TOPOLOGICAL DEGREE IN $\mathbf{R}^n$ (*)

by EDI ROSSET (in Trieste)(**)

SOMMARIO.- *Si fornisce una rappresentazione integrale del grado topologico in R^n che permette, in modo semplice e naturale, di costruire il grado e di derivarne le usuali proprietà, nonchè di estendere la nozione di numero di rotazione a mappe in spazi di Sobolev.*

SUMMARY.- *We give an integral representation of the topological degree in R^n. This approach allows to construct the degree itself and to derive its usual properties in a natural way, and also to extend the definition of winding number to maps in Sobolev spaces.*

0. Introduction.

In this paper we introduce the topological degree in R^n by means of an explicit integral formula which has two main advantages. The first one is to avoid the technical difficulties which usually arise in the case of singular values. In fact, while the definition which "counts" the preimages is available only for regular values, we have to deal with an integral which is defined for singular values as well as for regular ones, and which is continuous with respect to the value in R^n.

The second advantage is that this approach permits to give a reasonable definition of degree also for maps between manifolds, under suitable conditions which ensure the existence of a Green's function for the Laplacian in the target manifold.

In the third paragraph, since the degree of a map f in p may be regarded as the winding number around p of the restriction of f to the boundary, we extend the definition of winding number to $W^{1,n-1}$ maps.

1. Definition of the degree for C^2 maps.

Let Ω be a bounded open set in R^n, $n \geq 2$, such that its boundary $\partial\Omega$ is smooth. Given $p \in R^n$, let G_p be the fundamental solution of the Laplacian, singular in p, that is:

(*) Pervenuto in Redazione il 2 giugno 1989.

(**) Indirizzo dell'Autore: Dipartimento di Scienze Matematiche - Università degli Studi - 34100 Trieste (Italy).

$$G_p : \mathbf{R}^n \setminus \{p\} \to \mathbf{R}$$

$$G_p(x) = \begin{cases} \frac{1}{2\pi} \log |x-p| & n = 2 \\ \frac{1}{n(2-n)\omega_n} |x-p|^{2-n} & n \geq 3 \end{cases} \tag{1.1}$$

$G_p \in C^\infty(\mathbf{R}^n \setminus \{p\}, \mathbf{R})$ determines the following differential form of degree $n-1$ in $\mathbf{R}^n \setminus \{p\}$:

$$\omega_p = \sum_{i=1}^{n} (-1)^{i+1} \frac{\partial G_p}{\partial x^i} \, dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n \tag{1.2}$$

where the hat indicates cancellation. Since G_p is harmonic, ω_p is closed in $\mathbf{R}^n \setminus \{p\}$.

We denote with $C^k(\bar{\Omega},\mathbf{R}^n)$ the set of continuous functions in $\bar{\Omega}$, k times differentiable in Ω, with derivatives of order less or equal to k restrictions of continuous functions in $\bar{\Omega}$, endowed with the usual topology.

Let $f \in C^2(\bar{\Omega},\mathbf{R}^n)$, $p \notin f(\partial\Omega)$. We define the degree of f in p, denoted by $\deg(f,\Omega,p)$, as follows:

$$\deg(f,\Omega,p) = \int_{\partial\Omega} (f\,|_{\partial\Omega})^* \omega_p = \int_{\partial\Omega} i^* \circ f^* \omega_p \tag{1.3}$$

where i is the inclusion map $i : \partial\Omega \to \bar{\Omega}$, $*$ is the pull back of differential forms, the orientation of $\partial\Omega$ being that determined by the usual one in $\bar{\Omega}$.

REMARK 1.1 : The definition makes sense because $p \notin f(\partial\Omega)$ and it depends only on the values of f on the boundary.

Let p be a regular value for f. Then two cases occur:
a) $p \notin f(\Omega)$
b) $p \in f(\Omega)$.

In case a) Stokes' theorem gives:

$$0 = \int_\Omega f^*(d\omega_p) = \int_\Omega d(f^*\omega_p) = \int_{\partial\Omega} i^* \circ f^* \omega_p \tag{1.4}$$

so that $\deg(f,\Omega,p) = 0$ if $f(x) = p$ hasn't solution in Ω.

