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Almost PSH Functions on Calabi's Bundles

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A la mémoire de notre Professeur et Ami Thierry Aubin

Abstract. *We give an explicit lower bound for almost psh functions on some Fano manifolds. These manifolds generalize those introduced by Calabi in [5], and also proivde a generalization of the concept of the blowing-up of $\mathbb{P}_m\mathbb{C}$ at one point. To this end, we use a method introduced in [4], which consists of studying the behavior of psh functions along some well-chosen holomorphic curves.*

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1. Introduction and Statement of Results

1.1. The Manifold M Bundled in $\mathbb{P}_n\mathbb{C}$.

Let $\mathbb{P}_k\mathbb{C}$ be the complex projective space of complex dimension k, and let $[z_0, z_1, \ldots, z_k]$ denote the homogeneous coordinates in $\mathbb{P}_k\mathbb{C}$. We define M as the sub-manifold of $\mathbb{P}_{m-1}\mathbb{C} \times \mathbb{P}_{nm}\mathbb{C}$, where $m > 1$ and $n > 0$, consisting of the points

$$([Z], [z_m, z_{m+1}Z^a, \ldots, z_{m+n}Z^a]) \in \mathbb{P}_{m-1}\mathbb{C} \times \mathbb{P}_{nm}\mathbb{C},$$

where a is a positive integer, $Z = [z_0, z_1, \ldots, z_{m-1}] \in \mathbb{P}_{m-1}\mathbb{C}$, $[z_m, z_{m+1}, \ldots, z_{m+n}] \in \mathbb{P}_n\mathbb{C}$ and $Z^a = [z_0^a, z_1^a, \ldots, z_{m-1}^a]$. Note that $dim(M) = m + n - 1$, and that, in the above description, the point $[z_m, z_{m+1}, \ldots, z_{m+n}]$ of $\mathbb{P}_n\mathbb{C}$ depends on the choice of the coordinates $(z_0, z_1, \ldots, z_{m-1})$ of the basis point $[Z]$. An equivalent description is the following:

$$\begin{aligned} M \;=\; \Big\{ & ([z_0, z_1, \ldots, z_{m-1}], [z_m; z_{m+1}, \ldots, z_{2m}; \ldots; \\ & z_{nm+1}, \ldots, z_{(n+1)m}]) \in \mathbb{P}_{m-1}\mathbb{C} \times \mathbb{P}_{nm}\mathbb{C} \text{ s.t. } \forall p \in \{1, \ldots n\}, \\ & (z_{pm+1}, \ldots, z_{(p+1)m}) \text{ and } (z_0^a, z_1^a, \ldots, z_{m-1}^a) \text{ are } \mathbb{C}\text{-parallel} \Big\} \end{aligned}$$

We introduce two other coordinate systems, which will be more convenient for our later computations. We use the first, which we denote by S, when all components are not zero; in this case, the choice of homogeneous coordinates in the basis is immaterial, and S is given by

$$\begin{aligned} & ([z_1, \ldots, z_m], [1; z_1^a, \ldots, z_m^a; z_{m+1}(z_1^a, \ldots, z_m^a); \ldots; \\ & \qquad z_{m+n-1}(z_1^a, \ldots, z_m^a)]) \in \mathbb{P}_{m-1}\mathbb{C} \times \mathbb{P}_{nm}\mathbb{C}. \end{aligned}$$

The second coordinate system, which we denote S', is given, in the local chart $\{z_0 \neq 0,\, z_m \neq 0\}$, when we use the description

$$\begin{aligned} & ([z_0, z_1, \ldots, z_{m-1}], [z_m; z_{m+1}(z_0^a, z_1^a, \ldots, z_{m-1}^a); \ldots; \\ & \qquad z_{m+n}(z_0^a, z_1^a, \ldots, z_{m-1}^a)]) \in M, \end{aligned}$$

by

$$\begin{aligned} & ([1, z_1, \ldots, z_{m-1}], [1; z_{m+1}(1, z_1^a, \ldots, z_{m-1}^a); \ldots; \\ & \qquad z_{m+n}(1, z_1^a, \ldots, z_{m-1}^a)]) \in M. \end{aligned}$$

Thus, in order to make our proofs more readable, sometimes we shall work in S and sometimes in S'.

1.2. The Metric g on M

First, we endow $\mathbb{P}_k\mathbb{C}$ by the Fubini Study metric g_k whose components, in the chart $\{[z_0, z_1, \ldots, z_k] \in \mathbb{P}_k\mathbb{C} \text{ s.t. } z_0 \neq 0\}$, are given by

$$g_{\lambda\bar{\mu}} = \partial_{\lambda\bar{\mu}} \ln(1 + x_1 + \ldots + x_k)$$

where $x_i = \mid z_i \mid^2$ and $\partial_{\lambda\bar{\mu}} = \frac{\partial^2}{\partial z_\lambda \partial \bar{z}_\mu}$. Then, we consider the projections π_1 and π_2 of M respectively on $\mathbb{P}_{m-1}\mathbb{C}$ and $\mathbb{P}_{mn}\mathbb{C}$, and define the metric g on M by

$$g = \alpha \pi_1^* g_{m-1} + \beta \pi_2^* g_{mn}.$$

Its components in the local chart S' are given by

$$\begin{aligned} g_{\lambda\overline{\mu}} &= \alpha\partial_{\lambda\bar{\mu}}\ln(1+x_1+\ldots+x_{m-1}) \\ &\quad +\beta\partial_{\lambda\bar{\mu}}\ln\{1+x_{m+1}(1+x_1^a+\ldots+x_{m-1}^a) \\ &\quad +\ldots+x_{m+n}(1+x_1^a+\ldots+x_{m-1}^a)\}\,, \end{aligned}$$

where $x_i =\mid z_i \mid^2$ and $\lambda, \mu = 1, \ldots, m-1, m+1, \ldots, m+n$. In the coordinate system S, its components are given by

$$\begin{aligned} g_{\lambda\overline{\mu}} &= \alpha\partial_{\lambda\bar{\mu}}\ln(x_1+\ldots+x_m)+\beta\partial_{\lambda\bar{\mu}}\ln\{1+(x_1^a+\ldots+x_m^a) \\ &\quad +x_{m+1}(x_1^a+\ldots+x_m^a)+\ldots+x_{m+n-1}(x_1^a+\ldots+x_m^a)\}\,. \end{aligned}$$

We shall later prove

Proposition 1.1. *For $\alpha = m-na$ and $\beta = n+1$, the metric g belongs to the first Chern class $C_1(M)$; therefore, M is Fano.*

The metric g will be considered with $\alpha = m-na$ and $\beta = n+1$.

1.3. The Automorphisms Group G on M

Let us consider the automorphisms group G_{m-1} on $\mathbb{P}_{m-1}\mathbb{C}$ spanned by the automorphisms $\sigma_{i,j}$ and $\tau_{l,\theta}$ defined $\forall\ i,j \in \{0,1,\ldots,m-1\}$, $l \in \{0,\ldots,m-1\}$ and $\theta \in [0,2\pi]$ by

$$\begin{aligned} &\sigma_{i,j}([z_0,\ldots,z_i,\ldots,z_j,\ldots,z_k,\ldots,z_{m-1}]) \\ &\qquad\qquad = [z_0,\ldots,z_j,\ldots,z_i,\ldots,z_k,\ldots,z_{m-1}] \end{aligned}$$

and

$$\tau_{l,\theta}([z_0,\ldots,z_l,\ldots,z_{m-1}]) = [z_0,\ldots,z_l e^{i\theta},\ldots,z_{m-1}].$$

On $\mathbb{P}_{mn}\mathbb{C}$, we define another automorphisms group G_{mn}, spanned by:

1) $\varphi_{k,l}$, $k,l \in \{1,\ldots,n\}$ defined by

$$\begin{aligned} &\varphi_{k,l}([z_m, z_{m+1}Z^a,\ldots,z_{m+k}Z^a,\ldots,z_{m+l}Z^a,\ldots,z_{m+n}Z^a]) \\ &\quad = ([z_m, z_{m+1}Z^a,\ldots,z_{m+l}Z^a,\ldots,z_{m+k}Z^a,\ldots,z_{m+n}Z^a]) \end{aligned}$$

where $Z^a = (z_0^a,\ldots,z_{m-1}^a) \in \mathbb{C}^m$.

2) for $\theta \in [0, 2\pi]$, and $l \in \{0, \dots, n\}$,

$$\tau'_{l,\theta}([z_m, z_{m+1}Z^a, \dots, z_{m+l}Z^a, \dots, z_{m+n}Z^a]) = ([z_m, z_{m+1}Z^a, \dots, z_{m+l}e^{i\theta}Z^a, \dots, z_{m+n}Z^a]).$$

3) The above defined automorphisms $\sigma_{i,j}$ and $\tau_{l,\theta}$ of G_{m-1}, acting only on $Z = (z_0, \dots, z_{m-1}) \in \mathbb{C}^m$ in the description

$$([z_m, z_{m+1}Z^a, \dots, z_{m+k}Z^a, \dots, z_{m+l}Z^a, \dots, z_{m+n}Z^a]).$$

The groups G_{m-1} and G_{mn} generate a natural automorphisms group G on M, which we use later on.

