

COMPARISON OF HYPERTOPOLOGIES (*)

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SOMMARIO. - Sia $CL(X)$ la famiglia di tutti i sottoinsiemi chiusi non vuoti di uno spazio di Tychonoff X con una base d'uniformità compatibile \mathcal{V} e una prossimità compatibile δ . In questo lavoro si studiano le relazioni esistenti tra varie topologie di $CL(X)$, dette ipertopologie, e cioè: le ipertopologie di Fell, Wijsman, della palla prossimale, della palla, prossimale, localmente finita, prossimale localmente finita, di Hausdorff, di Vietoris, etc. Benchè il lavoro contenga un buon numero di risultati nuovi, esso si presenta anche come un lavoro di rassegna. La ricerca delle condizioni sotto le quali le suddette ipertopologie sono a due a due uguali, produce interessanti caratterizzazioni di proprietà topologiche ed uniformi di X . Alcune di queste proprietà sono la compattezza, la pseudocompattezza, la totale limitatezza, l'equinormalità, etc. Questi risultati generalizzano alcuni dei risultati contenuti in un recente lavoro di Beer, Lechicki, Levi e Naimpally intitolato "Distance functionals and suprema of hyperspace topologies".

SUMMARY. - Let $CL(X)$ denote the family of all nonempty closed subsets of a Tychonoff space X with a compatible uniformity base \mathcal{V} and a compatible proximity δ . In this paper a study is made of the relationships that exist among various topologies on $CL(X)$, called hypertopologies, viz: Fell, Wijsman, Proximal ball, Ball, Proximal, Locally finite, Proximal locally finite, Hausdorff, Vietoris, etc. Although the paper contains several new results, it is also a survey. Investigations of conditions under which the above hypertopologies are pairwise equal, yield interesting characterizations of topological and uniform properties of X . Some of these properties are compactness, pseudo-compactness, total boundedness, equinormality, etc. These results generalize some of the results contained in the recent paper "Distance functionals and suprema of hyperspace topologies" by Beer, Lechicki, Levi and Naimpally.

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1. Introduction. Let (X, τ_0) be a Tychonoff space with a compatible uniformity generated by a family of pseudometrics \mathcal{P} . For each $d \in \mathcal{P}$ and each $\varepsilon > 0$, we set

$$U(d, \varepsilon) = \{(x, y) \in X \times X : d(x, y) < \varepsilon\},$$

$$V(d, \varepsilon) = \{(x, y) \in X \times X : d(x, y) \leq \varepsilon\}.$$

Thus $\mathcal{U} = \{U(d, \varepsilon) : d \in \mathcal{P}, \varepsilon \in \mathbb{Q}^+\}$ and $\mathcal{V} = \{V(d, \varepsilon) : d \in \mathcal{P}, \varepsilon \in \mathbb{Q}^+\}$ are respectively open and closed bases for the uniformity generated by \mathcal{P} . In case X is metrizable with a compatible metric d , we choose $\mathcal{P} = \{d\}$ and clearly \mathcal{U} and \mathcal{V} are countable.

Let $\delta = \delta(\mathcal{U}) = \delta(\mathcal{V})$ denote the (EF) -proximity on X induced by \mathcal{U} or \mathcal{V} viz:

$$A\delta B \text{ iff for each } U \in \mathcal{U}, U[A] \cap B \neq \emptyset.$$

In the metric case, we set

$$d(A, B) = \inf \{d(a, b) : a \in A, b \in B\}$$

and note that

$$A\delta B \text{ iff } d(A, B) = 0.$$

We also write $A \ll B$ for $A \not\delta (X - B)$.

In addition to δ , we also consider the LO -proximity δ_0 on X , where

$$A\delta_0 B \text{ iff } Cl A \cap Cl B \neq \emptyset.$$

By Urysohn's Lemma, δ_0 is EF iff X is normal.