In case b), since p is regular, $f^{-1}(p) = \{x_1 \ldots. x_k\}$ and there exist $\varepsilon > 0$, $V_1 \ldots\ldots V_k$ open neighbourhoods of $x_1 \ldots\ldots x_k$ in Ω such that:

$$f\,|_{\overline{V}_i} : \overline{V}_i \to \overline{B(p,\varepsilon)} \qquad\qquad i = 1, \ldots\ldots. k$$

are diffeomorphisms. Applying Stokes' theorem to $\overline{\Omega} \setminus \bigcup_{i=1}^{k} V_i$, we obtain:

$$\int_{\partial\Omega} i^* \circ f^* \omega_p = \sum_{i=1}^{k} \int_{\partial V_i} i^* \circ f^* \omega_p = \sum_{i=1}^{k} \varepsilon_i \int_{S(p,\varepsilon)} \omega_p \tag{1.5}$$

where $\varepsilon_i = +1(-1)$ if $f|_{\partial V_i}$ preserves (reverses) the orientation (the orientations of ∂V_i and $S(p,\varepsilon)$ being those determined by the usual ones on V_i and $B(p,\varepsilon)$). It is easy to verify that a diffeomorphism between manifolds with boundary which preserves (reverses) the orientation, when restricted to the boundaries, also preserves (reverses) the induced orientation. Hence we have:

$$\varepsilon_i = \operatorname{sign} J_f(x_i) \qquad\qquad i = 1, \ldots\ldots k.$$

Let ν be the outward unit normal on $S(p,\varepsilon)$; then:

$$\int_{S(p,\varepsilon)} \omega_p = \int_{S(p,\varepsilon)} \langle \nabla G_p, \nu \rangle = 1 \ . \tag{1.6}$$

Therefore, if p is a regular value for f we have that:

$$\deg(f,\Omega,p) = \sum_{x_i \in f^{-1}(p)} \operatorname{sign} J_f(x_i) \in Z. \tag{1.7}$$

Let us recall:

SARD'S THEOREM: *Let Ω be a bounded open set in R^n, $f \in C^1(\Omega, R^n)$, $S = \{x \in \Omega : J_f(x) = 0\}$. Then $f(S)$ has measure zero.*

From Sard's theorem and (1.7) it follows that the *continuous* map

$$\deg(f,\Omega,\cdot) : R^n \setminus f(\partial\,\Omega) \to R$$

$$q \to \int_{\partial\Omega} i^* \circ f^* \omega_q$$

takes integer values in a dense set, so that it takes always values in $\boldsymbol{Z}$ and is therefore locally constant.

2. Extension to continuous maps and properties.

As an immediate consequence of Stokes' theorem, we have:

PROPOSITION 2.1: *Let* $f,g \in C^2(\bar{\Omega},\boldsymbol{R}^n)$ *and* $H \in C^2(\partial\Omega \times [0,1], \boldsymbol{R}^n)$ *s.t.* $H(\cdot,0) = f|_{\partial\Omega}$, $H(\cdot,1) = g|_{\partial\Omega}$, $p \notin H(\partial\Omega \times [0,1])$. *Then* $\deg(f,\Omega,p) = \deg(g,\Omega,p)$.

Given $f \in C^0(\bar{\Omega},\boldsymbol{R}^n)$, $p \notin f(\partial\Omega)$, $r = d(p,f(\partial\Omega))$, there exists $g_1 \in C^2(\bar{\Omega},\boldsymbol{R}^n)$ s.t. $\|g_1 - f\|_\infty < \frac{r}{2}$, so that $p \notin g_1(\partial\Omega)$. If g_2 is another map with the properties of g_1, the C^2 map $H(x,t) = tg_1(x) + (1-t)g_2(x)$, $x \in \bar{\Omega}$, $t \in [0,1]$, is an admissible homotopy between g_1 and g_2 in the sense that $p \notin H(\partial\Omega \times [0,1])$, and then from proposition 2.1 it follows that $\deg(g_1,\Omega,p) = \deg(g_2,\Omega,p)$.