1.4. The Extremal Function ψ on M

Let us consider the functions

$$\psi_1 = \ln\left\{ \frac{(|z_0^{(0)}| \dots |z_{m-1}^{(0)}|)^{\frac{2(m-an)}{m}}}{(|z_0^{(0)}|^2 + \dots + |z_{m-1}^{(0)}|^2)^{m-an}} \times |z_0^{(1)}|^{2(n+1)} \times \Big[|z_0^{(1)}|^2 + (|z_1^{(1)}|^2 + \dots + |z_m^{(1)}|^2) + \dots + (|z_{(n-1)m+1}^{(1)}|^2 + \dots + |z_{nm}^{(1)}|^2)\Big]^{-(n+1)} \right\}$$

and

$$\psi_2 = \ln\left\{ \frac{(|z_0^{(0)}| \dots |z_{m-1}^{(0)}|)^{\frac{2(m-an)}{m}}}{(|z_0^{(0)}|^2 + \dots + |z_{m-1}^{(0)}|^2)^{m-an}} \times\Big[(|z_1^{(1)}|\dots|z_m^{(1)}|)\dots(|z_{(n-1)m+1}^{(1)}|\dots|z_{nm}^{(1)}|)\Big]^{2(n+1)/nm} \times\Big[|z_0^{(1)}|^2 + (|z_1^{(1)}|^2 + \dots + |z_m^{(1)}|^2) + \dots + (|z_{(n-1)m+1}^{(1)}|^2 + \dots + |z_{nm}^{(1)}|^2)\Big]^{-(n+1)} \right\}$$

ψ_1 and ψ_2 are functions defined on

$$\left(\mathbb{C}^m \backslash \bigcup_i \{z_i^{(0)} = 0\}\right) \times \left(\mathbb{C}^{nm+1} \backslash \bigcup_j \{z_j^{(1)} = 0\}\right)$$

where $(z_i^{(0)})_{0\leq i\leq m-1}$ and $(z_j^{(1)})_{0\leq j\leq nm}$ are respectively the coordinates on $\mathbb{C}^m$ and $\mathbb{C}^{nm+1}$. They are homogeneous of degree zero in the variables de $\mathbb{C}^m$ and $\mathbb{C}^{nm+1}$ separately. Thus, they define two functions on $\mathbb{P}_{m-1}\mathbb{C}\times\mathbb{P}_{nm}\mathbb{C}$, and, by restriction on M, two functions on M, given by (keeping the same notations) :

$$\psi_1 = \ln\left\{\frac{(x_0\ldots x_{m-1})^{\frac{(m-an)}{m}}}{(x_0+\ldots+x_{m-1})^{m-an}}\times \frac{x_m^{n+1}}{[x_m+x_{m+1}(x_0^a+\ldots+x_{m-1}^a)+\ldots+x_{m+n}(x_0^a+\ldots+x_{m-1}^a)]^{(n+1)}}\right\} \tag{1}$$

and

$$\psi_2 = \ln\left\{\frac{(x_0\ldots x_{m-1})^{\frac{(m-an)}{m}}}{(x_0+\ldots+x_{m-1})^{m-an}}\times \frac{[(x_{m+1}x_0^a\ldots x_{m+1}x_{m-1}^a)\ldots(x_{m+n}x_0^a\ldots x_{n+m}x_{m-1}^a)]^{(n+1)/nm}}{[x_m+x_{m+1}(x_0^a+\ldots+x_{m-1}^a)+\ldots+x_{m+n}(x_0^a+\ldots+x_{m-1}^a)]^{(n+1)}}\right\}, \tag{2}$$

where $x_i =\mid z_i\mid^2$, and the points of M are described by their homogeneous coordinates, that is:

$$([z_0,\ldots,z_{m-1}],[z_m;z_{m+1}z_0^a,\ldots,z_{m+1}z_{m-1}^a;\ldots;\ z_{m+n}z_0^a,\ldots,z_{m+n}z_{m-1}^a]).$$

$\psi = \inf(\psi_1,\psi_2)$ is then an extremal function, in the sense of the following

Theorem 1.2. *The inequality $\varphi\geq\psi$ holds, for all g-admissible and G-invariant function $\varphi\in C^\infty(M)$ satisfying* $\sup\varphi=0$ *on M.*

Let us recall that φ is said to be g-admissible, when the matrix of terms $g_{\lambda\bar\mu}+\frac{\partial^2\varphi}{\partial z^\lambda\partial\bar z^\mu}$ is definite positive.

As an immediate consequence of theorem 1.2, we have:

Corollary 1.3. *A sequence $(\varphi_k)_{k\in\mathbb{N}}$ of g-admissible, G-invariant functions satisfying* $\sup\varphi_k=0$ *cannot go to $-\infty$ outside the boundaries of the usual charts (described above).*

Another consequence is:

THEOREM 1.4. *For all* $\alpha < \frac{1}{n+1}$, *the inequality*

$$\int_M \exp(-\alpha\varphi)dv \leq \text{Cst}$$

holds for all g-admissible and G-invariant functions $\varphi \in C^\infty(M)$, *satisfying* $\sup\varphi = 0$ *on* M. *(dv is the volume element on M with respect to the metric g).*

This implies that the Tian constant of M, $\alpha(M)$, is greater or equal to $\frac{1}{n+1}$. Consequently, we have the following

COROLLARY 1.5. *For all* $t < \frac{dim(M)+1}{dim(M)} \times \frac{1}{(n+1)}$, *there exists a metric* g_t *in* $c_1(M)$ *such that* $Ricci(g_t) > tg_t$.

The proof of corollary 1.5 uses the flow in t of the Monge-Ampère equations

$$\log\det(g'g^{-1}) = -t\varphi + f,$$

where ${g'}_{\lambda\overline{\mu}} = g_{\lambda\overline{\mu}} + \partial_{\lambda\overline{\mu}}\varphi$ is a Kähler change of metric, and f is a known geometric function, given by $Ricci(g) - g = i\partial\overline{\partial}f$. We proved in [3] that, when $\alpha(M) \geq C$, then for all $0 \leq t < C\frac{(dim(M)+1)}{dim(M)}$, the above Monge-Ampère equations do have solutions. We can prove this by a method different than the one used in [3], using Tian's method for the C^0 estimate, given in [8]. In our case, $\alpha(M) \geq \frac{1}{n+1}$, so we have solutions for $0 \leq t < \frac{m+1}{m(n+1)}$. Consequently, for these values of t,

$$\begin{aligned}
Ricci(g') &= -i\partial\overline{\partial}\log\det(g') \\
&= -i\partial\overline{\partial}\log\det(g'g^{-1}g) \\
&= -i\partial\overline{\partial}\log\det(g'g^{-1}) - i\partial\overline{\partial}\log\det(g) \\
&= -i\partial\overline{\partial}\log(g'g^{-1}) + Ricci(g) \\
&= -i\partial\overline{\partial}(-t\varphi + f) + g + i\partial\overline{\partial}f \\
&= -i\partial\overline{\partial}(-t\varphi) + (g' - i\partial\overline{\partial}\varphi) \\
&= (t-1)i\partial\overline{\partial}\varphi + g' \\
&= tg' + (1-t)g
\end{aligned}$$

and the result holds.

Finally, let us note that this type of manifolds are generally used to prevent the existence of Kähler-Einstein metrics. Indeed, when $a = 1$ and $n = 1$, M is nothing but the blowing-up of $\mathbb{P}_m\mathbb{C}$ at one point; and it is a well-known fact that it does not carry Kähler-Einstein metric because the Lie algebra of its holomorphic vector fields is not reductive (Lichnerowicz and Matsushima obstructions). If $a \neq 1$, M generalizes the manifolds introduced by Calabi in [5] and used by Futaki in [6] to give examples of manifolds which cannot carry Kähler-Einstein metrics, and yet, the Lie algebra of their holomorphic vector fields is reductive.

2. Proof of the Results

Proof of Proposition 1.1. Our goal is to find a condition on α and β such that the quantity

$$F_{0,m} = (1+ \mid z_1 \mid^2 + \ldots + \mid z_{m-1} \mid^2)^\alpha \times \left\{1+(\mid z_{m+1} \mid^2 +\ldots+ \mid z_{m+n} \mid^2)\times(1+ \mid z_1 \mid^{2a} +\ldots+ \mid z_{m-1} \mid^{2a})\right\}^\beta,$$

written in the local chart $\{z_0 \neq 0,\ z_m \neq 0\}$ (which justifies the reason for the notation $F_{0,m}$), is a metric on the line bundle $\Lambda^{m+n-1}T^*M$. Then, its Ricci will be exactly the metric g and will, by definition, belong to $c_1(M)$, so that M will be Fano. Let us write the conditions which make (3) intrinsic in $\Lambda^{mn}T^*M$. The first change of charts we consider is

$$\varphi_1(z_1, \ldots, z_{m-1}; z_{m+1}, \ldots, z_{m+n}) = \left(\frac{1}{z_1}, \frac{z_2}{z_1}, \ldots, \frac{z_{m-1}}{z_1}; z_{m+1}z_1^a, \ldots, z_{m+n}z_1^a\right),$$

its Jacobian J_1 verifies

$$\mid J_1 \mid^2 = \frac{1}{\mid z_1 \mid^{2(m-an)}}.$$

In the new chart, the expression of $F_{0,m}$ becomes

$$F_{1,m} = \frac{1}{\mid z_1 \mid^{2\alpha}} F_{0,m},$$

and the first condition, i.e. : $\alpha = m - an$, holds.
Now, let us consider the change of charts:

$$\varphi_2(z_1, \dots, z_{m-1}; z_{m+1}, \dots, z_{m+n}) = \left(z_1, \dots, z_{m-1}; \frac{1}{z_{m+1}}, \frac{z_{m+2}}{z_{m+1}}, \dots, \frac{z_{m+n}}{z_{m+1}} \right).$$

Its Jacobian J_2 verifies

$$| J_2 |^2 = \frac{1}{| z_{m+1} |^{2(n+1)}}$$

and $F_{0,m}$ becomes

$$F_{0,m+1} = \frac{1}{| z_{m+1} |^{2\beta}} F_{0,m} .$$

This yields the second condition, i.e. $\beta = n + 1$. We easily verify that these conditions also hold for the other changes of charts; thus, M is Fano.