Let $CL(X)$ denote the family of all nonempty closed subsets of X . For $A \subset X$ and $\mathcal{Q} \subset P(X)$ we set:

$$A^- = \{F \in CL(X) : F \cap A \neq \emptyset\}.$$

$$A^+ = \{F \in CL(X) : F \subset A\}.$$

$$A^{++} = \{F \in CL(X) : F \ll A\}.$$

$$\mathcal{Q}^- = \{F \in CL(X) : F \cap A \neq \emptyset \text{ for each } A \in \mathcal{Q}\}.$$

$$\mathcal{Q}^+ = \{F \in CL(X) : F \subset \cup \mathcal{Q}\}.$$

$$\mathcal{Q}^{++} = \{F \in CL(X) : F \ll \cup \mathcal{Q}\}.$$

Let K denote a compact subset of X . $CL(X)$ is often called a *hyperset* and any topology on $CL(X)$ is called a *hypertopology*. In this paper, we study the relationships that exist among the various hypertopologies on $CL(X)$ and their relationships with the topological, uniform and proximal properties of X .

Our references are: for topology [Ke], [WI]; for proximity [NW]; for hypertopologies [Mi], [BL] and [BLLN] where further references will be found.

2. Fell Hypertopology $\tau_F = \tau_{F^-} \vee \tau_{F^+}$.

Here τ_{F^-} is generated by the subbase $\{V^- : V \in \tau_0\}$ and τ_{F^+} is generated by the subbase $\{W^+ : W \in \tau_0 \text{ with } W^c \text{ compact}\}$. It is known that $(CL_0(X), \tau_F)$, where $CL_0(X)$ is the family of all closed subsets of X , is always compact and it is Hausdorff iff (X, τ_0) is locally compact. (see [Fe]). We note that τ_F depends only on τ_0 and not on \mathcal{U} or \mathcal{V} .

3. Wijsman Hypertopology $\tau_W = \tau_{W^-} \vee \tau_{W^+}$.

Originally $\tau_W = \tau_W(d)$ was defined on a metric space (X, d) as the weakest topology on $CL(X)$ such that for each $x \in X$, the map $A \rightarrow d(x, A)$ from $CL(X) \rightarrow \mathbf{R}$ is continuous. In terms of convergence, we can split τ_W into τ_{W^-} and τ_{W^+} .

(3.1) $A = (\tau_{W^-})\text{-lim } A_n$ iff for each $x \in X, \varepsilon > 0$ if $A \cap S_\varepsilon(x) \neq \emptyset$, then eventually $A_n \cap S_\varepsilon(x) \neq \emptyset$. It turns out that $\tau_{W^-} = \tau_{F^-}$.

(3.2) $A = (\tau_{W^+})\text{-lim } A_n$ iff for each $x \in X, \varepsilon > r > 0$ if $A \cap S_\varepsilon(x) = \emptyset$, then eventually $A_n \cap S_r(x) = \emptyset$.

In (3.2), one may use closed balls for open spheres. To generalize the Wijsman convergence to the uniform case, we set $\tau_{W^-} = \tau_{F^-}$ or replacing $S_\varepsilon(x)$ by $U(x)$ where $U \in \mathcal{U}$. In order to extend the τ_{W^+} convergence to the uniform case we need the following

(3.3) DEFINITION. For each $V \in \mathcal{V}$, we say that $V' \in \mathcal{V}$ is *composably contained* in V iff there is a $V'' \in \mathcal{V}$ such that $V' \circ V'' \subset V$.

In the sequel, (A_n) always denotes a net of elements $A_n \in CL(X)$ with n in a directed set D .

(3.4) $A = (\tau_{W^+})$ -lim A_n iff for each $x \in X, V \in \mathcal{V}$ and V' composably contained in V , if $A \cap V(x) = \emptyset$, then eventually $A_n \cap V'(x) = \emptyset$.

(3.5) $A = (\tau_W)$ -lim A_n iff $A = (\tau_{W^+})$ -lim A_n and $A = (\tau_{W^-})$ -lim A_n , (See [Wi]).

Alternatively, τ_W can be defined as the weakest topology τ on $CL(X)$ such that for each $x \in X, d \in \mathcal{P}, A \rightarrow d(x, A)$ is continuous. Thus τ_W is Tychonoff. In [LL] it is shown that even uniformly equivalent metrics on X may give rise to different plus Wijsman topologies. They have also shown that $\tau_W(d)^+ = \tau_W(\rho)^+$ for uniformly equivalent metrics if each d -ball $\neq X$ is totally bounded. It easy to extend this result to uniform spaces.