We can now define, for $f \in C^0(\bar{\Omega},\boldsymbol{R}^n)$, $p \notin f(\partial\Omega)$,

$$\deg(f,\Omega,p) = \deg(g,\Omega,p)$$

with $g \in C^2(\bar{\Omega},\boldsymbol{R}^n)$ s.t. $\|g-f\|_{\infty,\partial\Omega} < \frac{r}{2}$, $r = d(p,f(\partial\Omega))$.

PROPOSITION 2.2: *Let* $f \in C^0(\bar{\Omega},\boldsymbol{R}^n)$, $p \notin f(\partial\Omega)$. *Then* $\deg(f,\Omega,p) = \deg(f-p,\Omega,0)$.

Proof: If $f \in C^2(\bar{\Omega},\boldsymbol{R}^n)$

$$\deg(f,\Omega,p) = \int_{\partial\Omega} i^* \circ f^* \omega_p = \int_{\partial\Omega} (\omega_p \circ f)(df, \ldots, df).$$

$$\deg(f-p,\Omega,0) = \int_{\partial\Omega} i^* \circ (f-p)^* \omega_0 = \int_{\partial\Omega} (\omega_0 \circ (f-p))(d(f-p), \ldots, d(f-p)) =$$

$$= \int_{\partial\Omega} (\omega_p \circ f)(df, \ldots, df)$$

since $G_p(x) = G_0(x-p)$ and then $\omega_p(x) = \omega_0(x-p)$.

In the general case, let $g \in C^2(\bar{\Omega},R^n)$ s.t. $\|g - f\|_{\infty,\partial\Omega} < \frac{d(p,f(\partial\Omega))}{2}$. Then $\deg(f,\Omega,p) = \deg(g,\Omega,p) = \deg(g - p,\Omega,0) = \deg(f - p,\Omega,0)$.

PROPOSITION 2.3: *Let $f \in C(\bar{\Omega},R^n)$, $p \notin f(\partial\Omega)$. Then there exists a neighbourhood V of f in $C(\bar{\Omega},R^n)$ s.t. for each $g \in V$ $p \notin g(\partial\Omega)$ and $\deg(f,\Omega,p) = \deg(g,\Omega,p)$.*

Proof: Let $V' = \{h \in C(\bar{\Omega},R^n) : \|h - f\|_{\infty,\partial\Omega} < \frac{r}{4}\} \supset V = \{h \in C(\bar{\Omega},R^n) : \|h - f\|_{\infty,\bar{\Omega}} < \frac{r}{4}\}$, where $r = d(p,f(\partial\Omega))$. If $h \in V'$ we have that $p \notin h(\partial\Omega)$ and $d(p,h(\partial\Omega)) = r' \geq \frac{3}{4}r$. Let $g \in C^2(\bar{\Omega},R^n)$ s.t. $\|g - f\|_{\infty,\partial\Omega} < \frac{r}{8}$. Then $\deg(f,\Omega,p) = \deg(g,\Omega,p)$ and $\|g - h\|_{\infty,\partial\Omega} < \frac{r'}{2}$ so that $\deg(h,\Omega,p) = \deg(g,\Omega,p)$.

REMARK 2.4: More generally the thesis is true for a neighbourhood of f with respect to the seminorm $\|\cdot\|_{\infty,\partial\Omega}$.

PROPOSITION 2.5: *Let $H \in C(\bar{\Omega} \times [0,1],R^n)$, $p \notin H(\partial\Omega \times [0,1])$. Then* $\deg(H(\cdot,t),\Omega,p)$ *is constant in t.*

Proof: $H(\cdot,t)$: $[0,1] \to C(\bar{\Omega},R^n)$ is continuous and by proposition 2.3 also the degree map is continuous with respect to the uniform norm. Hence $\deg H(\cdot,t) : [0,1] \to Z$ is continuous and therefore constant.

PROPOSITION 2.6: *The degree is constant on the connected components of $R^n \backslash f(\partial\Omega)$.*

Proof: Let $q \notin f(\partial\Omega)$, $r = d(q,f(\partial\Omega))$, $q' \in R^n$ s.t. $d(q',q) < \frac{r}{2}$. Then $q' \notin f(\partial\Omega)$ and, from propositions 2.2 and 2.3, $\deg(f,\Omega,q) = \deg(f,\Omega,q')$, so that $\deg(f,\Omega,\cdot)$ is locally constant and, since it takes integer values, is constant on the connected components of $R^n \backslash f(\partial\Omega)$.