Proof of theorem 1.2. The proof requires four lemmas. In each step, we use the G-invariance of functions

$$\varphi([z_0, \dots, z_{m-1}], [z_m, z_{m+1}(z_0^a, \dots, z_{m-1}^a); \dots; z_{m+n}(z_0^a, \dots, z_{m-1}^a)]) ,$$

which allows us to consider them in the form

$$\varphi([x_0, \dots, x_{m-1}], [x_m, x_{m+1}(x_0^a, \dots, x_{m-1}^a); \dots; x_{m+n}(x_0^a, \dots, x_{m-1}^a)]),$$

where $x_i =| z_i |> 0$. Then, in S, we can write the function φ as:

$$\varphi([x_1, \dots, x_m], [1; (x_1^a, \dots, x_m^a); x_{m+1}(x_1^a, \dots, x_m^a); \dots; x_{m+n-1}(x_1^a, \dots, x_m^a)]).$$

Lemma 2.1. *Let $\varphi \in C^\infty(M)$, be a g-admissible G-invariant function. Then, for all $x_i =| z_i |> 0$,*

$$(\varphi - \psi)([x_1, \dots, x_m], [1; (x_1^a, \dots, x_m^a); x_{m+1}(x_1^a, \dots, x_m^a); \dots; x_{m+n-1}(x_1^a, \dots, x_m^a)]) \geq (\varphi - \psi)([1^{[m]}], [1; \zeta^{[m]}; x_{m+1}\zeta^{[m]}; \dots; x_{m+n-1}\zeta^{[m]}]), \quad (3)$$

where $h^{[m]} = (h, \dots, h) \in \mathbb{C}^m$ and $\zeta = (x_1 \dots x_m)^{a/m}$.

Proof. We proceed by induction. Assume that, for $1 \leq p < m$ and for all $(x_1, \dots, x_m) \in \mathbb{R}^{m-1}$ $(x_i > 0)$,

$$\begin{aligned}(\varphi - \psi)([x_1, \dots, x_m], [1; (x_1^a, \dots, x_m^a); x_{m+1}(x_1^a, \dots, x_m^a); \dots ;\\ x_{m+n-1}(x_1^a, \dots, x_m^a)])\\ \geq (\varphi - \psi)([(x_1 \dots x_p)^{a/p}, \dots, (x_1 \dots x_p)^{a/p}, x_{p+1}^a, \dots, x_m^a],\\ [1; ((x_1 \dots x_p)^{a/p}, \dots, (x_1 \dots x_p)^{a/p}, x_{p+1}^a, \dots, x_m^a);\\ x_{m+1}((x_1 \dots x_p)^{a/p}, \dots, (x_1 \dots x_p)^{a/p}, x_{p+1}^a, \dots, x_m^a); \dots ;\\ x_{m+n-1}((x_1 \dots x_p)^{a/p}, \dots, (x_1 \dots x_p)^{a/p}, x_{p+1}^a, \dots, x_m^a)]), \quad (4)\end{aligned}$$

which is obviously verified for $p = 1$. Now, assume that inequality (4) did not hold for $p + 1$. Then, there would be a point $(u_1, \dots, u_m) \in \mathbb{R}^m$, with $u_i > 0$ for all i, such that

$$\begin{aligned}(\varphi - \psi)([u_1, \dots, u_m], [1; (u_1^a, \dots, u_m^a); u_{m+1}(u_1^a, \dots, u_m^a); \dots ;\\ u_{m+n-1}(u_1^a, \dots, u_m^a)])\\ < (\varphi - \psi)([(u_1 \dots u_{p+1})^{a/p+1}, \dots, (u_1 \dots u_{p+1})^{a/p+1}, u_{p+2}^a, \dots, u_m^a],\\ [1; ((u_1 \dots u_{p+1})^{a/p+1}, \dots, (u_1 \dots u_{p+1})^{a/p+1}, u_{p+2}^a, \dots, u_m^a);\\ u_{m+1}((u_1 \dots u_{p+1})^{a/p+1}, \dots, (u_1 \dots u_{p+1})^{a/p+1}, u_{p+2}^a \dots, u_m^a); \dots ;\\ u_{m+n-1}((u_1 \dots u_{p+1})^{a/p+1}, \dots, (u_1 \dots u_{p+1})^{a/p+1}, u_{p+2}^a \dots, u_m^a)]).\\ (5)\end{aligned}$$

Using the G-invariance of φ, we can assume that $u_1 \leq \dots \leq u_m$. On the other hand, taking into account the G-invariance of φ and the induction assumption (4) at the points

$$\begin{aligned}([u_1, \dots, u_p, u_{p+1}, \dots, u_m], [1; (u_1^a, \dots, u_p^a, u_{p+1}^a, \dots, u_m^a);\\ u_{m+1}(u_1^a, \dots, u_p^a, u_{p+1}^a, \dots, u_m^a); \dots ;\\ u_{m+n-1}(u_1^a, \dots, u_p^a, u_{p+1}^a, \dots, u_m^a)])\end{aligned}$$

and

$$\begin{aligned}([u_2, \dots, u_{p+1}, u_1, u_{p+2}, \dots, u_m], [1; (u_2^a, \dots, u_{p+1}^a, u_1^a, u_{p+2}^a, \dots, u_m^a);\\ u_{m+1}(u_2^a, \dots, u_{p+1}^a, u_1^a, u_{p+2}^a, \dots, u_m^a); \dots ;\\ u_{m+n-1}(u_2^a, \dots, u_{p+1}^a, u_1^a, u_{p+2}^a, \dots, u_m^a)])\end{aligned}$$

of M, we can write

$$\begin{aligned}(\varphi-\psi)&([u_1,\dots,u_p,u_{p+1},\dots,u_m],[1;(u_1^a,\dots,u_p^a,u_{p+1}^a,\dots,u_m^a);\\&u_{m+1}(u_1^a,\dots,u_p^a,u_{p+1}^a,\dots,u_m^a);\dots;\\&u_{m+n-1}(u_1^a,\dots,u_p^a,u_{p+1}^a,\dots,u_m^a)])\\\geq(\varphi-\psi)&([(u_1\dots u_p)^{1/p},\dots,(u_1\dots u_p)^{1/p},u_{p+1},\dots,u_m],\\&[1;((u_1\dots u_p)^{a/p},\dots,(u_1\dots u_p)^{a/p},u_{p+1}^a,\dots,u_m^a);\\&u_{m+1}((u_1\dots u_p)^{a/p},\dots,(u_1\dots u_p)^{a/p},u_{p+1}^a,\dots,u_m^a);\dots;\\&u_{m+n-1}((u_1\dots u_p)^{a/p},\dots,(u_1\dots u_p)^{a/p},u_{p+1}^a,\dots,u_m^a)]),\end{aligned}\tag{6}$$

and

$$\begin{aligned}(\varphi-\psi)&([u_2,\dots,u_{p+1},u_1,u_{p+2},\dots,u_m],\\&([1;u_2^a,\dots,u_{p+1}^a,u_1^a,u_{p+2}^a,\dots,u_m^a);\\&u_{m+1}(u_2^a,\dots,u_{p+1}^a,u_1^a,u_{p+2}^a,\dots,u_m^a);\dots;\\&u_{m+n-1}(u_2^a,\dots,u_{p+1}^a,u_1^a,u_{p+2}^a,\dots,u_m^a)])\\\geq(\varphi-\psi)&([(u_2\dots u_{p+1})^{1/p},\dots,(u_2\dots u_{p+1})^{1/p},u_1,u_{p+2},\dots,u_m],\\&[1;((u_2\dots u_{p+1})^{a/p},\dots,(u_2\dots u_{p+1})^{a/p},u_1^a,u_{p+2}^a,\dots,u_m^a);\\&u_{m+1}((u_2\dots u_{p+1})^{a/p},\dots,(u_2\dots u_{p+1})^{a/p},u_1^a,u_{p+2}^a,\dots,u_m^a);\dots;\\&u_{m+n-1}((u_2\dots u_{p+1})^{a/p},\dots,(u_2\dots u_{p+1})^{a/p},u_1^a,u_{p+2}^a,\dots,u_m^a)]).\end{aligned}\tag{7}$$

