma}(J)}M$, we set:

$$\begin{aligned} &T^{\sigma(0,2)}(J)(X,Y) := \\ &= T^\sigma(JX, JY) - JT^\sigma(JX, Y) - JT^\sigma(X, JY) - T^\sigma(X,Y) \end{aligned}$$

and:

$$\begin{aligned} &T^{(0,2)}(J)(X,Y) := \\ &= T(JX, JY) - JT(JX, Y) - JT(X, JY) - T(X,Y). \end{aligned}$$

We can prove that $T^{(0,2)}$ is an **invariant of the conformal class** $[(\omega, \nabla)]$, in fact:

LEMMA 3.5. *The following relationships hold:*

$$\begin{aligned} &(i)\ T^\sigma = T + 2\omega A^\sigma \\ &(ii)\ T^{\sigma(0,2)} = T^{(0,2)}. \end{aligned}$$

Proof. (i) We have:

$$T^\sigma(X,Y) = \nabla^\sigma_X Y - \nabla^\sigma_Y X - [X,Y]$$
$$= T(X,Y) + X(\sigma)Y + Y(\sigma)X + \omega(X,Y)A^\sigma +$$
$$-Y(\sigma)X - X(\sigma)Y - \omega(Y,X)A^\sigma$$
$$= T(X,Y) + 2\omega(X,Y)A^\sigma.$$

(ii) follows from (i) and from the fact that any J is $\omega-$calibrated.

□

Thus we get:

REMARK 3.6. *Statement (ii) in Lemma* 6 *says in particular that* $T^{(0,2)}$ *is an* ***invariant of the conformal class*** $[(\omega,\nabla)]$. *Furthermore the horizontal part of the Nijenhuis tensor of* $\mathbb{J}^+$ *and* $\mathbb{J}^-$ *is an invariant of the conformal class* $[(\omega,\nabla)]$.

Also we can compute the relationship between the curvature R^σ of ∇^σ and R of ∇. We have following Lemmas whose proofs are in Appendix:

LEMMA 3.7. *The following formula holds:*

$$\begin{aligned} R^\sigma(X,Y)Z \;=\;& R(X,Y)Z + Z(\sigma)(T(X,Y) + 2\omega(X,Y)A^\sigma) + \\ & +\omega(T(X,Y),Z)A^\sigma + (Y(\sigma)Z(\sigma) - \omega(\nabla_Y A^\sigma, Z))X + \\ & (\omega(\nabla_X A^\sigma, Z) - X(\sigma)Z(\sigma))Y + (Y(\sigma)\omega(X,Z) + \\ & -X(\sigma)\omega(Y,Z))A^\sigma + \\ & +\omega(Y,Z)\nabla_X A^\sigma - \omega(X,Z)\nabla_Y A^\sigma. \end{aligned}$$

Moreover if we pose:

$$R^{\sigma(0,2)}(J)(X,Y) :=$$
$$= R^\sigma(JX,JY) - JR^\sigma(JX,Y) - JR^\sigma(X,JY) - R^\sigma(X,Y)$$

and:

$$R^{(0,2)}(J)(X,Y) :=$$
$$= R(JX,JY) - JR(JX,Y) - JR(X,JY) - R(X,Y),$$

we have:

LEMMA 3.8. $R^{\sigma(0,2)} = R^{(0,2)} + \omega(A^\sigma, \cdot)T^{0,2} + \Lambda^{(0,2)}$,
where:

$$\begin{aligned}\Lambda(X,Y)Z \ \ : &= \omega(T(X,Y),Z)A^\sigma - \omega(\nabla_Y A^\sigma, Z)X + \omega(\nabla_X A^\sigma, Z)Y + \\ &+(Y(\sigma)\omega(X,Z) - X(\sigma)\omega(Y,Z))A^\sigma + \\ &+\omega(Y,Z)\nabla_X A^\sigma - \omega(X,Z)\nabla_Y A^\sigma,\end{aligned}$$

and:

$$\begin{aligned}&\Lambda^{(0,2)}(J)(X,Y) := \\ &= \Lambda(JX,JY) - J\Lambda(JX,Y) - J\Lambda(X,JY) - \Lambda(X,Y).\end{aligned}$$

Moreover:

LEMMA 3.9. *The following formula holds:*

$$\begin{aligned}\widehat{R^{\sigma(0,2)}(J)} \ \ = \ \ & \widehat{R^{(0,2)}(J)} + \omega(A^\sigma, \cdot)\widehat{T^{0,2}(J)} + \\ & -(\omega(JT^{(0,2)}, \cdot)A^\sigma + \omega(T^{(0,2)}, \cdot)JA^\sigma)\frac{\partial}{\partial J}.\end{aligned}$$

From previous lemmas we get immediately the following:

PROPOSITION 3.10. *The integrability condition* (1), *that can be expressed as:*

$$\left\{ \begin{array}{l} T^{0,2} = 0 \\ \widehat{R^{0,2}} = 0 \end{array} \right. , \tag{2}$$

is an invariant of the conformal class $[(\omega, \nabla)]$.

