UPPER BOUNDS FOR ORTHOGONAL INVARIANTS OF SOME POSITIVE LINEAR OPERATORS (*)

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SOMMARIO. - *Viene indicato un metodo per l'approssimazione per eccesso degli invarianti ortogonali di taluni operatori lineari positivi. Fra essi rientrano gli operatori differenziali ellittici autoaggiunti con, ad esempio, condizioni di Dirichlet al contorno.*

SUMMARY. - *A method is given for the upper approximation of orthogonal invariants of certain positive linear operators. These operators include elliptic differential self-adjoint operators with, for instance, Dirichlet boundary conditions.*

Let S be a complex separable Hilbert space where the scalar product is denoted by $(.\,,.)$. Let L be a linear operator with domain the linear variety $\mathcal{D}_L$ of S. Let V be a linear sub-variety of L.

The following hypothesis be satisfied:

I) *There exists a strictly positive compact operator* G *of the space* S *such that*: i) *the range* $G(S)$ *of* G *is contained in* V; ii) $GL = LG = I$ ($I =$ identity operator).

The following eigenvalue problem

$$Lv - \lambda v = 0 \qquad v \in V \tag{1}$$

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is equivalent to the following one

$$Gu - \mu u = 0 \qquad u \in S,$$

where $\mu = \lambda^{-1}$. It follows that all the eigenvalues of (1) constitute a non-decreasing sequence tending to $+\infty$

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots .$$

If G has some power with a finite Hilbert-Schmidt trace, then it has been shown that it is possible to construct an increasing sequence converging to λ_k, provided at least one orthogonal invariant $\mathcal{J}_s^{\nu}(G)$ of G is known [1].

For the actual computation of $\mathcal{J}_s^{\nu}(G)$ it suffices to know a decreasing sequence of PCO's (Positive Compact Operators) uniformly converging to G. In [2], [3] it has been shown how to construct this sequence when L is a linear elliptic self-adjoint differential operator and V a submanifold of the function space of admissible functions defined by some boundary conditions, i.e. Dirichlet boundary conditions. However this construction requires the explicit knowledge of a fundamental solution operator, which practically restricts the method to operators with constant coefficients. We want now to show how the variable coefficients case can be handled and practically reduced to the constant coefficient case, by using a procedure due to N. Aronszajn [1] for the construction of *intermediate operators*. Anyway, differently from the method of intermediate problems, we shall not need the explicit knowledge of a *base operator*, i.e. the explicit spectral resolution of an operator L_0 such that $L_0 < L$ in V.

Let us assume that

II) *There exists an operator L_0 defined on V and a (strictly) PCO Γ defined on S, satisfying with respect to L_0 the same hypothesis* I) *satisfied by G with respect to L.*

III) *For any non zero $v \in V$*

$$(L_0 v, v) < (Lv, v). \tag{2}$$

We suppose that Γ is such that

(1) See [2], [3].

IV) *For any* $u, w \in S$ *we are able to compute* $(\Gamma^\nu u, w)$ (ν *positive integer*) *to any prescribed degree of accuracy. Moreover some orthogonal invariant of* Γ — *say* $\mathcal{J}_s^\nu(\Gamma)$ — *is known.* (2)

Let us consider as an example the case of a linear elliptic formally self-adjont operator. We restrict ourselves to the 2nd order case. Anyway the extension to the higher order case is rather simple.

Let A be a properly regular bounded domain (see [3], p. 21) of the cartesian space X^r. Let S be the Hilbert space of complex valued, square-integrable functions on A, with the usual scalar product

$$(u, v) = \int_A u\, v\, dx \qquad (dx = dx_1 \dots dx_r).$$

We may restrict ourselves to consider only real valued functions of S.

Let $\mathcal{D}_L$ be the space H_2 (H_m = space of functions with square-integrable derivatives, in the sense of Friedrichs and Sobolev, of order $\leq m$). Let V be the subspace of all the functions of H_2 vanishing on the boundary ∂A of A.

Set

$$Lv = -\sum_{i,j}^{1,r} \frac{\partial}{\partial x_i}\left[a_{ij}(x)\frac{\partial v}{\partial x_j}\right] + c(x)\, v,$$

where the $a_{ij}(x)$ are continuously differentiable real valued functions in $\overline{A}$, such that $a_{ij}(x) = a_{ji}(x)$ and being the real quadratic form $\sum_{ij}^{1,r} a_{ij}(x)\, \xi_i\, \xi_j$ positive definite for any $x \in \overline{A}$. The function $c(x)$ is supposed continuous and non-negative in $\overline{A}$.

It is well known that hypothesis I) is satisfied and G is the *Green operator* for L with the Dirichlet boundary condition.

Let p_0 be a positive constant such that for any $x \in \overline{A}$ we have for any non-zero real r-vector $\xi = (\xi_1 \dots \xi_r)$

$$\sum_{i,j}^{1,r} a_{ij}(x)\, \xi_i\, \xi_j > p_0 \sum_{i=1}^{r} \xi_i^2 .$$

We have no loss if ve assume $p_0 = 1$.