Now, let us consider the curve C, of equation

$$t^p x_{p+1}=u_1\dots u_{p+1}\,,$$

in the real plane

$$\begin{aligned}\{([t,\dots,t,x_{p+1},u_{p+2},\dots,u_m],[1;(t^a,\dots,t^a,x_{p+1}^a,u_{p+2}^a,\dots,u_m^a);\\u_{m+1}(t^a,\dots,t^a,x_{p+1}^a,u_{p+2}^a,\dots,u_m^a);\dots;\\u_{m+n-1}(t^a,\dots,t^a,x_{p+1}^a,u_{p+2}^a,\dots,u_m^a)])\},\end{aligned}$$

where t and x_{p+1} are variables. The points

$$\begin{aligned}P_1=(&[(u_1\dots u_p)^{1/p},\dots,(u_1\dots u_p)^{1/p},u_{p+1},\dots,u_m],\\&[1;((u_1\dots u_p)^{a/p},\dots,(u_1\dots u_p)^{a/p},u_{p+1}^a,\dots,u_m^a);\\&u_{m+1}((u_1\dots u_p)^{a/p},\dots,(u_1\dots u_p)^{a/p},u_{p+1}^a,\dots,u_m^a);\dots;\\&u_{m+n-1}((u_1\dots u_p)^{a/p},\dots,(u_1\dots u_p)^{a/p},u_{p+1}^a,\dots,u_m^a)])\end{aligned}$$

and

$$\begin{aligned}P_2 = ([&(u_2 \dots u_{p+1})^{1/p}, \dots, (u_2 \dots u_{p+1})^{1/p}, u_1, u_{p+2}, \dots, u_m],\\ &[1; ((u_2 \dots u_{p+1})^{a/p}, \dots, (u_2 \dots u_{p+1})^{a/p}, u_1^a, u_{p+2}^a, \dots, u_m^a);\\ &u_{m+1}((u_2 \dots u_{p+1})^{a/p}, \dots, (u_2 \dots u_{p+1})^{a/p}, u_1^a, u_{p+2}^a, \dots, u_m^a); \dots;\\ &u_{m+n-1}((u_2 \dots u_{p+1})^{a/p}, \dots, (u_2 \dots u_{p+1})^{a/p}, u_1^a, u_{p+2}^a, \dots, u_m^a)]),\end{aligned}$$

belong to this curve. Note that we cannot have $u_1 = \dots = u_{p+1}$, for, otherwise, (5) would be an equality.

Taking into account that we have chosen $u_1 \leq \dots \leq u_{p+1}$, the points P_1 and P_2 (which are different) are on different sides of the diagonal $t = x_{p+1}$ of the plane described above.

Note that the curve C intersects this diagonal at the point

$$\begin{aligned}P_3 = ([&(u_1 \dots u_{p+1})^{1/p+1}, \dots, (u_1 \dots u_{p+1})^{1/p+1}, u_{p+2}, \dots, u_m], \qquad (8)\\ &[1; ((u_1 \dots u_{p+1})^{a/p+1}, \dots, (u_1 \dots u_{p+1})^{a/p+1}, u_{p+2}^a, \dots, u_m^a);\\ &u_{m+1}((u_1 \dots u_{p+1})^{a/p+1}, \dots, (u_1 \dots u_{p+1})^{a/p+1}, u_{p+2}^a, \dots, u_m^a); \dots;\\ &u_{m+n-1}((u_1 \dots u_{p+1})^{a/p+1}, \dots, (u_1 \dots u_{p+1})^{a/p+1}, u_{p+2}^a, \dots, u_m^a)]),\end{aligned}$$

which appears in inequality (5). On the other hand, using relations (5), (6) and (7), we obtain that

$$(\varphi - \psi)(P_3) > (\varphi - \psi)(P_1) \text{ et } (\varphi - \psi)(P_3) > (\varphi - \psi)(P_2),$$

which proves that the function $(\varphi - \psi)$ reaches a local maximum on the curve C. Consequently, the restriction of the G-invariant function $(\varphi - \psi)$ to the holomorphic curve (that we denote again by C) $\xi^p z = u_1 \dots u_{p+1}$ of the complex dimensional 2-plane

$$\begin{aligned}\{([&\xi, \dots, \xi, z, u_{p+2}, \dots, u_m], [1; (\xi^a, \dots, \xi^a, z^a, u_{p+2}^a, \dots, u_m^a); \dots;\\ &u_{m+1}(\xi^a, \dots, \xi^a, z^a, u_{p+2}^a, \dots, u_m^a);\\ &u_{m+n-1}(\xi^a, \dots, \xi^a, z^a, u_{p+2}^a, \dots, u_{m-1}^a)])\},\end{aligned}$$

reaches a local maximum at a point $P = C(\zeta)$. Let us set

$$\begin{aligned}C(\zeta) &= ([1, C^1(\zeta), \dots, C^{m-1}(\zeta)], [1,\\ &\qquad C^{m+1}(\zeta)(C^1(\zeta)^a, \dots, C^{m-1}(\zeta)^a); \dots;\\ &\qquad C^{m+n}(\zeta)(C^1(\zeta)^a, \dots, C^{m-1}(\zeta)^a)]),\\ \dot{C}^\lambda(\xi) &= \frac{dC^\lambda}{d\xi}(\xi) \qquad \text{and} \qquad \dot{C}^{\overline{\mu}}(\xi) = \overline{\dot{C}^\mu(\xi)}.\end{aligned}$$

Note that, by the continuity of $(\varphi - \psi)$, we can always choose the point

$$([u_1, \dots, u_m], [1; (u_1^a, \dots, u_m^a); u_{m+1}(u_1^a, \dots, u_m^a); \dots; \\ u_{m+n-1}(u_1^a, \dots, u_m^a)]),$$

in inequality (5), so that

$$(u_1 \dots u_m)^{a/m}(u_{m+1} \dots u_{m+n-1})^{1/n} \neq 1.$$

Thus, the equation of C, as well as the definition of ψ_1 and ψ_2 (given by (1) and (2)), show that every point of the curve C satisfies

$$\begin{aligned} &\psi_1([\xi, \dots, \xi, z, u_{p+2}, \dots, u_m], [1; (\xi^a, \dots, \xi^a, z^a, u_{p+2}^a, \dots, u_m^a); \\ &\quad u_{m+1}(\xi^a, \dots, \xi^a, z^a, u_{p+2}^a, \dots, u_m^a); \dots; \\ &\qquad u_{m+n-1}(\xi^a, \dots, \xi^a, z^a, u_{p+2}^a, \dots, u_m^a)]) \\ &\quad \neq \psi_2([\xi, \dots, \xi, z, u_{p+2}, \dots, u_m], [1, (\xi^a, \dots, \xi^a, z^a, u_{p+2}^a, \dots, u_m^a); \\ &\qquad\quad u_{m+1}(\xi^a, \dots, \xi^a, z^a, u_{p+2}^a, \dots, u_m^a); \dots; \\ &\qquad\qquad u_{m+n-1}(\xi^a, \dots, \xi^a, z^a, u_{p+2}^a, \dots, u_m^a)]). \end{aligned} \tag{9}$$

Consequently, we can assume that $\psi = \psi_1$ in a neighborhood of P, the proof being exactly the same if we assume $\psi = \psi_2$ in a neighborhood of P. Therefore,

$$\frac{\partial^2}{\partial \xi \partial \overline{\xi}} \{(\varphi - \psi_1)(C(\zeta))\} = \frac{\partial^2 (\varphi - \psi_1)}{\partial z_\lambda \partial \overline{z}_\mu}(C(\zeta)) \dot{C}^\lambda(\zeta) \dot{C}^{\overline{\mu}}(\zeta) \leq 0$$

Since

$$-\frac{\partial^2 \psi_1}{\partial z_\lambda \partial \overline{z}_\mu} = g_{\lambda\overline{\mu}}\,,$$

the previous inequality expresses the fact that the Hermitian form of the matrix

$$\left(g_{\lambda\overline{\mu}} + \frac{\partial^2 \varphi}{\partial z_\lambda \partial \overline{z}_\mu} \right)_{\lambda,\mu} = \left(\frac{\partial^2 (\varphi - \psi_1)}{\partial z_\lambda \partial \overline{z}_\mu} \right)_{\lambda,\mu}$$

is negative at $P = C(\zeta)$. This contradicts the g-admissibility of φ at P. It follows that inequality (4) holds also for $p + 1$, and lemma 2.1 is proven. □

In the next lemma, it is more convenient, for our computations, to use the chart given by $\{z_0 \neq 0\}$ and $\{z_m \neq 0\}$ in the parametrization

$$[z_0, z_1, \ldots, z_{m-1}], [z_m; z_{m+1}(z_0^a, z_1^a, \ldots, z_{m-1}^a); \ldots; \\ z_{m+n}(z_0^a, z_1^a, \ldots, z_{m-1}^a)] .$$

□

Lemma 2.2. *Let $\varphi \in C^\infty(M)$, be a g-admissible G-invariant function. Then, for all $x_i =| z_i |> 0$,*

$$\begin{aligned}(\varphi - \psi)([1, x_1, \ldots, x_{m-1}], [1; x_{m+1}(1, x_1^a, \ldots, x_{m-1}^a); \ldots; \\ x_{m+n}(1, x_1^a, \ldots, x_{m-1}^a)]) \\ \geq (\varphi - \psi)([1, x_1, \ldots, x_{m-1}], [1; \lambda(1, x_1^a, \ldots, x_{m-1}^a); \ldots; \\ \lambda(1, x_1^a, \ldots, x_{m-1}^a)]),\end{aligned} \tag{10}$$

where $\lambda = (x_{m+1} \ldots x_{m+n})^{1/n}$.