Moreover we can prove that the almost complex structure $\mathbb{J}^+$ on $Z(M,\omega)$ is an **invariant of the conformal class** $[(\omega,\nabla)]$, in fact we have the following:

PROPOSITION 3.11. $(Z(M,\omega), \mathbb{J}^+_\nabla)$ *is biholomorphic to* $(Z(M, e^{2\sigma}\omega)$, $\mathbb{J}^+_{\nabla^\sigma})$.

Proof. Let $J \in Z(M,\omega)$, let $T_J Z_\omega = H_J^\nabla \oplus V_J$ and let $H_J^\nabla = \left\{\widehat{E_1}, ..., \widehat{E_{2n}}\right\}$, $\widehat{E_i} = E_i - [\Gamma_i, J]\frac{\partial}{\partial J}$, where $\omega(E_i,E_j) = -(J_n)^i_j$, $(J_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix})$.

Let $T_J Z_{e^{2\sigma}\omega} = H_J^{\nabla^\sigma} \oplus V_J$, then $H_J^{\nabla^\sigma} = \left\{ \widehat{E_1^\sigma}, ..., \widehat{E_{2n}^\sigma} \right\}$, $\widehat{E_i^\sigma} = e^{-\sigma} E_i - [(\Gamma^\sigma)_i, J] \frac{\partial}{\partial J}$.

Let us compute Cristoffel's symbols $(\Gamma^\sigma)_{ij}^k$ of ∇^σ :

$$\nabla^\sigma_{e^{-\sigma}E_i} e^{-\sigma} E_j = e^{-\sigma} (\Gamma^\sigma)_{ij}^k E_k$$
$$= e^{-\sigma}(E_i(e^{-\sigma})E_j + e^{-\sigma} \nabla^\sigma_{E_i} E_j)$$
$$= e^{-2\sigma}(\nabla_{E_i} E_j + E_j(\sigma) E_i + (-J_n)_j^i A^\sigma)$$
$$= e^{-2\sigma}(\Gamma_{ij}^k E_k + E_j(\sigma) E_i - E_l(\sigma)(J_n)_j^i (J_n)_k^l E_k);$$

then:

$$(\Gamma^\sigma)_{ij}^k = e^{-\sigma}(\Gamma_{ij}^k + E_j(\sigma)\delta_{ik} - E_l(\sigma)(J_n)_j^i (J_n)_k^l)$$

where δ_{ik} is Kronecher's symbol.

Thus we have:

$$\widehat{E_i^\sigma} = e^{-\sigma}(\widehat{E_i} - [L_i, J] \frac{\partial}{\partial J})$$

where: $L_{ij}^k = e^{-\sigma}(E_j(\sigma)\delta_{ik} - E_l(\sigma)(J_n)_j^i (J_n)_k^l)$.

Now is:

$$\mathbb{J}^+_{\nabla^\sigma} \widehat{E_i^\sigma} = J_i^k \widehat{E_k^\sigma}$$

and:

$\mathbb{J}^+_{\nabla} \widehat{E_i^\sigma} = e^{-\sigma}(J_i^k \widehat{E_k} - J[L_i, J]\frac{\partial}{\partial J}) = J_i^k \widehat{E_k^\sigma} + e^{-\sigma}(J_i^k [L_k, J]\frac{\partial}{\partial J} - J[L_i, J]\frac{\partial}{\partial J})$.

Let us compute:

$$J_i^k [L_k, J] - J[L_i, J] =$$
$$= J_i^k L_k J - J_i^k J L_k - J L_i J - L_i =$$
$$= J_i^k L_{ks}^r J_h^s - J_i^k J_s^r L_{kh}^s - J_s^r L_{il}^s J_h^l - L_{ih}^r =$$
$$= e^{-\sigma} J_i^k J_h^s (E_s(\sigma)\delta_{kr} - E_\nu(\sigma)(J_n)_s^k (J_n)_r^\nu) - e^{-\sigma} J_i^k J_s^r (E_h(\sigma)\delta_{ks} +$$
$$- E_\nu(\sigma)(J_n)_h^k (J_n)_s^\nu) - e^{-\sigma} J_s^r J_h^l (E_l(\sigma)\delta_{si} - E_\nu(\sigma)(J_n)_l^i (J_n)_s^\nu) +$$
$$- e^{-\sigma}(E_h(\sigma)\delta_{ir} - E_\nu(\sigma)(J_n)_h^i (J_n)_r^\nu) = 0,$$

that is:

$$\mathbb{J}^+_{\nabla^\sigma} \widehat{E_i^\sigma} = \mathbb{J}^+_{\nabla} \widehat{E_i^\sigma}. \qquad \square$$