(3.6) REMARKS. Let (X, d) be a metric space and $d_M = \min\{d, M\}$, $M > 0$ be the uniformly equivalent metric on X . If d is unbounded, then d_M has fewer spheres than d and so $\tau_W(d_M) \subset \tau_W(d)$ and they are equal iff d is B - TB ([LL]). Hence

$$\tau_W(d) = \sup\{\tau_W(d_M) : M > 0\}.$$

4. The Ball Hypertopology (see [FLL], [Be]).

Here $\tau_{B^-} = \tau_{W^-} = \tau_{F^+}$ and τ_{B^+} is generated by $\{[V(x)^c]^+ : V \in \mathcal{V}, x \in X\}$. $\tau_B = \tau_B(d) = \tau_{B^-} \vee \tau_{B^+}$. It follows from [LL] that $\tau_B(d)^+ = \tau_B(\rho)^+$ for two uniformly equivalent metrics if each d -ball $\neq X$ is totally bounded. This result can be extended to uniformities.

5. Vietoris Hypertopology (see [Vi]).

Here $\tau_{V^-} = \tau_{B^-} = \tau_{W^-} = \tau_{F^-}$ and τ_{V^+} is generated by $\{G^+ : G \in \tau_0\}$. $\tau_V = \tau_{V^-} \vee \tau_{V^+}$ depends only on τ_0 and not on \mathcal{U} or \mathcal{V} . τ_V , discovered

by Vietoris in the early twenties, is one of the earliest hypertopologies along with the Hausdorff metric hypertopology (see Section 11) and has been extensively studied, see [Mi], [Ke], [BL].

6. Locally finite hypertopology.

τ_{LF^-} is generated by $\{\mathcal{Q}^-$ where \mathcal{Q} is a locally finite subfamily of $\tau_0\}$. $\tau_{LF^+} = \tau_{V^+}$. $\tau_{LF} = \tau_{LF^-} \vee \tau_{V^+}$ depends upon τ_0 only and not on \mathcal{U} or \mathcal{V} . This has been studied by [Ma], [BHPV], [NS], [DNS]. It is known that (X, τ_0) is normal iff its fine uniformity generates τ_{LF} via its Hausdorff uniformity ([NS]). In the case of a metric space (X, d) , τ_{LF} is the sup of all Hausdorff metric topologies corresponding to compatible metrics on X ([BHPV]).

7. Fell, Wijsman, Ball, Vietoris, locally finite hypertopologies.

The following is true:

(7.1) THEOREM. $\tau_F \subset \tau_W \subset \tau_B \subset \tau_V \subset \tau_{LF}$.

Proof. Each of the above inclusions, except perhaps the first one, is obvious. To show that $\tau_F \subset \tau_W$ we need only to prove $\tau_{F^+} \subset \tau_{W^+}$. Suppose $A = (\tau_{W^+})\text{-lim } A_n$ and $A \in [K^c]$, K compact. Then there is a $V \in \mathcal{V}$ such that $V^2[A] \cap K = \emptyset$ and since K is compact, $K \subset \{\cup V(x_i) : 1 \leq i \leq n\}$, $x_i \in K$. Since $A = (\tau_{W^+})\text{-lim } A_n$ and $A \cap V^2[K] = \emptyset$, eventually $A_n \cap V(x_i) = \emptyset$ for each i , $1 \leq i \leq n$. Hence eventually $A_n \in [K^c]^+$ i.e. $A = (\tau_{F^+})\text{-lim } A_n$.

We now provide examples to show that every inclusion can be strict.

(7.2) EXAMPLE. $\tau_F \neq \tau_W$.

$X = 1_\infty$, $A = \{2e_1\}$, $A_n = \{2e_1, e_n\}$. $d(\theta, A) = 2$, $d(\theta, A_n) = 1$ and so $A \neq (\tau_W)\text{-lim } A_n$. Suppose $A \in \{\cap G_i^- : 1 \leq i \leq m\} \cap [K^c]^+$, where $G_i \in \tau_0$, K is compact. Then $A_n \in (\cap G_i^- : 1 \leq i \leq m)$. Since

$\{e_n\}$ has no cluster point, $\{e_n\}$ is not frequently in K i.e. it is eventually in K^c and so $A_n \in [K^c]^+$ eventually. Thus $A = (\tau_F)\text{-lim } A_n$.