PROPOSITION 2.7: *Let $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \emptyset$, Ω_1, Ω_2 open sets in R^n, $f \in C(\bar{\Omega},R^n)$, $p \notin f(\partial\Omega_1) \cup f(\partial\Omega_2)$. Then* $\deg(f,\Omega,p) = \deg(f,\Omega_1,p) + \deg(f,\Omega_2,p)$.

Proof: Since R^n is locally connected, the equality $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ holds and we can reduce to the case $f \in C^2(\bar{\Omega},R^n)$. The thesis follows from the additivity of the integral.

COROLLARY 2.8: *Let id:* $\bar{\Omega} \to \mathbf{R}^n$ *the identity map and* $p \notin \partial\Omega$. *Then*

$$\deg(id,\Omega,p) = \begin{cases} 1 & p \in \Omega \\ 0 & p \notin \Omega . \end{cases} \tag{2.1}$$

Proof: $\deg(\mathrm{id},\Omega,p) = \int_{\partial\Omega} \omega_p$.

If $p \notin \Omega$, Stokes'theorem gives

$$\int_{\partial\Omega} \omega_p = \int_{\Omega} d\omega_p = 0 .$$

If $p \in \Omega$ we can apply Stokes' theorem to $\bar{\Omega}\backslash B(p,\varepsilon)$, for convenient $\varepsilon > 0$, and obtain

$$\int_{\partial\Omega} \omega_p = \int_{S(p,\varepsilon)} \omega_p = 1 .$$

COROLLARY 2.9: *If* $p \notin f(\bar{\Omega})$, *then* $\deg(f,\Omega,p) = 0$.

Proof: If $f \in C^2(\bar{\Omega},\mathbf{R}^n)$, it is an immediate consequence of Stokes' theorem; in the general case let $r = d(p,f(\bar{\Omega})) > 0$ and $g \in C^2(\bar{\Omega},\mathbf{R}^n)$ s.t. $\|g - f\|_{\infty,\bar{\Omega}} < \frac{r}{2}$. Then $\deg(f,\Omega,p) = \deg(g,\Omega,p) = 0$.

COROLLARY 2.10: $\deg(f,\Omega,p) \neq 0 \Rightarrow \exists\, x_0 \in \Omega$ s.t. $f(x_0) = p$.

COROLLARY 2.11: *If* $\deg(f,\Omega,p) \neq 0$, *then* $f(\Omega)$ *is a neighbourhood of* p.

COROLLARY 2.12: *If* $f(\Omega)$ *is contained in a proper subspace of* $\mathbf{R}^n$, *then* $\deg(f,\Omega,p) = 0$, $\forall p \in \mathbf{R}^n \backslash f(\partial\Omega)$.

PROPOSITION 2.13: *Let* K *be a closed subset of* $\bar{\Omega}$, $p \notin f(K), p \notin f(\partial\Omega)$. *Then*

$$\deg(f,\Omega,p) = \deg(f,\Omega\backslash K,p). \tag{2.2}$$

Proof: As for proposition 2.7, since $\partial(\Omega\backslash K) = \partial\Omega \cup \partial K$.

PROPOSITION 2.14: *Let* $\{\Omega_i\}_{i\in I}$ *be a collection of pairwise disjoint open subsets of* Ω, $p \in \mathbf{R}^n$ *s.t.* $f^{-1}(p) \subset \bigcup_{i\in I}\Omega_i$. *Then*

$\deg(f,\Omega_i,p) = 0$ *except for finitely many indices*

$$\deg(f,\Omega,p) = \sum_{i\in I} \deg(f,\Omega_i,p). \tag{2.3}$$

Proof: It follows from proposition 2.13 and from the compactness of $f^{-1}(p)$.

PROPOSITION 2.15: *Let* $f,g \in C(\bar{\Omega},R^n)$, $H \in C(\partial\Omega \times [0,1], R^n)$ *s.t.* $p \notin H(\partial\Omega \times [0,1])$, $H(\cdot,0) = f|_{\partial\Omega}$ *and* $H(\cdot,1) = g|_{\partial\Omega}$. *Then* $\deg(f,\Omega,p) = \deg(g,\Omega,p)$.