REMARK 3.12. *Previous proof, repeated on* $\mathbb{J}^-_\nabla$*, shows that* $\mathbb{J}^-_\nabla$ *is not an invariant of the conformal class* $[(\omega, \nabla)]$.

REMARK 3.13. *It is known that some conformal symplectic property can be characterized in terms of the existence of special connections on* M*, ([8],[9]), so we expect to find a relationship between* ***"conformal symplectic properties"*** *of* (M, ω) *and* ***"complex properties"*** *of* $Z(M, \omega)$*, passing trough special connections. We will investigate this in the following.*

4. Locally symplectic conformal manifolds

Let (M, ω) be an almost symplectic manifold, in [7] two conformal curvature tensors are introduced in the following way: let $\{E_1, ..., E_{2n}\}$ be a local frame on M, let $\omega_{ij} = \omega(E_i, E_j)$ and let ω^{ij} be the inverse matrix, we define $H \in \Lambda^1(M)$ by:

$$H(X) := d\omega(X, E_i, E_j)\omega^{ij}$$

for any vector field X; then we set:

DEFINITION 4.1. $B(X,Y) := dH(X,Y)$ *is called the* ***first conformal curvature tensor****;*

$$C(X,Y,Z) := d\omega(X,Y,Z) +$$

$$-\frac{1}{2n-2}\{H(Z)\omega(X,Y) + H(X)\omega(Y,Z) + H(Y)\omega(Z,X)\}$$

is called the ***second conformal curvature tensor****.*

DEFINITION 4.2. *An almost symplectic manifold* (M, ω) *is called* ***locally conformal symplectic*** *if for any* $x \in M$ *there exists an open neighborhood* $U(x)$ *and a map* $\sigma \in C^\infty(U, \mathbb{R})$ *such that* $d(e^{2\sigma}\omega_{|U}) = 0$*. If* $U = M$ *is called* ***globally conformally symplectic****.*

In [7] it is proved that (M, ω) is locally conformal symplectic if and only if there exists $\alpha \in \Lambda^1(M)$, $d\alpha = 0$, such that $d\omega = \alpha \wedge \omega$, and (M, ω) is globally conformal symplectic if and only if α is exact. Moreover the following characterization in terms of conformal curvature tensors, is proved:

Proposition 4.3. *If $n > 1$ then (M, ω) is globally conformal symplectic if and only if it results:*

$$\begin{cases} B = 0 \\ C = 0 \end{cases} .$$

We set:

Definition 4.4. *A linear connection ∇ on M is said to be **conformal almost symplectic** if there exists $\alpha \in \Lambda^1(M)$ such that $\nabla_X \omega = \alpha(X)\omega$ for any vector field X.*

We have:

Lemma 4.5. *Let ∇ be a conformal almost symplectic connection and let $\nabla_X \omega = \alpha(X)\omega$, then ∇' defined by:*

$$\nabla'_X Y := \nabla_X Y + \frac{1}{2}\alpha(X)Y$$

is a symplectic connection.

Proof. We have:

$$(\nabla'_X \omega)(X, Y) = X\omega(Y, Z) - \omega(\nabla'_X Y, Z) - \omega(Y, \nabla'_X Z) =$$
$$= (\nabla_X \omega)(Y, Z) - \alpha(X)\omega(Y, Z) = 0. \qquad \square$$

Moreover if we denote by T and T' the torsion of ∇ and ∇', defined as before, we get:

Lemma 4.6. $T'^{(0,2)} = T^{(0,2)}$.