(7.3) EXAMPLE. $\tau_W \neq \tau_B$. ([Be])

$X = \{\theta\} \cup F \cup \{e_1 + e_n : n > 1\} \subset 1_\infty$, where $F = \{(n+1)/n e_1 + 1/2 e_n : n > 1\}$.

$F_n = F \cup \{e_1 + e_k : k > n\}$. $F \cap B_1(\theta) = \emptyset$, but $F_n \cap B_1(\theta) \neq \emptyset$, so $F \neq (\tau_B)\text{-lim } F_n$. But $F = (\tau_W)\text{-lim } F_n$.

(7.4) EXAMPLE. $\tau_B \neq \tau_V$.

$X = \mathbf{R}$, $A = \{0\}$, $A_n = \{0, n\}$. Here $A = (\tau_B)\text{-lim } A_n$, but $A \neq (\tau_V)\text{-lim } A_n$.

(7.5) EXAMPLE. $\tau_V \neq \tau_{LF}$.

$X = \mathbf{R}$. In every τ_V -nbhd of $X \in CL(X)$, there is a finite subset of X . On the other hand if \mathcal{Q} is an infinite locally finite open cover of X which has no finite subcover, then there is no finite set that is in \mathcal{Q}^- which is a τ_{LF} -nbhd of X .

8. Proximal Ball Hypertopology $\tau_{B\delta} = \tau_{B\delta}(\mathcal{V})$.

This is a new hypertopology patterned after τ_B . We set $\tau_{B\delta^-} = \tau_{F^-} = \tau_{W^-} = \tau_{B^-} = \tau_{V^-}$ and $\tau_{B\delta^+}$ is generated by $\{[V(x)^c]^{++} : x \in X, V \in \mathcal{V}\}$. Clearly, $\tau_{B\delta}$ depends upon \mathcal{V} , a fact that will be pursued in some detail here.

If (X, d) is a metric space, and for $M > 0$, $d_M = \min\{d, M\}$ is the uniformly equivalent bounded metric on X , then from [LL] and Section 13 it follows:

- (8.1) $\tau_W(d_M) \subset \tau_W(d) \subset \tau_{B\delta}(d) \subset \tau_\delta$.
- (8.2) $\tau_W(d_M) \subset \tau_{B\delta}(d_M) \subset \tau_{B\delta}(d) \subset \tau_\delta$.
- (8.3) $\tau_W(d) \not\subset \tau_{B\delta}(d_M)$ for some $M > 0$ if d is not $B\text{-}TB$.
- (8.4) $\tau_{B\delta}(d) = \{\sup \tau_{B\delta}(d_M) : M > 0\}$.
- (8.5) If d is $B\text{-}TB$, then $\tau_W(d_M) = \tau_W(d) = \tau_{B\delta}(d_M) = \tau_{B\delta}(d)$.
- (8.6) If for all $M > 0$, $\tau_W(d_M) = \tau_W(d)$ or $\tau_{B\delta}(d_M) = \tau_{B\delta}(d)$, then d is $B\text{-}TB$.

9. Proximal Hypertopology $\tau_\delta = \tau_\delta(\mathcal{V}) = \tau_{\delta^-} \vee \tau_{\delta^+}$.

This was introduced in [Na] and then studied in [BLLN], [DNS] and [BDN]. We set $\tau_{\delta^-} = \tau_{B\delta^-} = \tau_{F^-} = \tau_{W^-} = \tau_{B^-} = \tau_{V^-}$ and τ_{δ^+} is generated by $\{G^{++} : G \in \tau_0\}$. Clearly τ_δ is the same for all proximally equivalent uniformities. We note that $\tau_V = \tau_{\delta_0}$ and if $\delta_1 < \delta_2$, then $\tau_{\delta_1} \subset \tau_{\delta_2}$ if δ_1 is *EF* ([DNS]).