(2) The statement « the orthogonal invariant $\mathcal{J}_s^\nu(\Gamma)$ is known » must be understood in the sense that we are able to construct a non-increasing sequence of positive numbers converging to $\mathcal{J}_s^\nu(\Gamma)$.

Let us assume $L_0 = -\Delta_2 \equiv -\sum_{i=1}^{r} \frac{\partial^2}{\partial x_i^2}$. Hypotheses II) and III) are then satisfied. We need only to prove (2). For any $v \in V$ we have

$$(Lv, v) = \sum_{i,j}^{1,r} \int_A a_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \int_A c(x) v^2 dx \geq \sum_{i=1}^{r} \int_A \left(\frac{\partial v}{\partial x_i}\right)^2 dx = (L_0 v, v).$$

By using results of [2] and [3] we can construct explicitly the Green operator Γ of L_0. To this end we introduce the operators R^* and R

$$R^* u = \sum_{i=1}^{r} \int_A \frac{\partial u}{\partial y_i} \frac{\partial}{\partial y_i} s(x, y)\, dy, \quad Ru = \int_A u(y)\, s(x, y)\, dy,$$

where

$$s(x, y) \begin{cases} = \dfrac{1}{2\pi} \log |x - y| & \text{for } r = 2 \\[2ex] = -\dfrac{1}{(r-2)\,\omega_r} |x - y|^{2-r} & \text{for } r > 2 \end{cases}$$

($\omega_r =$ measure of the unit sphere $|x| = 1$).

The operator R^* maps the space H_1 into the space S, while the operator R maps S into H_1. Let Ω be the subspace of H_1 of all the harmonic functions belonging to H_1. Let P be the projector of H_1 onto Ω. Then [see [3], p. 169] we have

$$\Gamma u = R^* Ru - R^* PRu.$$

It follows

$$\| \Gamma u \| = \| R^* (I - P) Ru \| \leq \| R^* \| \| R \| \| u \|.$$

It is very easy to give an explicit upper bound M to $\| R^* \| \| R \|$.

Let now $\{z_s\}$ be a system of functions belonging to the range of Γ^ν and such that $\{\Delta_2^\nu z_s\}$ be complete in S. For any given $\varepsilon > 0$ and for $w \neq 0$, let us determine the constants $c_s^{(\varepsilon)}$ $(s = 1, \ldots, q_\varepsilon)$ such that

$$\left\| u - \sum_{s=1}^{q_\varepsilon} c_s^{(\varepsilon)} \Delta_2^\nu z_s \right\| < \frac{\varepsilon}{M^\nu \| w \|}.$$

Set $\zeta_\varepsilon = \sum_{s=1}^{q_\varepsilon} c_s^{(\varepsilon)} z_s$. We have

$$(\Gamma^\nu u, w) = (\zeta_\varepsilon, w) + (\Gamma^\nu (u - \Delta_2^\nu \zeta_\varepsilon), w)$$

and

$$|(\Gamma^\nu (u - \Delta_2^\nu \zeta_s), w)| \leq M^\nu \| u - \Delta_2^\nu \zeta_\varepsilon \| \| w \| < \varepsilon.$$

Hypothesis IV) is then satisfied.

Let us now set $L_1 = L - L_0$ and introduce, following Aronszajn [1], this new scalar product in V

$$((v, w)) = (L_1 v, w).$$

Let $\alpha_1, \dots, \alpha_n$ be linearly independent vectors of V. Consider the Gramian matrix

$$\{a_{ij}\} = \{(L_1 \alpha_i, \alpha_j)\} \qquad i, j = 1, \dots, n$$

and be $\{b_{ij}\}$ the inverse matrix of $\{a_{ij}\}$. Set

$$P_n u = \sum_{i,j}^{1,n} b_{ij} ((u, \alpha_i)) \alpha_j$$

and

$$L_n = L_0 + L_1 P_n.$$

The operator L_n is symmetric in V in the sense that $(L_n v, w) = (v, L_n w)$, for any pair v, w of elements of V, as it is readily seen.

We have for $v \in V$

$$(L_n v, v) = (L_0 v, v) + ((P_n v, P_n v)).$$

If follows that the problem

$$L_n v = u \qquad v \in V \tag{3}$$

has at most, for any given $u \in S$, one solution. Let us prove that solution of problem (3) exists. We can write equation (3) as follows

$$L_0 v + \sum_{i,j}^{1,n} b_{ij} (v, L_1 \alpha_i) L_1 \alpha_j = u \qquad v \in V$$

which is equivalent to the following problem

$$v = \Gamma u - \sum_{i,j}^{1,n} b_{ij} \, c_i \, \varphi_j , \tag{4}$$

$$c_k = (\Gamma u, L_1 \alpha_k) - \sum_{i,j}^{1,n} b_{ij} \, c_i \, (\varphi_j , L_1 \alpha_k) \qquad (k = 1, \dots , n) \tag{5}$$

where

$$\varphi_j = \Gamma L \alpha_j - \alpha_j .$$

If we set $\gamma_j = \sum_{i=1}^{n} b_{ij} \, c_i$, we see that the algebraic system (5) is equivalent to the following one

$$\sum_{j=1}^{n} (\Gamma L \alpha_j , L_1 \alpha_k) \, \gamma_j = (u, \varphi_k) .$$

The $n \times n$ matrix

$$\{(\Gamma L \alpha_j , L_1 \alpha_k)\} \tag{6}$$

is non-singular, otherwise problem (3) should have more than one solution.