Proof. We have:

$$\begin{aligned} T'(X, Y) &= \nabla'_X Y - \nabla'_Y X - [X, Y] = \\ &= T(X, Y) + \frac{1}{2}\alpha(X)Y - \frac{1}{2}\alpha(Y)X \end{aligned}$$

then:

$$\begin{aligned} &T'^{(0,2)}(J)(X, Y) = \\ &T'(JX, JY) - JT'(JX, Y) - JT'(X, JY) - T(X, Y) = \\ &= T^{(0,2)}(J)(X, Y). \end{aligned}$$

$\square$

In [11] the following is proved:

PROPOSITION 4.7. *Let (M,ω) be an almost symplectic manifold, there exists a conformal almost symplectic connection such that its torsion tensor, T, is defined by:*

$$\omega(3T(X,Y),Z) = C(X,Y,Z). \tag{3}$$

Then we have:

PROPOSITION 4.8. *If (M,ω) is a locally conformal symplectic manifold then there exists a symplectic connection, θ, such that the Nijenhuis tensor of the associate almost complex structures $\mathbb{J}^{\pm}_{\theta}$ on the twistor bundle $Z(M,\omega)$ has no horizontal component.*

Proof. Let ∇ be the covariant derivative of a conformal almost symplectic connection whose torsion is defined by (3), we get:

$$\begin{aligned}
&\omega(3T^{(0,2)}(X,Y),Z) = C(JX,JY,Z) + C(JX,Y,JZ) + \\
&+C(X,JY,JZ) - C(X,Y,Z); \\
&= d\omega(JX,JY,Z) + d\omega(JX,Y,JZ) + d\omega(X,JY,Z) - d\omega(X,Y,Z) + \\
&-\frac{1}{2n-2}(H(JX)\omega(JY,Z) + H(JY)\omega(Z,JX) + \\
&+H(Z)\omega(JX,JY) + H(JX)\omega(Y,JZ) + H(Y)\omega(JZ,JX) + \\
&+H(JZ)\omega(JX,Y) + H(X)\omega(JY,JZ) + H(JY)\omega(JZ,X) + \\
&+H(JZ)\omega(X,JY) - H(X)\omega(Y,Z) - H(Y)\omega(Z,X) + \\
&-H(Z)\omega(X,Y)) = \\
&= d\omega(JX,JY,Z) + d\omega(JX,Y,JZ) + d\omega(X,JY,Z) - d\omega(X,Y,Z).
\end{aligned}$$

Using the fact that (M,ω) is locally conformal symplectic, posed $d\omega = \alpha \wedge \omega$, we have:

$\omega(3T^{(0,2)}(X,Y),Z) = 0$, for all X,Y,Z,

then:

$T^{(0,2)} = 0$.

Thus, using lemmas 4.5 and 4.6, there exists a symplectic connection such that its torsion has vanishing (0,2)-part. ☐

5. Local sections

Let (M,ω) be an almost symplectic manifold, let $U \subset M$ be an open set and let J be an almost complex structure on U, ω-calibrated, that is ω_x-calibrated for any $x \in U$. J defines a local section of $Z(M,\omega)$ over U by $J(x) := J_x \in p_\omega^{-1}(x)$.

Let ∇ be a symplectic connection, in [10] we exploited conditions on J and ∇ under which $(J(U),J)$ is an almost complex local submanifold of $(Z(M,\omega),\mathbb{J}^+_\nabla)$. Regarding $\mathbb{J}^-_\nabla$ we have the following :

PROPOSITION 5.1. *Following facts are equivalent:*

(i) $(J(U),J)$ *is an almost complex local submanifold of* $(Z(M,\omega),\mathbb{J}^-_\nabla)$;

(ii) $P(X,Y) := (\nabla_{JX}J)Y + J(\nabla_X J)Y = 0$
for all tangent vector fields X,Y *on* U;

(iii) $(d^\nabla J)(X,Y) := (\nabla_X J)Y - (\nabla_Y J)X = 0$
for all tangent vector fields X,Y *on* U.

Proof. Let $\{E_1,...,E_{2n}\}$ be a local symplectic frame, we have:

$$J_*(E_l) = E_l + E_l(J)\frac{\partial}{\partial J} = \widehat{E_l} + \widehat{\Gamma}_l + \frac{1}{2}\widehat{JE_l(J)},$$

then $\mathbb{J}^-_\nabla \circ J_* = J_* \circ J$ if and only if, for all $l = 1,...2n$, it results:

$$\mathbb{J}^-_\nabla(\widehat{E_l} + \widehat{\Gamma}_l + \frac{1}{2}\widehat{JE_l(J)}) = J_l^k(\widehat{E_k} + \widehat{\Gamma}_k + \frac{1}{2}\widehat{JE_k(J)}),$$

that is:

$$J\widehat{\Gamma}_l - \frac{1}{2}\widehat{E_l(J)}) + J_l^k\widehat{\Gamma}_l + \frac{1}{2}J_l^k\widehat{E_k(J)} = 0;$$

on the other hand we have:

$$\begin{aligned}
&(\nabla_{JE_l}J)E_k + J(\nabla_{E_l}J)E_k = \\
&= \nabla_{JE_l}JE_k - J\nabla_{JE_l}E_k + J\nabla_{E_l}JE_k + \nabla_{E_l}E_k \\
&= J_l^r\nabla_{E_r}J_k^hE_h - J_l^r\Gamma^i_{rk}J_i^jE_j + J\nabla_{E_l}J_k^hE_h + \Gamma^j_{lk}E_j \\
&= (J_l^rJ_k^h\Gamma^j_{rh} + J_l^rE_r(J_k^j) - J_l^r\Gamma^i_{rk}J_i^j + J_k^h\Gamma^r_{lh}J_r^j + E_l(J_k^h)J_h^j + \Gamma^j_{lk})E_j \\
&= (J_l^r[\Gamma_r,J] + J_l^rE_r(J) + J\Gamma_lJ + JE_l(J) + \Gamma_l)^j_kE_j \\
&= J_l^k\widehat{\Gamma}_l + \frac{1}{2}J_l^k\widehat{E_k(J)} + J\widehat{\Gamma}_l - \frac{1}{2}\widehat{E_l(J)}).
\end{aligned}$$

Thus we have that (i) and (ii) are equivalent.
Now we have:

$$(\nabla_Y J)JX = -J(\nabla_Y J)X,$$

then:

$$\begin{aligned} P(X,Y) &= (\nabla_{JX} J)Y + J(\nabla_X J)Y \\ &= (\nabla_{JX} J)Y - (\nabla_Y J)JX - J(\nabla_Y J)X + J(\nabla_X J)Y \\ &= (d^\nabla J)(JX,Y) - J(d^\nabla J)(Y,X) \end{aligned}$$

and:

$$P(X,Y) - P(JX,Y) = 2(d^\nabla J)(X,Y).$$

That is (ii) and (iii) are equivalent. □

6. Special Kähler manifolds

Special geometry is of relevant interest in theoretical physics and in differential geometry [1, 3, 2, 5, 6]. We recall the definition of **special Kähler manifold**:

Definition 6.1. *A* ***special Kähler manifold*** (M,ω,J,∇) *is a symplectic manifold* (M,ω) *with a* $\omega-$*calibrated complex structure* J *and a flat, torsion free, symplectic connection* ∇ *such that* $(\nabla_X J)Y = (\nabla_Y J)X$ *for all vector fields* X, Y.

If, in previous definition, we don't require J to be $\omega-$calibrated we have the concept of special symplectic manifold introduced in [1].

Special Kähler manifolds and more generally special symplectic manifolds turn out to be of great interest for us because they provide examples of manifolds whose twistor bundle with natural structures, $(Z(M,\omega), \mathbb{J}^+_\nabla, \mathbb{G}^\nabla)$, is a Kähler manifold [10].

Using Proposition 5.1 we get the following:

Proposition 6.2. *If* (M,ω,J,∇) *is a special Kähler manifold then the submanifold* $J(M)$ *of* $Z(M,\omega)$*, defined naturally by the complex structure* J*, is* $\mathbb{J}^-_\nabla-$ *almost complex.*

On the other hand we have:

PROPOSITION 6.3. *Let (M, ω, J, ∇) be a special Kähler manifold, the submanifold $J(M)$ of $Z(M, \omega)$, defined naturally by J, is $\mathbb{J}^+_\nabla-$ almost complex if and only if ∇ is the Levi Civita connection.*

Proof. We have that $\mathbb{J}^+_\nabla \circ J_* = J_* \circ J$ if and only if it results:

$$(\nabla_{JX}J)Y - J(\nabla_X J)Y = 0,$$

now we get:

$$\begin{aligned}(\nabla_{JX}J)Y &= (\nabla_Y J)JX = -J(\nabla_Y J)X = -J(\nabla_X J)Y \\ &= J(\nabla_X J)Y\end{aligned}$$

and then if and only if $\nabla J = 0$. As ∇ is symplectic and torsion free this is equivalent to be the Levi Civita connection of the Kähler metric $g(X, Y) := \omega(X, JY)$. □

REMARK 6.4. *From previous proposition we have that special Kähler manifolds provide examples of manifolds for which there exist integrable complex structures such that the corresponding section on the twistor bundle $(\frac{P(M,Sp(2n))}{U(n)}, \mathbb{J}^+_\nabla)$ is not almost complex. This doesn't happen in the Riemannian case [4, 10].*

7. Appendix

In this section we report details of computations that give proofs of Lemmas 3.7, 3.8 and 3.9.