In [BLLN] it was shown that $\tau_\delta(d)$ is the sup of all $\tau_W(\rho)$ where ρ is uniformly equivalent to d . From Section 8 it follows that

$$(9.1) \quad \tau_\delta(d) = \sup \{\tau_{B\delta}(\rho) : \rho \simeq d\}.$$

10. Fell, Wijsman, Proximal Ball, Proximal, Vietoris.

The following is true:

$$(10.1) \text{ THEOREM. } \tau_F \subset \tau_W \subset \tau_{B\delta} \subset \tau_\delta \subset \tau_V \subset \tau_{LF}.$$

Proof. We need prove only $\tau_W \subset \tau_{B\delta}$ and $\tau_\delta \subset \tau_V$.

(i) Suppose $A = (\tau_{B\delta^+})\text{-}\lim A_n$, $A \cap V(x) = \emptyset$ and $V' \circ V'' \subset V$. $A \in [V''(x)^c]^{++}$ and so eventually $A_n \in [V''(x)^c]^{++}$ i.e. $A_n \cap V''(x) = \emptyset$ i.e. $A = (\tau_{W^+})\text{-}\lim A_n$.

(ii) Suppose $A \in G^{++} \in \tau_\delta$, $G \in \tau_0$ i.e. $A \ll G$. There is $G' \in \tau_0$ such that $A \ll G' \ll G$.

Then $A \in G'^+ \subset G^{++}$ and so $G^{++} \in \tau_V$.

We now provide examples to show that every inclusion can be strict.

$$(10.2) \text{ EXAMPLE. } \tau_W \neq \tau_{B\delta}$$

$$X = \{\theta\} \cup \{(n+1)/n e_{2n} : n \in \mathbb{N}\} \cup \{(n+1)/n e_{2n+1} : n \in \mathbb{N}\} \cup \\ \cup \{e_{2n+1} : n \in \mathbb{N}\} \subset 1_2.$$

$$A = \{(n+1)/n e_{2n} : n \in \mathbb{N}\}, A_n = \{(k+1)/k e_{2k+1} : k \geq n\} \cup A.$$

$$D(A, B_1(\theta)) = 1 \text{ and so } A \in [B_1(\theta)^c]^{++}.$$

But $D(A_n, B_1(\theta)) = 0$ and so $A_n \notin [B_1(\theta)^c]^{++}$, i.e. $A \neq (\tau_{B\delta^+})\text{-}\lim A_n$.

However, $A_n \cap B_1(\theta) = \emptyset$ for each $n \in \mathbf{N}$ and it can be verified that $A = (\tau_w)\text{-lim } A_n$. We also note that $A = (\tau_B)\text{-lim } A_n$.

(10.3) EXAMPLE. $\tau_{B\delta} \neq \tau_\delta$.

$X = \mathbf{R}$, $A = \{0\}$, $A_n = \{0, n\}$. Here $A = (\tau_{B\delta})\text{-lim } A_n$, but $A \neq (\tau_\delta)\text{-lim } A_n$.

(10.4) EXAMPLE. $\tau_\delta \neq \tau_V$.

$X = \mathbf{R}$, $A = \mathbf{N}$, $G^c = \{n - 1/n : n \in \mathbf{N}\}$. $A \in G^+$ and $G^+ \notin \tau_\delta$. Suppose not, and $A \in H^{++} \subset G^+$. But $A\delta G^c$ and $G^c \subset H^c$ and hence $A\delta H^c$, contradiction.

11. Hausdorff uniform hypertopology $\tau_{\mathcal{H}}(\mathcal{V}) = \tau_{\mathcal{H}}$.

In this case we may use either \mathcal{U} or \mathcal{V} . For each $U \in \mathcal{U}$ we set

(11.1) $\mathbf{U} = \{(A, B) \in CL(X) \times CL(X) : A \subset U[B] \text{ and } B \subset U[A]\}$, and

(11.2) $\mathbf{U} = \{U : U \in \mathcal{U}\}$

\mathbf{U} is a uniformity base on $CL(X)$ and $\tau_{\mathcal{H}}(\mathbf{U}) = \tau_{\mathcal{H}}$ is the topology induced by \mathbf{U} on $CL(X)$ and so is always Tychonoff. \mathbf{U} is called the *Hausdorff uniformity* on $CL(X)$ induced by \mathcal{U} . In terms of convergence we write $\tau_{\mathcal{H}} = \tau_{\mathcal{H}^-} \vee \tau_{\mathcal{H}^+}$, where

(11.3) $A = (\tau_{\mathcal{H}^-})\text{-lim } A_n$ iff for each $U \in \mathcal{U}$, eventually $A \subset U[A_n]$,

(11.4) $A = (\tau_{\mathcal{H}^+})\text{-lim } A_n$ iff for each $U \in \mathcal{U}$, eventually $A_n \subset U[A]$.