Proof: As for proposition 2.5, using remark 2.4.

3. Winding number.

As we have seen, $\deg(f,\Omega,p)$ depends on the homotopy class of $f\circ i : \partial\Omega \to R^n$ only. More precisely, when $f \in C^2(\bar{\Omega},R^n)$, $p \notin f(\partial\Omega)$, $\deg(f,\Omega,p)$ may be interpreted as the winding number of $f\circ i$ around p (see [1], [2]), that is the degree of the map

$$u : \partial\Omega \to S^{n-1}$$

$$u(x) = \frac{f(x)-p}{|f(x)-p|}$$

where $\deg(u) = \int_{\partial\Omega} u^* \varpi$, ϖ a nowhere vanishing differential form of degree $n-1$ in S^{n-1} such that $\int_{S^{n-1}} \varpi = 1$. In fact, we can consider the following factorization

$$\begin{array}{ccccccccc} \partial\Omega & \overset{f\circ i}{\to} & R^n\backslash\{p\} & \overset{T-p}{\to} & R^n\backslash\{0\} & \overset{\pi}{\to} & S^{n-1} & \overset{i}{\to} & R^n\backslash\{0\} \\ i^*\circ f^*\omega_p & & (T-p)^*\omega_0 = \omega_p & & (i\circ\pi)^*\omega_0 = \omega_0 & & i^*\omega_0 & & \omega_0. \end{array}$$

It is easy to verify that $(i\circ\pi^*)\omega_0 = \omega_0$ and that $i^*\omega_0$ is a nowhere vanishing differential form of degree $n-1$ on S^{n-1} s.t. $\int_{S^{n-1}} i^*\omega_0 = 1$. Then

$$\deg(u) = \int_{\partial\Omega} u^*(i^*\omega_0)$$

and $u^*(i^*\omega_0) = (f \circ i)^*(T-p)^*(i \circ \pi)^*\omega_0 = (f \circ i)^*\omega_p = i^* \circ f^*\omega_p$ so that $\deg(u) = \deg(f,\Omega,p)$. As we shall see it makes sense to consider $\int_{\partial\Omega}\hat{f}^*\omega_p$ for a map $\hat{f} \in W^{1,n-1}(\partial\Omega, R^n \setminus V)$, V open neighbourhood of p and this integral is continuous with respect to the $W^{1,n-1}$ topology.

PROPOSITION 3.1: *Let $\hat{f} \in W^{1,n-1}(\partial\Omega, R^n \setminus V)$, V open neighbourhood of p. Then $\hat{f}^*\omega_p$ is integrable on $\partial\Omega$* and, if $\hat{f}_k \in W^{1,n-1}(\partial\Omega, R^n \setminus V)$ *is a sequence converging to $\hat{f}$ in the $W^{1,n-1}$ topology, then $\int_{\partial\Omega}\hat{f}_k^*\omega_p \to \int_{\partial\Omega}\hat{f}^*\omega_p$.*

Proof: Since $\partial\Omega$ is compact, it is sufficient to verify the local integrability of $\hat{f}^*\omega_p$, where

$$\hat{f}^*\omega_p = \sum_{i=1}^{n}(-1)^{i+1}\frac{\partial G_p}{\partial x^i} \circ \hat{f} dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n (d\hat{f}, \ldots . d\hat{f}) .$$

For each $p \in \partial\Omega$, there exists an open neighbourhood U of p, where is defined an orthonormal basis of tangent vectors $e_i = e_i(q)$, $q \in U$, $i = 1 \ldots . n-1$. We will indicate its dual basis of differential forms of degree 1 with $\omega_i = \omega_i(q)$, $q \in U$, $i = 1 \ldots . n-1$. If we write

$$d\hat{f}(e_i) = \sum_{\alpha=1}^{n} a_i^\alpha \frac{\partial}{\partial x^\alpha}$$

$$a_i^\alpha = a_i^\alpha(q) \quad q \in U$$

an easy calculation shows that in U

$$\hat{f}^*\omega_p = \sum_{i=1}^{n}(-1)^{i+1}\frac{\partial G_p}{\partial x^i} \circ \hat{f} \ \det(A_i)\omega_1 \wedge \ldots \wedge \omega_{n-1}$$

where A_j is the minor obtained by removing the j^{th} column of the matrix A:

$$A = (a_i^\alpha) = \begin{bmatrix} a_1^1 & \ldots\ldots\ldots\ldots & a_1^n \\ & & \\ a_{n-1}^1 & \ldots\ldots & a_{n-1}^n \end{bmatrix} .$$