Proof of Lemma 3.7. $R^\sigma(X, Y)Z = [\nabla^\sigma_X, \nabla^\sigma_Y]Z - \nabla^\sigma_{[X,Y]}Z =$

$$= \nabla^\sigma_X(\nabla_Y Z + Y(\sigma)Z + Z(\sigma)Y + \omega(Y, Z)A^\sigma) +$$
$$-\nabla^\sigma_Y(\nabla_X Z + X(\sigma)Z + Z(\sigma)X + \omega(X, Z)A^\sigma) +$$
$$-\nabla_{[X,Y]}Z - [X, Y](\sigma)Z - Z(\sigma)[X, Y] - \omega([X, Y], Z)A^\sigma =$$
$$= \nabla_X\nabla_Y Z + \nabla_X(Y(\sigma)Z) + \nabla_X(Z(\sigma)Y) + \nabla_X(\omega(Y, Z)A^\sigma) +$$
$$+X(\sigma)(\nabla_Y Z + Y(\sigma)Z + Z(\sigma)Y + \omega(Y, Z)A^\sigma) +$$
$$+((\nabla_Y Z)(\sigma) + Y(\sigma)Z(\sigma) + Z(\sigma)Y(\sigma) + \omega(Y, Z)A^\sigma(\sigma))X +$$
$$+\omega(X, \nabla_Y Z + Y(\sigma)Z + Z(\sigma)Y + \omega(Y, Z)A^\sigma)A^\sigma +$$

$$
\begin{aligned}
&-(\nabla_Y\nabla_X Z + \nabla_Y(X(\sigma)Z) + \nabla_Y(Z(\sigma)X) + \nabla_Y(\omega(X,Z)A^\sigma)+\\
&+Y(\sigma)(\nabla_X Z + X(\sigma)Z + Z(\sigma)X + \omega(X,Z)A^\sigma)+\\
&+((\nabla_X Z)(\sigma) + X(\sigma)Z(\sigma) + Z(\sigma)X(\sigma) + \omega(X,Z)A^\sigma(\sigma))Y+\\
&+\omega(Y, \nabla_X Z + X(\sigma)Z + Z(\sigma)X + \omega(X,Z)A^\sigma)A^\sigma)+\\
&-\nabla_{[X,Y]}Z - [X,Y](\sigma)Z - Z(\sigma)[X,Y] - \omega([X,Y],Z)A^\sigma =\\
&= R(X,Y)Z + XY(\sigma)Z + Y(\sigma)\nabla_X Z + (\nabla_X Y)Z(\sigma) + XZ(\sigma)Y+\\
&+\nabla_X(\omega(Y,Z)A^\sigma) + X(\sigma)\nabla_Y Z + X(\sigma)Z(\sigma)Y + X(\sigma)\omega(Y,Z)A^\sigma+\\
&+(\nabla_Y Z)(\sigma)X + 2Y(\sigma)Z(\sigma)X + \omega(X,\nabla_Y Z)A^\sigma+\\
&+Y(\sigma)\omega(X,Z)A^\sigma + Z(\sigma)\omega(X,Y)A^\sigma + \omega(Y,Z)\omega(X,A^\sigma)A^\sigma+\\
&-(YX(\sigma)Z + X(\sigma)\nabla_Y Z + (\nabla_Y X)Z(\sigma) + YZ(\sigma)X+\\
&+\nabla_Y(\omega(X,Z)A^\sigma) + Y(\sigma)\nabla_X Z + Y(\sigma)Z(\sigma)X + Y(\sigma)\omega(X,Z)A^\sigma+\\
&+(\nabla_X Z)(\sigma)Y + 2X(\sigma)Z(\sigma)Y + \omega(Y,\nabla_X Z)A^\sigma+\\
&+X(\sigma)\omega(Y,Z)A^\sigma + Z(\sigma)\omega(Y,X)A^\sigma + \omega(X,Z)\omega(Y,A^\sigma)A^\sigma)+\\
&-\omega(Y,X)Z(\sigma)A^\sigma - [X,Y](\sigma)Z - Z(\sigma)[X,Y] - \omega([X,Y],Z)A^\sigma =\\
&= R(X,Y)Z + Z(\sigma)(T(X,Y) + 2\omega(X,Y)A^\sigma)+\\
&+((\nabla_Y Z)(\sigma) + Y(\sigma)Z(\sigma) - YZ(\sigma))X - ((\nabla_X Z)(\sigma) + X(\sigma)Z(\sigma)+\\
&-XZ(\sigma))Y + (\omega(\nabla_X Y, Z) - \omega(\nabla_Y X, Z) - \omega([X,Y],Z)+\\
&-X(\sigma)\omega(Y,Z) + Y(\sigma)\omega(X,Z))A^\sigma+\\
&+\omega(Y,Z)\nabla_X A^\sigma - \omega(X,Z)\nabla_Y A^\sigma =\\
&= R(X,Y)Z + Z(\sigma)(T(X,Y) + 2\omega(X,Y)A^\sigma) + \omega(T(X,Y)Z)A^\sigma+\\
&+(Y(\sigma)Z(\sigma) - \omega(\nabla_Y A^\sigma, Z))X + (\omega(\nabla_X A^\sigma, Z) - X(\sigma)Z(\sigma))Y+\\
&+(Y(\sigma)\omega(X,Z) - X(\sigma)\omega(Y,Z))A^\sigma + \omega(Y,Z)\nabla_X A^\sigma+\\
&-\omega(X,Z)\nabla_Y A^\sigma. \qquad \square
\end{aligned}
$$