In case (X, τ_0) is metrizable, let \mathcal{D} denote the set of all compatible metrics on X . For $d \in \mathcal{D}$, the *Hausdorff metric* \mathcal{H}_d on $CL(X)$ is defined by

$$(11.5) \quad \begin{aligned} \mathcal{H}_d(A, B) &= \sup \{|d(x, A) - d(x, B)| : x \in X\}, \text{ or} \\ &= \inf \{\varepsilon > 0 : A \subset S_\varepsilon(B), B \subset S_\varepsilon(A)\} \\ &= \infty \text{ if no such } \varepsilon \text{ exists.} \end{aligned}$$

A comparison with Wijsman convergence shows that $A = (\tau_W)\text{-lim } A_n$ iff for each $x \in X$, $d(x, A_n) \rightarrow d(x, A)$ pointwise whereas $A = (\tau_{\mathcal{H}})\text{-lim } A_n$ iff $d(x, A_n) \rightarrow d(x, A)$ uniformly.

It is known that if d_1, d_2 are two elements of \mathcal{D} which are not uniformly equivalent, then $\tau_{\mathcal{H}}(d_1) \neq \tau_{\mathcal{H}}(d_2)$. Also if \mathcal{U}_1 is TB and \mathcal{U}_2 is not TB , then $\tau_{\mathcal{H}}(\mathcal{U}_1) \neq \tau_{\mathcal{H}}(\mathcal{U}_2)$. (See [NW]). It is also known that $\tau_{LF} = \sup\{\tau_{\mathcal{H}}(d) : d \in \mathcal{D}\}$, (See [BHPV]).

12. Proximal locally finite hypertopology $\tau_{LF\delta}$.

This was introduced in [DNS]. Here

$$\tau_{LF\delta^-} = \tau_{LF^-}$$

$$\tau_{LF\delta^+} = \tau_{\delta^+}.$$

13. Fell, Wijsman, proximal ball, proximal, Hausdorff, proximal locally finite, locally finite.

The following is true:

(13.1) THEOREM. $\tau_F \subset \tau_W \subset \tau_{B\delta} \subset \tau_{\delta} \subset \tau_{\mathcal{H}} \subset \tau_{LF\delta} \subset \tau_{LF}$.

All except $\tau_{\delta} \subset \tau_{\mathcal{H}}$ and $\tau_{\mathcal{H}} \subset \tau_{LF\delta}$ are obvious. We note that $\tau_{LF} = \tau_{LF\delta_0}$.

(i) $\tau_{\delta} \subset \tau_{\mathcal{H}}$. Since $A \in G^{++}$ iff there is a $U \in \mathcal{U}$ such that $U[A] \subset G$, it follows that $\tau_{\delta^+} = \tau_{\mathcal{H}^+}$. If $A \in G^-$, $A \cap G \neq \emptyset$, then there is $U \in \mathcal{U}$ such that $U[a] \subset G$ where $a \in A \cap G$. Suppose $A = (\tau_{\mathcal{H}^-})\text{-lim } A_n$, then eventually $A \subset U[A_n]$ and so eventually $A_n \cap U[a] \neq \emptyset$, i.e. $A_n \in G^-$. Hence $\tau_{\delta^-} \subset \tau_{\mathcal{H}^-}$. We note that $\tau_{\delta} = \tau_V \vee \tau_{\mathcal{H}^+}$, $\tau_V \subset \tau_{\mathcal{H}^-}$ and $\tau_{\mathcal{H}^+} \subset \tau_V$.