By definition of $|df|^2$, we have

$$|df|^2 = <df,df> = \sum_{i=1}^{n-1} \sum_{\alpha=1}^{n} (a_i^\alpha)^2$$

$$|df|^{n-1} = \left[\sum_{i=1}^{n-1} \sum_{\alpha=1}^{n} (a_i^\alpha)^2 \right]^{\frac{n-1}{2}} .$$

Since $\hat{f} \in W^{1,n-1}(\partial\Omega)$, $a_i^\alpha \in L^{n-1}(U)$ and hence $\det(A_i) \in L^1(U), i = 1 \dots n$. Since $\hat{f}(\partial\Omega)$ is far from p, $\frac{\partial G_p}{\partial x^i} \circ \hat{f}$ is bounded and so $\hat{f}^* \omega_p$ is integrable in U.

Let $\hat{f}_k \in W^{1,\, n-1}(\partial\Omega, R^n \backslash V)$ be a sequence converging to $\hat{f}$ in the $W^{1,\, n-1}$ topology; then locally we have that $a_i^\alpha(k) \overset{L^{n-1}}{\rightarrow} a_i^\alpha$, $\det(A_i(k)) \overset{L^1}{\rightarrow} \det(A_i)$, while $\frac{\partial G_p}{\partial x^i} \circ \hat{f}_k$, $\frac{\partial G_p}{\partial x^i} \circ \hat{f}$ are uniformly bounded in $L^\infty(U)$ and therefore, by passing to a subsequence, $\int_{\partial\Omega} \hat{f}_k^* \omega_p \rightarrow \int_{\partial\Omega} \hat{f}^* \omega_p$.

DEFINITION 3.2: Let $\hat{f} \in W^{1,\, n-1}(\partial\Omega, R^n \backslash V)$, V open neighbourhood of p. The winding number of $\hat{f}$ around p, denoted by $W(\hat{f}, p)$, is defined as follows:

$$W(\hat{f}, p) = \int_{\partial\Omega} \hat{f}^* \omega_p .$$

In order to justify this definition we will prove the following:

PROPOSITION 3.3: *Let $\hat{f} \in W^{1,\, n-1}(\partial\Omega, R^n \backslash V)$, V open neighbourhood of p. Then $W(\hat{f}, p) = \int_{\partial\Omega} \hat{f}^* \omega_p$ is an integer number and if $\hat{f} \in C^0 \cap W^{1,\, n-1}$ $(\partial\Omega, R^n \backslash \{p\})$ then*

$$W(\hat{f}, p) = \deg(f, \Omega, p)$$

for any continuous extension of $\hat{f}$ to $\bar{\Omega}$.

Proof: If $\hat{f} \in W^{1,n-1}(\partial\Omega, R^n \setminus V)$, there exist $\hat{f}_k \in C^\infty(\partial\Omega, R^n \setminus V)$ s.t. $\hat{f}_k \xrightarrow{W^{1,n-1}} \hat{f}$. It is well known that $\hat{f}_k$ may be extended to some $f_k \in C^\infty(R^n, R^n)$ and so $\int_{\partial\Omega} \hat{f}^* \omega_p = \lim_{k\to\infty} \int_{\partial\Omega} \hat{f}_k^* \omega_p = \deg(f_k, \Omega, p) \in Z$ so that $W(\hat{f}, p)$ is an integer.

Let $\hat{f} \in C^0 \cap W^{1,n-1}(\partial\Omega, R^n \setminus \{p\})$. We claim that there exists a sequence $\varphi_k \in C^\infty(\partial\Omega, R^n)$ converging to $\hat{f}$ both in uniform and in $W^{1,n-1}$ norm. In fact for each $p \in \partial\Omega$ there exists a local coordinate system (U_p, φ_p), U_p open, φ_p diffeomorphism on $\varphi_p(U_p) \subseteq R^{n-1}$, $U_p \supset \overline{V}_p \supset V_p$, V_p open neighbourhood of p.