Proof of Lemma 3.8. Set:

$$\Psi(X,Y)Z := Y(\sigma)Z(\sigma)X - X(\sigma)Z(\sigma)Y,$$

and:

$$\Psi^{(0,2)}(J)(X,Y) :=$$
$$= \Psi(JX,JY) - J\Psi(JX,Y) - J\Psi(X,JY) - \Psi(X,Y),$$

we have immediately that $\Psi^{(0,2)} = 0$. □

Proof of Lemma 3.9. We have:

$$\Lambda^{(0,2)}(J)(X,Y)(Z) =$$
$$= \Lambda(JX,JY)Z - J\Lambda(JX,Y)Z - J\Lambda(X,JY)Z - \Lambda(X,Y)Z$$
$$= \omega(T(JX,JY),Z)A^\sigma - \omega(\nabla_{JY}A^\sigma,Z)JX + \omega(\nabla_{JX}A^\sigma,Z)JY +$$
$$+(JY(\sigma)\omega(JX,Z) - JX(\sigma)\omega(JY,Z))A^\sigma + \omega(JY,Z)\nabla_{JX}A^\sigma +$$
$$-\omega(JX,Z)\nabla_{JY}A^\sigma - (\omega(T(JX,Y),Z)JA^\sigma + \omega(\nabla_Y A^\sigma,Z)X +$$
$$+\omega(\nabla_{JX}A^\sigma,Z)JY + Y(\sigma)\omega(JX,Z)JA^\sigma - JX(\sigma)\omega(Y,Z)JA^\sigma +$$
$$+\omega(Y,Z)J\nabla_{JX}A^\sigma - \omega(JX,Z)J\nabla_Y A^\sigma + \omega(T(X,JY),Z)JA^\sigma +$$
$$-\omega(\nabla_{JY}A^\sigma,Z)JX - \omega(\nabla_X A^\sigma,Z)Y + JY(\sigma)\omega(X,Z)JA^\sigma +$$
$$-X(\sigma)\omega(JY,Z)JA^\sigma + \omega(JY,Z)J\nabla_X A^\sigma - \omega(X,Z)J\nabla_{JY}A^\sigma +$$
$$+\omega(T(X,Y),Z)A^\sigma - \omega(\nabla_Y A^\sigma,Z)X + \omega(\nabla_X A^\sigma,Z)Y +$$
$$+Y(\sigma)\omega(X,Z)A^\sigma - X(\sigma)\omega(Y,Z))A^\sigma + \omega(Y,Z)\nabla_X A^\sigma +$$
$$-\omega(X,Z)\nabla_Y A^\sigma =$$
$$= (\omega(T(JX,JY),Z) + JY(\sigma)\omega(JX,Z) - JX(\sigma)\omega(JY,Z) +$$
$$-\omega(T(X,Y),Z) + Y(\sigma)\omega(X,Z) - X(\sigma)\omega(Y,Z))A^\sigma +$$
$$-(\omega(T(JX,Y),Z) + Y(\sigma)\omega(JX,Z) - JX(\sigma)\omega(Y,Z) +$$
$$+\omega(T(X,JY),Z) + JY(\sigma)\omega(X,Z) - X(\sigma)\omega(JY,Z))JA^\sigma +$$
$$+\omega(JY,Z)(\nabla_{JX}A^\sigma - J\nabla_X A^\sigma) - \omega(JX,Z)(\nabla_{JY}A^\sigma - J\nabla_Y A^\sigma) +$$
$$-\omega(Y,Z)(J\nabla_{JX}A^\sigma + \nabla_X A^\sigma) + \omega(X,Z)(J\nabla_{JY}A^\sigma + \nabla_Y A^\sigma).$$