(ii) $\tau_{\mathcal{H}} \subset \tau_{LF\delta}$. Suppose $A = U[A] = \{B \in CL(X) : A \subset U[B] \text{ and } B \subset U[A]\} \in \tau_{\mathcal{H}}$, $U \in \mathcal{U}$. Let $U' \in \mathcal{U}$ such that $U'^3 \subset U$. By Zorn's Lemma, there exists a maximal set $Q \subset A$ such that for $x, y \in Q$, $x \neq y$ implies $(x, y) \notin U'$. Let $\mathcal{Q} = \{U'(x) : x \in Q\}$ is a discrete family of open sets. Let $G = \cup\{U'^2(x) : x \in Q\}$.

Claim $A \in \mathcal{Q}^- \cap G^{++} \subset U[A]$. (Details are in [DNS]).

We now give examples to show that every inclusion can be strict.

(13.2) EXAMPLE. $\tau_\delta \neq \tau_{\mathcal{H}}$.

$X = \mathbf{R}$, $A = \mathbf{N}$, $A_n = \{m \in \mathbf{N} \mid m \leq n\}$, $A = (\tau_\delta^-)\text{-lim } A_n$ and $A = (\tau_\delta)\text{-lim } A_n$, but $A \neq (\tau_{\mathcal{H}}^-)\text{-lim } A_n$ and $A \neq (\tau_{\mathcal{H}})\text{-lim } A_n$.

(13.3) EXAMPLE. $\tau_{\mathcal{H}} \neq \tau_{LF\delta}$.

$X = \mathbf{R}$, $A = [0, \infty)$. For each $n \in \mathbf{N}$, let A_n be a maximal $1/n$ discrete subset of A . Then $A = (\tau_{\mathcal{H}})\text{-lim } A_n$. For each $n \in \mathbf{N} \cup \{0\}$, let \mathcal{Q}_n be a finite open cover of $[n, n+1]$ each member of which has diameter less than $1/(n+1)^2$. Then $\mathcal{Q} = \cup \mathcal{Q}_n$ is a locally finite open cover of A and clearly for each n , $A_n \not\subseteq \mathcal{Q}^-$, and so $A \neq (\tau_{LF\delta})\text{-lim } A_n$. Actually we have shown that $\tau_{\mathcal{H}}^- \neq \tau_{LF\delta}^-$.

(13.4) EXAMPLE. $\tau_{LF\delta} \neq \tau_{LF}$.

Example (10.4) shows that $G^+ \notin \tau_{LF\delta}$ but $G^+ \in \tau_{LF}$.

14. Non comparable hypertopologies.

In this section we give examples to show that the following pairs are not comparable

(i) $\tau_{B\delta}$, τ_B ; (ii) τ_B , τ_δ ; (iii) τ_B , $\tau_{\mathcal{H}}$; (iv) τ_B , $\tau_{LF\delta}$; (v) τ_V , $\tau_{\mathcal{H}}$; (vi) τ_V , $\tau_{LF\delta}$.

(14.1) EXAMPLES. $\tau_{B\delta}$, τ_B .

(a) $\tau_{B\delta} \not\subseteq \tau_B$.

Example (10.2) shows that $A = (\tau_B)\text{-lim } A_n$ but $A \neq (\tau_{B\delta})\text{-lim } A_n$.

(b) $\tau_B \not\subseteq \tau_{B\delta}$ [Be].

$X = \{\theta\} \cup F \cup \{e_n : n \in \mathbf{N}\} \subset 1_\infty$, where $F = \{(n+1)/n e_n : n \in \mathbf{N}\}$, $F_n = \{(j+1)/j e_j : j \leq n\} \cup \{e_j : j > n\}$. Here F_n converges to F in $\tau_{\mathcal{H}}$, τ_δ , $\tau_{B\delta}$, τ_w but $F \neq (\tau_B)\text{-lim } F_n$. We note that since $\tau_\delta \subset \tau_V$ one would expect that $\tau_{B\delta} \subset \tau_B$ but surprizingly it is not true!

(14.2) EXAMPLES. τ_B , τ_δ .

(a) Since $\tau_{B\delta} \subset \tau_\delta$, (14.1) (a) shows $\tau_\delta \not\subseteq \tau_B$.

(b) (14.1) (b) shows $\tau_B \not\subseteq \tau_\delta$.