Proof. As in lemma 2.1, we proceed by induction. Assume that, for $1 \leq p < n$ and for all $(x_{m+1}, \ldots, x_{m+n}) \in \mathbb{R}^{m-1}$ $(x_i > 0)$,

$$\begin{aligned}(\varphi - \psi)([1, x_1, \ldots, x_{m-1}], [1; x_{m+1}(1, x_1^a, \ldots, x_{m-1}^a); \ldots; \\ x_{m+n}(1, x_1^a, \ldots, x_{m-1}^a)]) \\ \geq (\varphi - \psi)([1, x_1, \ldots, x_{m-1}], [1; \\ (x_{m+1} \ldots x_{m+p})^{1/p}(1, x_1^a, \ldots, x_{m-1}^a); \ldots; \\ (x_{m+1} \ldots x_{m+p})^{1/p}(1, x_1^a, \ldots, x_{m-1}^a), \\ x_{m+p+1}(1, x_1^a, \ldots, x_{m-1}^a); \ldots; \\ x_{m+n}(1, x_1^a, \ldots, x_{m-1}^a)]),\end{aligned} \tag{11}$$

which is obviously verified for $p = 1$. Assume that inequality (11) did not hold for $p + 1$. Then, there would exist a point $(u_1, \ldots, u_{m+1}, \ldots, u_{m+n}) \in \mathbb{R}^n$, with $u_i^0 > 0$ for all i, such that

$$\begin{aligned}&(\varphi - \psi)([1, u_1, \ldots, u_{m-1}], \\ &[1; u_{m+1}(1, u_1^a, \ldots, u_{m-1}^a); \ldots; u_{m+n}(1, u_1^a, \ldots, u_{m-1}^a)]) \\ &< (\varphi - \psi)([1, u_1, \ldots, u_{m-1}], [1; \\ &(u_{m+1} \ldots u_{m+p+1})^{1/p+1}(1, u_1^a, \ldots, u_{m-1}^a); \ldots; \\ &(u_{m+1} \ldots u_{m+p+1})^{1/p+1}(1, u_1^a, \ldots, u_{m-1}^a), \\ &u_{m+p+2}(1, u_1^a, \ldots, u_{m-1}^a); \ldots; u_{m+n}(1, u_1^a, \ldots, u_{m-1}^a)]).\end{aligned} \tag{12}$$

Using the G-invariance of φ, we can assume that $u_{m+1} \leq \ldots \leq u_{m+n}$. On the other hand, taking into account the G-invariance of φ, and the induction assumption (11) at the points

$$([1, u_1, \ldots, u_{m-1}], [1; u_{m+1}(1, u_1^a, \ldots, u_{m-1}^a); \ldots; u_{m+p}(1, u_1^a, \ldots, u_{m-1}^a); u_{m+p+1}(1, u_1^a, \ldots, u_{m-1}^a); u_{m+n}(1, u_1^a, \ldots, u_{m-1}^a)])$$

and

$$([1, u_1, \ldots, u_{m-1}], [1; u_{m+2}(1, u_1^a, \ldots, u_{m-1}^a); \ldots; u_{m+p+1}(1, u_1^a, \ldots, u_{m-1}^a); u_{m+1}(1, u_1^a, \ldots, u_{m-1}^a); u_{m+p+2}(1, u_1^a, \ldots, u_{m-1}^a); \ldots; u_{m+n}(1, u_1^a, \ldots, u_{m-1}^a)])$$

of M, we obtain

$$\begin{aligned}(\varphi - \psi)&([1, u_1, \ldots, u_{m-1}], [1; u_{m+1}(1, u_1^a, \ldots, u_{m-1}^a); \ldots; u_{m+p}(1, u_1^a, \ldots, u_{m-1}^a); u_{m+p+1}(1, u_1^a, \ldots, u_{m-1}^a); \ldots; u_{m+n}(1, u_1^a, \ldots, u_{m-1}^a)]) \\ &\geq (\varphi - \psi)([1, u_1, \ldots, u_{m-1}], [1; (u_{m+1} \ldots u_{m+p})^{1/p}(1, u_1^a, \ldots, u_{m-1}^a); \ldots; (u_{m+1} \ldots u_{m+p})^{1/p}(1, u_1^a, \ldots, u_{m-1}^a), \\ &\qquad u_{m+p+1}(1, u_1^a, \ldots, u_{m-1}^a); \ldots; u_{m+n}(1, u_1^a, \ldots, u_{m-1}^a)]),\end{aligned} \tag{13}$$

and

$$\begin{aligned}(\varphi - \psi)&([1, u_1, \ldots, u_{m-1}], [1; u_{m+2}(1, u_1^a, \ldots, u_{m-1}^a); \ldots; u_{m+p+1}(1, u_1^a, \ldots, u_{m-1}^a); u_{m+1}(1, u_1^a, \ldots, u_{m-1}^a); \\ &\qquad u_{m+p+2}(1, u_1^a, \ldots, u_{m-1}^a); \ldots; u_{m+n}(1, u_1^a, \ldots, u_{m-1}^a)]) \\ &\geq (\varphi - \psi)([1, u_1, \ldots, u_{m-1}], [1; (u_{m+2} \ldots u_{m+p+1})^{1/p}(1, u_1^a, \ldots, u_{m-1}^a); \ldots; \\ &\qquad (u_{m+2} \ldots u_{m+p+1})^{1/p}(1, u_1^a, \ldots, u_{m-1}^a), u_{m+1}(1, u_1^a, \ldots, u_{m-1}^a), \\ &\qquad u_{m+p+2}(1, u_1^a, \ldots, u_{m-1}^a); \ldots; u_{m+n}(1, u_1^a, \ldots, u_{m-1}^a)]).\end{aligned} \tag{14}$$

As in the previous lemma, we consider the curve C (we keep the same notation), given by

$$t^p x = u_{m+1} \dots u_{m+p+1}$$

of the real plane

$$\{([1, u_1, \dots, u_{m-1}], [1; t(1, u_1^a, \dots, u_{m-1}^a); \dots; \\ t(1, u_1^a, \dots, u_{m-1}^a); x(1, u_1^a, \dots, u_{m-1}^a); u_{m+p+2}(1, u_1^a, \dots, u_{m-1}^a); \dots; \\ u_{m+n}(1, u_1^a, \dots, u_{m-1}^a)])\},$$

parameterized by (t, x). The points

$$\begin{aligned} Q_1 \quad = \quad & ([1, u_1, \dots, u_{m-1}], [1; \\ & (u_{m+1} \dots u_{m+p})^{1/p}(1, u_1^a, \dots, u_{m-1}^a); \dots; \\ & (u_{m+1} \dots u_{m+p})^{1/p}(1, u_1^a, \dots, u_{m-1}^a); u_{m+p+1}(1, u_1^a, \dots, u_{m-1}^a); \dots; \\ & u_{m+n}(1, u_1^a, \dots, u_{m-1}^a)]) \end{aligned}$$

and

$$\begin{aligned} Q_2 \quad = \quad & ([1, u_1, \dots, u_{m-1}], [1; \\ & (u_{m+2} \dots u_{m+p+1})^{1/p}(1, u_1^a, \dots, u_{m-1}^a); \dots; \\ & (u_{m+2} \dots u_{m+p+1})^{1/p}(1, u_1^a, \dots, u_{m-1}^a); u_{m+1}(1, u_1^a, \dots, u_{m-1}^a); \\ & u_{m+p+2}(1, u_1^a, \dots, u_{m-1}^a); \dots; u_{m+n}(1, u_1^a, \dots, u_{m-1}^a)]), \end{aligned}$$

belong to this curve, and we cannot have $u_{m+1} = \dots = u_{m+p+1}$, for, otherwise, (12) would be an equality. Since $u_{m+1} \leq \dots \leq u_{m+p+1}$, the two different points Q_1 and Q_2 are from different sides of the diagonal $t = x$ of the above described plane, and the curve C intersects this diagonal at the point

$$\begin{aligned} Q_3 \quad = \quad & ([1, u_1, \dots, u_{m-1}], \\ & [1; (u_{m+1} \dots u_{m+p+1})^{1/p+1}(1, u_1^a, \dots, u_{m-1}^a); \dots; \\ & (u_{m+1} \dots u_{m+p+1})^{1/p+1}(1, u_1^a, \dots, u_{m-1}^a); \\ & u_{m+p+2}(1, u_1^a, \dots, u_{m-1}^a); \dots; u_{m+n}(1, u_1^a, \dots, u_{m-1}^a)]) \end{aligned} \tag{15}$$

of inequality (12). On the other hand, using relations (12), (13) and (14), we obtain that

$$(\varphi - \psi)(Q_3) > (\varphi - \psi)(Q_1) \text{ et } (\varphi - \psi)(Q_3) > (\varphi - \psi)(Q_2)\,,$$