Moreover:

$$\Lambda^{(0,2)}\widehat{(J)(}X,Y) = \left[\Lambda^{(0,2)}(X,Y),J\right]\frac{\partial}{\partial J}$$
$$= (\Lambda^{(0,2)}(X,Y)J - J\Lambda^{(0,2)}(X,Y))\frac{\partial}{\partial J}$$
$$= ((\omega(T(JX,JY),J\cdot) + JY(\sigma)\omega(JX,J\cdot) - JX(\sigma)\omega(JY,J\cdot) +$$

$$-\omega(T(X,Y),J\cdot)-Y(\sigma)\omega(X,J\cdot)+X(\sigma)\omega(Y,J\cdot))A^\sigma+$$
$$-(\omega(T(JX,Y),J\cdot)+Y(\sigma)\omega(JX,J\cdot)-JX(\sigma)\omega(Y,J\cdot)+$$
$$+\omega(T(X,JY),J\cdot)+Y(\sigma)\omega(X,J\cdot)-X(\sigma)\omega(JY,J\cdot))JA^\sigma+$$
$$+\omega(JY,J\cdot)(\nabla_{JX}A^\sigma-J\nabla_XA^\sigma)-\omega(JX,J\cdot)(\nabla_{JY}A^\sigma-J\nabla_YA^\sigma)+$$
$$-\omega(Y,J\cdot)(J\nabla_{JX}A^\sigma+\nabla_XA^\sigma)+\omega(X,J\cdot)(J\nabla_{JY}A^\sigma+\nabla_YA^\sigma)+$$
$$-(\omega(T(JX,JY),\cdot)+JY(\sigma)\omega(JX,\cdot)-JX(\sigma)\omega(JY,\cdot)+$$
$$-\omega(T(X,Y),\cdot)-Y(\sigma)\omega(X,\cdot)+X(\sigma)\omega(Y,\cdot))JA^\sigma+$$
$$-(\omega(T(JX,Y),\cdot)+Y(\sigma)\omega(JX,\cdot)-JX(\sigma)\omega(Y,\cdot)+$$
$$+\omega(T(X,JY),\cdot)+Y(\sigma)\omega(X,\cdot)-X(\sigma)\omega(JY,\cdot))A^\sigma+$$
$$-\omega(JY,\cdot)(J\nabla_{JX}A^\sigma+\nabla_XA^\sigma)+\omega(JX,\cdot)(J\nabla_{JY}A^\sigma+\nabla_YA^\sigma)+$$
$$+\omega(Y,\cdot)(-\nabla_{JX}A^\sigma+J\nabla_XA^\sigma)-\omega(X,\cdot)(-\nabla_{JY}A^\sigma+J\nabla_YA^\sigma))\frac{\partial}{\partial J}$$
$$=(\omega(T^{(0,2)}(X,Y),J\cdot)A^\sigma-\omega(T^{(0,2)}(X,Y),\cdot)JA^\sigma)\frac{\partial}{\partial J}. \qquad \square$$

References

[1] D.V. Alekseevsky, V. Cortés, and C. Devchand, *Special complex manifolds*, math.DG/9910091.

[2] V. Cortés, *Special Kähler manifolds: a survey*, math.DG/0112114.

[3] V. Cortés, *Homogeneous special geometry*, Transform. Groups **1** (1996), no. 4, 337–373.

[4] P. de Bartolomeis and A. Nannicini, *Introduction to differential geometry of twistor spaces*, Geometric theory of singular phenomena in partial differential equations, Simp. Math., vol. XXXVIII, 1998, pp. 91–160.

[5] B. de Wit and A. Van Proyen, *Potentials and symmetries of general gauged N=2 supergravity-Yang-Mills models*, Nucl. Phys. **B245** (1984), 89–117.

[6] D.S. Freed, *Special Kähler manifolds*, Commun. Math. Phys. **203** (1999), no. 1, 31–52.

[7] H.C. Lee, *A kind of even-dimensional differential geometry and its application to esterior differential calculus*, Amer. J. Math. **65** (1943), 433–438.

[8] R. Miron, *Connexions compatibles aux structures conformes presques symplectiques*, C. R. Acad. Sci. Paris **265** (1967), 685–687.
[9] R. Miron, *Espaces à structure conforme presque symplectique*, Colloq. Math. (1972), 207–215.
[10] A. Nannicini, *Twistor bundles of almost symplectic manifolds*, Rend. Istit. Mat. Univ. Trieste **XXX** (1998), 91–106.
[11] V. Oproiu, *Some remarks on the conformal almost symplectic connections*, An. Sti. Univ. "Al. I. Cuza" Iasi Sect. I a Mat. **15** (1969), 151–157.
[12] I. Vaisman, *On locally conformal almost Kähler manifolds*, Israel J. of Math. **24** (1976), no. 3-4, 338–351.

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