Let $\{\sigma_{ij}\}$ be the inverse matrix of matrix (6). Problem (3) has the following solution

$$v = \Gamma u - \sum_{j,k}^{1,n} \sigma_{jk} \, (u, \varphi_k) \, \varphi_j \equiv \Gamma_n \, u .$$

We have so constructed the Green operator Γ_n of problem (3).

If we consider the eigenvalue problem

$$L_n \, v - \lambda v = 0 \qquad v \in V ,$$

it admits a non decreasing sequence of positive eigenvalues

$$\lambda_{n,1} \leq \lambda_{n,2} \leq \dots \leq \lambda_{n,k} \leq \dots ,$$

satisfying, for any k, the following inequalities

$$\lambda_{0,k} \leq \lambda_{n,k} \leq \lambda_{n+1,k} \leq \lambda_k ,$$

where $\lambda_{0,k}$ is the k-th eigenvalue of the problem

$$L_0 \, v - \lambda v = 0 \qquad v \in V .$$

It follows (see [3] p. 145) that

$$\mathcal{J}_s^{\nu}(\Gamma) \geq \mathcal{J}_s^{\nu}(\Gamma_n) \geq \mathcal{J}_s^{\nu}(\Gamma_{n+1}) \geq \mathcal{J}_s^{\nu}(G).$$

Since an upper bound $\mathcal{J}_s^{\nu}(\Gamma_n)$ for $\mathcal{J}_s^{\nu}(G)$ is known, by applying the theory expounded is [2] and [3] we may construct lower bounds for any eigenvalue λ_k of problem (1).

Let us write here the explicit formulas for the orthogonal invariants $\mathcal{J}_1^1(\Gamma_n)$, $\mathcal{J}_1^2(\Gamma_n)$, $\mathcal{J}_1^3(\Gamma_n)$. To this end we must use formula (4.2) of [2]. We obtain

$$\mathcal{J}_1^1(\Gamma_n) = \mathcal{J}_1^1(\Gamma) - \sum_{j,k}^{1,n} \sigma_{jk}(\varphi_j, \varphi_k)$$

$$\mathcal{J}_1^2(\Gamma_n) = \mathcal{J}_1^2(\Gamma) - 2\sum_{j,k}^{1,n} \sigma_{jk}(\varphi_j, \Gamma\varphi_k) + \sum_{i,k}^{1,n}\sum_{j,h}^{1,n} \sigma_{ik}\sigma_{jh}(\varphi_k, \varphi_j)(\varphi_h, \varphi_i)$$

$$\mathcal{J}_1^3(\Gamma_n) = \mathcal{J}_1^3(\Gamma) - 3\sum_{i,k}^{1,n} \sigma_{ik}(\varphi_i, \Gamma^2\varphi_k) + 3\sum_{i,k,r,s}^{1,n} \sigma_{ik}\sigma_{rs}(\varphi_k, \varphi_r)(\varphi_s, \Gamma\varphi_i)$$
$$- \sum_{i,k,j,h,r,s}^{1,n} \sigma_{ik}\sigma_{jh}\sigma_{rs}(\varphi_k, \varphi_j)(\varphi_h, \varphi_r)(\varphi_s, \varphi_i).$$

Hypothesis IV) insures that the scalar products (φ_i, φ_k), $(\varphi_i, \Gamma\varphi_k)$, $(\varphi_i, \Gamma^2\varphi_k)$, ... can be computed to any prescribed degree of accuracy.

The same can be said for the entries σ_{ik} of the inverse matrix of the $n \times n$ matrix (6). In fact, according to hypothesis IV), we can write any entry of matrix (6) as follows

$$q_{ij} + \varepsilon_{ij}$$

where ε_{ij} is an error term as small as we wish. Denote by Q the matrix $\{q_{ij}\}$ and by H the matrix $\{\varepsilon_{ij}\}$. We may suppose that Q is non-singular and that the Frobenius modulus $|H|$ of H is less than $(2|Q^{-1}|)^{-1}$.

Then we have

$$\{\sigma_{ik}\} \equiv (Q + H)^{-1} = \sum_{k=0}^{\infty} (-1)^k (Q^{-1}H)^k Q^{-1}.$$

Let $\tilde{\sigma}_{ik}$ be the entries of Q^{-1}. We have

$$|\tilde{\sigma}_{ik} - \sigma_{ik}| < 2|Q^{-1}|^2 |H|.$$

Since $|H|$ can be made arbitrarily small (hypothesis IV)), it follows that we can compute the σ_{ih} to any prescribed degree of accuracy.

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