which proves that the function $(\varphi - \psi)$ reaches a local maximum on the curve C. Consequently, the restriction of the G-invariant function $(\varphi - \psi)$ to the holomorphic curve (again denoted by C) $\xi^p z = u_{m+1} \dots u_{m+p+1}$ of the complex dimensional 2-plane

$$\{([1, u_1, \dots, u_{m-1}], [1; \xi(1, u_1^a, \dots, u_{m-1}^a); \dots ; \xi(1, u_1^a, \dots, u_{m-1}^a); z(1, u_1^a, \dots, u_{m-1}^a); u_{m+p+2}(1, u_1^a, \dots, u_{m-1}^a); \dots ; u_{m+n}(1, u_1^a, \dots, u_{m-1}^a)])\},$$

reaches a local maximum at a point $Q = C(\zeta)$. By the continuity of $(\varphi - \psi)$, we can choose the point

$$([1, u_1, \dots, u_{m-1}], [1; u_{m+1}(1, u_1^a, \dots, u_{m-1}^a); \dots ; u_{m+n}(1, u_1^a, \dots, u_{m-1}^a)])$$

in inequality (12), so that

$$(u_1 \dots u_{m-1})^{a/m}(u_{m+1} \dots u_{m+n})^{1/n} \neq 1.$$

Thus, the equation of C, as well as the definition of ψ_1 and ψ_2 (given by (1) and (2)), yield that

$$\begin{aligned} &\psi_1([1, u_1, \dots, u_{m-1}], [1; \xi(1, u_1^a, \dots, u_{m-1}^a); \dots ; \xi(1, u_1^a, \dots, u_{m-1}^a); \\ &\quad z(1, u_1^a, \dots, u_{m-1}^a); u_{m+p+2}(1, u_1^a, \dots, u_{m-1}^a); \dots ; u_{m+n}(1, u_1^a, \dots, u_{m-1}^a)]) \\ &\neq \psi_2([1, u_1, \dots, u_{m-1}], [1; \xi(1, u_1^a, \dots, u_{m-1}^a); \dots ; \xi(1, u_1^a, \dots, u_{m-1}^a); \\ &\quad z(1, u_1^a, \dots, u_{m-1}^a); u_{m+p+2}(1, u_1^a, \dots, u_{m-1}^a); \dots ; u_{m+n}(1, u_1^a, \dots, u_{m-1}^a)]) \end{aligned} \tag{16}$$

on C. Then, without loss of generality, we can assume that $\psi = \psi_1$ in a neighborhood of Q. We conclude then as in lemma 2.1, reaching a contradiction with the g-admissibility of φ at Q. □

As a consequence of lemmas 2.2 and 2.1, we have

Lemma 2.3. *Let $\varphi \in C^\infty(M)$, be a g-admissible G-invariant function. Then, for all $x_i = | z_i | > 0$,*

$$\begin{aligned} (\varphi - \psi)([1, x_1, \dots, x_{m-1}], [1; x_{m+1}(1, x_1^a, \dots, x_{m-1}^a); \dots ; \\ x_{m+n}(1, x_1^a, \dots, x_{m-1}^a)]) \\ \geq (\varphi - \psi)([1^{[m]}], [1; \mu^{[nm]}]), \end{aligned} \tag{17}$$

where $\mu = (x_{m+1} \dots x_{m+n})^{1/n}(x_1 \dots x_{m-1})^{a/m}$

Proof. Inequality (10) of lemma 2.2, followed by inequality (3) of lemma 2.1 leads to

$$\begin{aligned}
&(\varphi-\psi)([1,x_1,\dots,x_{m-1}],[1;x_{m+1}(1,x_1^a,\dots,x_{m-1}^a);\dots;\\
&\quad x_{m+n}(1,x_1^a,\dots,x_{m-1}^a)])\\
&\quad\geq\ (\varphi-\psi)([1,x_1,\dots,x_{m-1}],\\
&\qquad\qquad [1;\lambda(1,x_1^a,\dots,x_{m-1}^a);\dots;\lambda(1,x_1^a,\dots,x_{m-1}^a)])\\
&\quad=\ (\varphi-\psi)([\lambda^{1/a}(1,x_1,\dots,x_{m-1})],\\
&\qquad\qquad [1;\lambda(1,x_1^a,\dots,x_{m-1}^a);\dots;\lambda(1,x_1^a,\dots,x_{m-1}^a)])\\
&\quad=\ (\varphi-\psi)([y_1,\dots,y_m],[1;(y_1^a,\dots,y_m^a);\dots;(y_1^a,\dots,y_m^a)])\\
&\quad\geq\ (\varphi-\psi)([1^{[m]}],[1;\mu^{[m]};\mu^{[m]};\dots;\mu^{[m]}])\,,
\end{aligned}$$

where

$$\begin{aligned}
\lambda &= (x_{m+1}\dots x_{m+n})^{1/n},\\
y_1 &= \lambda^{1/a}, \qquad y_2=\lambda^{1/a}x_1,\dots, \qquad y_m=\lambda^{1/a}x_{m-1},
\end{aligned}$$

and

$$\begin{aligned}
\mu &= (y_1\dots y_m)^{a/m}\\
&= \lambda(x_1\dots x_{m-1})^{a/m}\\
&= (x_{m+1}\dots x_{m+n})^{1/n}(x_1\dots x_{m-1})^{a/m}
\end{aligned}$$

□

Finally, we claim:

LEMMA 2.4. *Let* $\varphi \in C^\infty(M)$ *be a* g*-admissible,* G*-invariant function, verifying* $\sup\varphi = 0$ *on* M*. Then,* $\forall\mu > 0$*,*

$$(\varphi-\psi)([1^{[m]}],[1;\mu^{[nm]}]) \geq 0\,. \tag{18}$$

Proof. Consider the point $R_0 \in \mathbb{P}_m\mathbb{C}$ where φ reaches its maximum (equal to zero). Using the G-invariance of φ, we can write R_0 as

$$\begin{aligned}
R_0 = ([v_0,\dots,v_{m-1}],[v_m;v_{m+1}(v_0^a,\dots,v_{m-1}^a);\dots;&\\
v_{m+n}(v_0^a,\dots,v_{m-1}^a)]),&
\end{aligned}$$

where the positive reals v_i verify $v_0 \geq v_1 \geq \ldots \geq v_{m-1}$ and $v_{m+1} \geq v_{m+2} \geq \ldots \geq v_{m+n}$. We have two separate cases, according to whether $v_m \neq 0$, or $v_m = 0$.

Case A : $v_m \neq 0$. In this case, we use the coordinates system M given in $\{v_0 \neq 0, v_m \neq 0\}$ by fixing $v_0 = 1$ and $v_m = 1$; thus, R_0 is of the form

$$R_0 = ([1, u_1 \ldots, u_{m-1}], [1; u_{m+1}(1, u_1^a \ldots, u_{m-1}^a); \ldots; \\ u_{m+n}^0(1, u_1^a \ldots, u_{m-1}^a)]),$$

where the reals u_i are such that $1 \geq u_1 \geq \ldots \geq u_{m-1}$ and $x_{m+1}^0 \geq \ldots \geq x_{m+n}^0$. Proceeding by contradiction, assume there is a point

$$R_1 = ([1^{[m]}], [1; \zeta_0^{[nm]}]) ,$$

such that $\zeta_0 > 0$ and

$$(\varphi - \psi)(R_1) < 0. \tag{19}$$

We separately consider the two following sub-cases: $u_{m+1} < \zeta_0$ and $u_{m+1} \geq \zeta_0$.

- $\underline{u_{m+1} \leq \zeta_0}$.

We introduce the auxiliary function $\psi_{0,m}$, given by

$$\psi_{0,m} = \ln\Bigg\{ \frac{x_0^{m-an}}{(x_0 + \ldots + x_{m-1})^{m-an}} \times \\ x_m^{n+1}[x_m + (x_{m+1}x_0^a + \ldots + x_{m+1}x_{m-1}^a) + \ldots + \\ (x_{m+n}x_0^a + \ldots + x_{n+m}x_{m-1}^a)]^{-(n+1)}\Bigg\}.$$

Since φ is a non positive function, we obtain that

$$(\varphi - \psi_{0,m})([1, 0^{[m-1]}], [1; 0^{[mn]}]) = \varphi([1, 0^{[m-1]}], [1; 0^{[mn]}]) \leq 0. \tag{20}$$

On the other hand, the identities $\varphi(R_0) = 0$ and $\psi_{0,m} \leq 0$ yield

$$(\varphi - \psi_{0,m})(R_0) \geq 0. \tag{21}$$

If $R_0 \neq ([1, 0^{[m-1]}], [1; 0^{[mn]}])$, then $\psi_{0,m}(R_0) < 0$, and inequality (21) is strict. If $R_0 = ([1, 0^{[m-1]}], [1; 0^{[mn]}])$, we can choose another point