Let us consider the map $\hat{f} \circ \varphi^{-1} : \varphi(U) \subseteq R^{n-1} \to R^n$, where $\varphi(U) \supset \overline{\varphi(V)} = \varphi(\overline{V}) \supset \varphi(V)$. Let ρ_k be a regularizing sequence in R^{n-1}; since $\varphi(V) \subset\subset \varphi(U)$, for k large enough $g_k = \rho_k * (\hat{f} \circ \varphi^{-1}) \in C^\infty(\varphi(V), R^n)$ and g_k converges to $\hat{f} \circ \varphi^{-1}$ in $\varphi(V)$ both in uniform and in $W^{1,n-1}$ norm. Then $\varphi_k = g_k \circ \varphi \in C^\infty(V, R^n)$ converges to $\hat{f}$ both in uniform and in $W^{1,n-1}$ norm. There exists a C^∞ partition od unity $\{h_j\}$ subordinate to the open cover $\{V_p\}$; from the compactness of $\partial\Omega$, it follows that $\{h_j\}$ is a finite collection $\{h_j\}_{j=1,\dots m}$ and for each $j = 1, \dots m$ there exists an index $i(j)$ s.t. $\text{supp}(h_j) \subset V_{i(j)}$. Let W_j denote $V_{i(j)}$; for each j, we have a sequence $\varphi_k^j \in C^\infty(W_j, R^n)$ converging to $\hat{f}$ in W_j both in uniform and in $W^{1,n-1}$ norm. Let $\varphi_k : \partial\Omega \to R^n$, $\varphi_k(x) = \sum_{j=1}^m h_j(x)\varphi_k^j(x)$. Then $\varphi_k \in C^\infty(\partial\Omega, R^n)$ converges to $\hat{f}$ both in uniform and in $W^{1,n-1}$ norm.

Let f_k be a smooth extension of φ_k to all of R^n. Then, if f is a continuous extension of $\hat{f}$, for k large enough:

$$\deg(f, \Omega, p) = \deg(f_k, \Omega, p) = \int_{\partial\Omega} i^* \circ f_k^* \omega_p = \int_{\partial\Omega} \varphi_k^* \omega_p \to \int_{\partial\Omega} \hat{f}^* \omega_p = W(\hat{f}, p).$$

REMARK 3.4: It is easy to verify that if $f \in C^1(\overline{\Omega}, R^n)$, $p \notin f(\partial\Omega)$, p regular value, then

$$\deg(f, \Omega, p) = \int_{\partial\Omega} i^* \circ f^* \omega_p = \sum_{x_i \in f^{-1}(p)} \text{sign}\, J_f(x_i)$$

that is the usual definition of topological degree.

REMARK 3.5: In defining the winding number we have avoid the explicit use of the projection $\pi : R^n \setminus \{0\} \to S^{n-1}$, introducing the differential form ω_p associated with the fundamental solution of the Laplacian. This approach is employable also in spaces where the projection is not available. More precisely, given an n-dimensional oriented compact Riemannian manifold with boundary N and $p \in N \setminus \partial N$, there exists in $N \setminus \{p\}$ the Green's function relative to the Laplacian and to the boundary ∂N, singular in p, which we indicate with G_p. Then $\omega_p = *dG_p$ defines a closed differential form of degree $n-1$ in $N \setminus \{p\}$, where $*$ denotes the Hodge's $*$-operator. If M is an n-dimensional manifold with boundary, the same arguments we have seen above allow to define the degree in p of a map $f \in C^2(M, N \setminus \{p\})$ or equivalently the winding number around p of its restriction to ∂M and again it is possible to extend the definition of winding number to C^0 and $W^{1,n-1}$ maps.

REFERENCES

[1] B.A. DUBROVIN, A.T. FOMENKO, S.P. NOVIKOV, *Modern geometry-methods and applications*, New York, Springer Verlag, 1984.

[2] V. GUILLEMIN, A. POLLACK, *Differential topology*, Englewood Cliffs, Prentice Hall, 1974.

[3] P.H. RABINOWITZ, *Théorie du degré topologique et applications a des problèmes aux limites non linéaires*, rédigé par H. Berestycki (1975).