R in the neighborhood of R_0, such that $(\varphi - \psi_{0,m})(R) > 0$. Indeed, if $(\varphi - \psi_{0,m}) \leq 0$ in any neighborhood of R_0, then, since $(\varphi - \psi_{0,m})(R_0) = 0$, $(\varphi - \psi_{0,m})$ reaches a local maximum local at R_0, and this contradicts the admissibility of φ at this point (recall that $\partial_{\lambda\overline{\mu}}(\varphi - \psi_{0,m})(R_0) = (g_{\lambda\overline{\mu}} + \partial_{\lambda\overline{\mu}}\varphi)(R_0))$. In conclusion, we deduce that there exists a point R_0' given by

$$([1, a_1, \ldots, a_{m-1}], [1; a_{m+1}(1, a_1^a, \ldots, a_{m-1}^a); \ldots; a_{m+n}(1, a_1^a, \ldots, a_{m-1}^a)])$$

satisfying

$$(\varphi - \psi_{0,m})(R_0') > 0. \tag{22}$$

By the continuity and G-invariance of φ, we have the additional conditions $1 > a_1 > \ldots > a_{m-1} > 0$ and $\zeta_0 > a_{m+1} > \ldots > a_{m+n} > 0$. On the other hand, the inequality (19), as well as the definitions of R_1, $\psi_{0,m}$, ψ_1, and $\psi = \inf(\psi_1, \psi_2)$ imply that

$$(\varphi - \psi_{0,m})(R_1) = (\varphi - \psi_1)(R_1) \leq (\varphi - \psi)(R_1) < 0. \tag{23}$$

Consider now the curve

$$\begin{aligned}[0,1] \ni t \to c(t) = (&[1, t, t^{(\ln a_2)/(\ln a_1)}, \ldots, t^{(\ln a_{m-1})/(\ln a_1)}],\\ &[1; \zeta_0 t^{\frac{\ln(a_{m+1}/\zeta_0)}{\ln a_1}}, \zeta_0 t^{\frac{\ln(a_{m+1}a_1^a/\zeta_0)}{\ln a_1}}, \ldots, \zeta_0 t^{\frac{\ln(a_{m+1}a_{m-1}^a/\zeta_0)}{\ln a_1}}; \ldots;\\ &\zeta_0 t^{\frac{\ln(a_{m+n}/\zeta_0)}{\ln a_1}}, \zeta_0 t^{\frac{\ln(a_{m+n}a_1^a/\zeta_0)}{\ln a_1}}, \ldots, \zeta_0 t^{\frac{\ln(a_{m+n}a_{m-1}^a/\zeta_0)}{\ln a_1}}]).\end{aligned}$$

It is easy to verify that this is a curve in M and that, because of our assumption, all its components are positive . We have that $c(0) = ([1, 0^{[m-1]}], [1; 0^{[nm]}])$, $c(a_1) = R_0'$ and, finally, $c(1) = R_1$. At these points, using respectively (20), (22) and (23), we deduce that $(\varphi - \psi_{0,m})$ is respectively negative, positive, and negative. The invariance by $\exp(i\theta)$ allows us to deduce that $(\varphi - \psi_{0,m})$ reaches a maximum on the holomorphic curve given by the complexified version of the above described curve. This is in contradiction with the admissibility of φ.

- $\underline{u_{m+1} > \zeta_0}$.

In this case, we need another auxiliary function, given by

$$\begin{aligned}\psi_{0,m+1} &= \ln \frac{x_0^{m-an}}{(x_0+\ldots+x_{m-1})^{m-an}} \times \\ &\quad (x_0^a x_{m+1})^{n+1}[x_m + (x_{m+1}x_0^a + \ldots + x_{m+1}x_{m-1}^a) + \ldots \\ &\quad +(x_{m+n}x_0^a + \ldots + x_{n+m}x_{m-1}^a)]^{-(n+1)}.\end{aligned}$$

We have

$$(\varphi - \psi_{0,m+1})(R_0) > 0\,. \tag{24}$$

By the continuity of $(\varphi-\psi_{0,m+1})$, we can assume, as in the preceding sub-case, that there is a point R_0' whose components a_i are strictly positive and close to the u_i. For $i \in \{0,\ldots,m-1\}, k \in \{1,\ldots,n\}$, let us set $\beta_{k,i} = \frac{\ln(a_{m+k}a_i^a/\zeta_0)}{\ln a_1}$ where $a_0 = 1$. The conditions we chose (as allowed by the G-invariance of the functions), that is, $1 > a_1 > \ldots > a_{m-1}$ and $a_{m+1} > \ldots > a_{m+n}$, show that $\forall k, i, -\beta_{k,i} \leq -\beta_{1,0} = -\frac{\ln(a_{m+1}/\zeta_0)}{\ln a_1}$. On the other hand, the condition $u_{m+1} > \zeta_0$ (near a_{m+1}) shows that at least $-\beta_{1,0}$ is positive. Setting

$$\begin{aligned}R_\varepsilon &= c(\varepsilon) \\ &= ([1,\varepsilon,\varepsilon^{(\ln a_2)/(\ln a_1)},\ldots,\varepsilon^{(\ln a_{m-1})/(\ln a_1)}],[1;\zeta_0\varepsilon^{\frac{\ln(a_{m+1}/\zeta_0)}{\ln a_1}}, \\ &\quad \zeta_0\varepsilon^{\frac{\ln(a_{m+1}a_1^a/\zeta_0)}{\ln a_1}},\ldots,\zeta_0\varepsilon^{\frac{\ln(a_{m+1}a_{m-1}^a/\zeta_0)}{\ln a_1}};\ldots;\zeta_0\varepsilon^{\frac{\ln(a_{m+n}/\zeta_0)}{\ln a_1}}, \\ &\quad \zeta_0\varepsilon^{\frac{\ln(a_{m+n}a_1^a/\zeta_0)}{\ln a_1}},\ldots,\zeta_0\varepsilon^{\frac{\ln(a_{m+n}a_{m-1}^a/\zeta_0)}{\ln a_1}}])\end{aligned}$$

we have that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\psi_{0,m+1}(R_\varepsilon) = \lim_{\varepsilon\to 0}\ln\Bigg\{ \\ &\quad \frac{1}{(1+\varepsilon^2+\varepsilon^{(2\ln a_2)/(\ln a_1)}+\ldots+\varepsilon^{(2\ln a_{m-1})/(\ln a_1)})^{m-an}} \times \\ &\quad \frac{\zeta_0^{2(n+1)}\varepsilon^{2(n+1)\beta_{1,0}}}{[1+\zeta_0^2\varepsilon^{2\beta_{1,0}}+\ldots\zeta_0^2\varepsilon^{2\beta_{1,m-1}}+\ldots+\zeta_0^2\varepsilon^{2\beta_{n,0}}+\ldots\zeta_0^2\varepsilon^{2\beta_{n,m-1}}]^{n+1}}\Bigg\} \\ &= \ln\lim_{t\to\infty}\frac{t^{2(n+1)(-\beta_{1,0})}}{[1+t^{2(n+1)(-\beta_{1,0})}+\ldots+t^{2(n+1)(-\beta_{n,m-1})}]^{n+1}} \\ &= \ln 1 = 0,\end{aligned}$$

$(-\beta_{1,0})$ being the larger of the positive powers in the fraction above. Since $\varphi(R_\varepsilon) \leq 0$, taking into account (24), we deduce that there exists ε_0 such that

$$(\varphi - \psi_{0,m+1})(R_{\varepsilon_0}) \leq -\psi_{0,m+1}(R_{\varepsilon_0}) < (\varphi - \psi_{0,m+1})(R_0). \qquad (25)$$

On the other hand, the inequality (19), and the definitions of R_1, $\psi_{0,m+1}$, ψ_2 and $\psi = \inf(\psi_1, \psi_2)$ imply that

$$(\varphi - \psi_{0,m+1})(R_1) = (\varphi - \psi_2)(R_1) \leq (\varphi - \psi)(R_1) < 0. \qquad (26)$$

By virtue of (25), (24) and (26), we deduce that $(\varphi - \psi_{0,m+1})$ reaches a local maximum on the curve

$$\begin{aligned}[\varepsilon_0, 1] \ni t \to c(t) = ([1, t, t^{(\ln a_2)/(\ln a_1)}, \ldots, t^{(\ln a_{m-1})/(\ln a_1)}],\\ [1; \zeta_0 t^{\frac{\ln(a_{m+1}/\zeta_0)}{\ln a_1}}, \zeta_0 t^{\frac{\ln(a_{m+1}a_1^a/\zeta_0)}{\ln a_1}}, \ldots, \zeta_0 t^{\frac{\ln(a_{m+1}a_{m-1}^a/\zeta_0)}{\ln a_1}}; \ldots;\\ \zeta_0 t^{\frac{\ln(a_{m+n}/\zeta_0)}{\ln a_1}}, \zeta_0 t^{\frac{\ln(a_{m+n}a_1^a/\zeta_0)}{\ln a_1}}, \ldots, \zeta_0 t^{\frac{\ln(a_{m+n}a_{m-1}^a/\zeta_0)}{\ln a_1}}])\end{aligned}$$

(because $c(\varepsilon_0) = R_{\varepsilon_0}$, $c(a_1) = R_0$ and $c(1) = R_1$). This is in contradiction with the admissibility of φ.

Case B : $u_m = 0$. In this case, we work in the domain of the chart of M, given by $\{z_0 \neq 0, z_{m+1} \neq 0\}$, where the points are written as

$$\begin{aligned}([1, z_1, \ldots, z_{m-1}], [z_m; (1, z_1^a, \ldots, z_{m-1}^a); z_{m+2}(1, z_1^a, \ldots, z_{m-1}^a); \ldots;\\ z_{m+n}(1, z_1^a, \ldots, z_{m-1}^a)]).\end{aligned}$$

Then, the point R_0 where φ reaches its maximum (equal to zero) can be written as

$$\begin{aligned}R_0 = ([1, u_1, \ldots, u_{m-1}], [0; (1, u_1^a, \ldots, u_{m-1}^a);\\ u_{m+2}(1, u_1^a, \ldots, u_{m-1}^a); \ldots; u_{m+n}(1, u_1^a, \ldots, u_{m-1}^a)]).\end{aligned}$$

Using the G-invariance of φ, we can also assume that $1 \geq u_1 \geq \ldots \geq u_{m-1}$ and $1 \geq u_{m+2} \geq \ldots \geq u_{m+n}$. We shall prove an equivalent version of lemma 2.4, that is

$$(\varphi - \psi)([1^{[m]}], [\zeta, 1^{[nm]}]) \geq 0 \qquad (27)$$

for all $\zeta > 0$.

Proceeding by contradiction, assume there exists a point

$$R_{m+1} = ([1^{[m]}], [\zeta_0; 1^{[nm]}])$$

of M with $\zeta_0 > 0$ and

$$(\varphi - \psi)(R_{m+1}) < 0. \tag{28}$$

Consider the auxiliary function $\psi_{0,m+1}$ introduced above. Since φ is negative, we obtain that

$$\begin{aligned}(\varphi - \psi_{0,m+1})([1, 0^{[m-1]}], [0; 1, 0^{[mn-1]}]) \\ = \varphi([1, 0^{[m-1]}], [0; 1, 0^{[mn-1]}]) \leq 0.\end{aligned} \tag{29}$$

On the other hand, since $\varphi(R_0) = 0$ and $\psi_{0,m+1} \leq 0$,

$$(\varphi - \psi_{0,m+1})(R_0) = -\psi_{0,m+1}(R_0) \geq 0, \tag{30}$$

this inequality being strict as soon as

$$R_0 \neq ([1, 0^{[m-1]}], [0; 1, 0^{[mn-1]}])\,.$$

If $R_0 = ([1, 0^{[m-1]}], [0; 1, 0^{[mn-1]}])$, it suffices to consider a point close to R_0 on which the inequality is strict. Indeed, when $\varphi - \psi_{0,m+1} \leq 0$ in a neighborhood of R_0, then $\varphi - \psi_{0,m+1}$ admits a local maximum at R_0, which is in contradiction with the admissibility of φ at R_0. So, as in case A, there exists a point

$$\begin{aligned}R'_0 = ([1, c_1, \ldots, c_{m-1}], [c_m; (1, c_1^a, \ldots, c_{m-1}^a); \\ c_{m+2}(1, c_1^a, \ldots, c_{m-1}^a); \ldots; c_{m+n}(1, c_1^a, \ldots, c_{m-1}^a)])\end{aligned}$$

satisfying

$$(\varphi - \psi_{0,m+1})(R'_0) > 0. \tag{31}$$

By the continuity and G-invariance of φ, and since c_m is close to $u_m = 0$, we can assume that $\zeta_0 > c_m > 0$, $1 > c_1 > \ldots > c_{m-1} > 0$ and $1 > c_{m+2} > \ldots > c_{m+n} > 0$. On the other hand, the inequality (28) and the definitions of R_{m+1}, $\psi_{0,m+1}$, ψ_2, and $\psi = \inf(\psi_1, \psi_2)$ imply that

$$(\varphi - \psi_{0,m+1})(R_{m+1}) = (\varphi - \psi_2)(R_{m+1}) \leq (\varphi - \psi)(R_{m+1}) < 0. \tag{32}$$

We now introduce another curve γ on M, defined by

$$\begin{aligned}
[0,1] \ni t \to \gamma(t) = ([1, t, t^{(\ln c_2)/(\ln c_1)}, \dots, t^{(\ln c_{m-1})/(\ln c_1)}],\\
[\zeta_0 t^{\frac{\ln(c_m/\zeta_0)}{\ln c_1}}; (1, t^a, t^{(\ln c_2^a)/(\ln c_1)}, \dots, t^{(\ln c_{m-1}^a)/(\ln c_1)});\\
(t^{(\ln c_{m+2})/(\ln c_1)}, t^{(\ln c_{m+2} c_1^a)/(\ln c_1)}, \dots, t^{(\ln c_{m+2} c_{m-1}^a)/(\ln c_1)}); \dots;\\
(t^{(\ln c_{m+n})/(\ln c_1)}, t^{(\ln c_{m+n} c_1^a)/(\ln c_1)}, \dots, t^{(\ln c_{m+n} c_{m-1}^a)/(\ln c_1)})]).
\end{aligned}$$

All the exponents appearing in this curve are positive, so that $\gamma(0) = ([1, 0^{[m-1]}], [0; 1, 0^{[nm-1]}])$, $\gamma(c_1) = R_0$ and $\gamma(1) = R_{m+1}$. Then, by (29), (31) and (32), we deduce that $(\varphi - \psi_{0,m+1})$ is respectively negative, positive and negative. Again, the invariance by $\exp(i\theta)$ allows us to conclude that $(\varphi - \psi_{0,m+1})$ reaches a maximum on the holomorphic curve given by the complexified version of γ. This is in contradiction with the admissibility of φ. It follows that (27) holds and lemma 2.4 is proven. □

2.1. Proof of Theorem 1.4

Let $\varphi \in C^\infty(M)$ be a g-admissible and G-invariant function with a null supremum on M. According to theorem 1.2, $\varphi \geq \psi$; therefore, for all $\alpha \geq 0$,

$$\int_M \exp(-\alpha\varphi) dv \leq \int_M \exp(-\alpha\psi) dv.$$

To obtain the values of α for which the last integral converges, we estimate $\int_M \exp(-\alpha\psi_1) dv$ and $\int_M \exp(-\alpha\psi_2) dv$. Indeed,

$$\begin{aligned}
\int_M \exp(-\alpha\psi) dv &= \int_{\psi_1 \leq \psi_2} \exp(-\alpha\psi) dv + \int_{\psi_2 \leq \psi_1} \exp(-\alpha\psi) dv\\
&= \int_{\psi_1 \leq \psi_2} \exp(-\alpha\psi_1) dv + \int_{\psi_2 \leq \psi_1} \exp(-\alpha\psi_2) dv\\
&\leq \int_{\psi_1 \leq \psi_2} \exp(-\alpha\psi_1) dv + \int_{\psi_2 \leq \psi_1} \exp(-\alpha\psi_2) dv\\
&\leq \int_M \exp(-\alpha\psi_1) dv + \int_M \exp(-\alpha\psi_2) dv\,,
\end{aligned}$$

and

$$\int_M \exp(-\alpha\psi_1) dv + \int_M \exp(-\alpha\psi_2) dv \leq 2 \int_M \exp(-\alpha\psi) dv\,.$$

We mention that we can avoid the very hard computation of the element volume dv (or equivalently of $\det(g)$), by means of the following remark. If we write $g_{\lambda\overline{\mu}}$ in the form $g_{\lambda\overline{\mu}} = \partial_{\lambda\overline{\mu}} \log K$, the quantity $[K \det(g)]$ is intrinsic since we chose the metric g in $c_1(M)$ (same proof as in proposition 1.1). Thus, we can deduce that there exist two constants C_1 and C_2, such that

$$\frac{C_1}{K} \leq \det(g) \leq \frac{C_2}{K}\,.$$

Using the preceding notations (with $d = m + n - 1$), and setting

$$r = x_1 + \ldots + x_m, \quad s = 1 + (x_1^a + \ldots + x_m^a) \times (1 + x_{m+1} + \ldots + x_d),$$

we obtain that

$$dv \simeq \frac{C dx_1 \wedge \ldots \wedge dx_d}{r^{m-an} s^{n+1}}.$$

Then,

$$\begin{aligned} I_1 &= \int_M \exp(-\alpha\psi_1) dv \\ &\simeq \int_{\mathbb{R}_+^d} \frac{dx_1 \wedge \ldots \wedge dx_d}{(x_1 \ldots x_m)^{\frac{d}{m}(m-an)} r^{(m-an)(1-\alpha)} s^{(n+1)(1-\alpha)}}, \end{aligned}$$

which converges for $\alpha < \frac{1}{n+1}$, and

$$\begin{aligned} I_2 &= \int_M \exp(-\alpha\psi_2) dv \\ &\simeq \int_{\mathbb{R}_+^d} \frac{dx_1 \wedge \ldots \wedge dx_d}{(x_1 \ldots x_m)^{\alpha\frac{m+a}{m}} (x_{m+1} \ldots x_d)^{\alpha\frac{n+1}{n}} r^{(m-an)(1-\alpha)} s^{(n+1)(1-\alpha)}}, \end{aligned}$$

which converges for $\alpha < \frac{n}{n+1}$.

In conclusion, $\int_M \exp(-\alpha\psi) dv$ exists for $\alpha < 1/(n+1)$.

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