(14.3) EXAMPLES. $\tau_B, \tau_{\mathcal{H}}$.

(a) Since $\tau_{B\delta} \subset \tau_{\mathcal{H}}$, (14.1)(a) shows $\tau_{\mathcal{H}} \not\subset \tau_B$.

(b) (14.1) (b) shows that $\tau_B \not\subset \tau_{\mathcal{H}}$.

(14.4) EXAMPLES. $\tau_B, \tau_{LF\delta}$.

(a) Example (7.4) shows that $A = (\tau_B)\text{-lim } A_n$ but $A \neq (\tau_{LF\delta})\text{-lim } A_n$. Hence $\tau_{LF\delta} \not\subset \tau_B$. Also (7.5).

(b) We now briefly describe the space ψ (Example 1 N, Page 62, [PW]). Let \mathcal{M} a maximal infinite family of infinite subsets of \mathbf{N} such that the intersection of any two is finite.

Let $\psi = \mathbf{N} \cup \mathcal{M}$ and let $\mathcal{B} = \{(n) : n \in \mathbf{N} \cup \{M\} \cup S : M \in \mathcal{M} \text{ and } S \text{ is a cofinite subset of } M\}$ be an open base. The space ψ is pseudo-compact but zero-sets and closed sets are not separated by (bounded) real valued continuous functions and consequently is not δ -normally separated (l.c. Page 65). Hence in the uniformity \mathcal{V} generated by $C(X)$, ψ is not B -equinormal. Thus $\tau_{LF\delta} = \tau_{B\delta} \not\subset \tau_B$.

(14.5) EXAMPLES. $\tau_{\mathcal{V}}, \tau_{\mathcal{H}}$.

(a) $\tau_{\mathcal{V}} \not\subset \tau_{\mathcal{H}}$: Example (14.1) (b).

(b) $\tau_{\mathcal{H}} \not\subset \tau_{\mathcal{V}}$: Example (13.2).

(14.6) EXAMPLES. $\tau_{\mathcal{V}}, \tau_{LF\delta}$

(a) $\tau_{\mathcal{V}} \not\subset \tau_{LF\delta}$. The space T_{∞} , Example 87 [SS], or (14.4) (b).

(b) $\tau_{LF\delta} \not\subset \tau_{\mathcal{V}}$. Example (13.2) shows that $A = (\tau_{\mathcal{V}})\text{-lim } A_n$ but $A \neq (\tau_{LF\delta})\text{-lim } A_n$.

15. B-total boundedness.

(15.1) DEFINITION. \mathcal{U} is B - TB iff for each $U \in \mathcal{U}$, $U(x) \neq X$ implies $U(x)$ is TB . (We also may replace \mathcal{U} by \mathcal{V}).

Obviously $TB \Rightarrow B$ - TB but $\mathbf{Q} \subset \mathbf{R}$ is B - TB but not TB nor B -compact. If d is a pseudometric on X , we set $d_M = \min\{M, d\}$ for each $M > 0$. Then d_M is a bounded metric which is uniformly equivalent to d . From [LL] Theorems (4.1), (4.2), have the following for a metric space (X, d) .

(15.2) THEOREM. *The following assertions are pairwise equivalent*

- (a) (X, d) is B - TB .
- (b) $\tau_w(d) = \tau_w(d_M)$ for each $M > 0$.
- (c) $\tau_w(d) = \tau_w(\rho)$ for each $\rho \simeq d$.
- (d) $\tau_{B\delta}(d) = \tau_{B\delta}(d_M)$ for each $M > 0$.
- (e) $\tau_{B\delta}(d) = \tau_{B\delta}(\rho)$ for each $\rho \simeq d$.

The above result extends to uniform spaces by considering the family $\{d_M\}$ associated with $\{d : d \in \mathcal{P}\}$.

(15.3) DEFINITION. \mathcal{V} has B - SP (Strong ball separation property) iff $A \in CL(X)$, $x \in X$, $V \in \mathcal{V}$, $A \not\delta V(x)$, there exists a $V' \in \mathcal{V}$ such that $A \not\delta V' \circ V(x)$.

(15.4) (a) RIVER EXAMPLE. $X \subset \mathbb{R}^2$, $X = A \cup B$, $A = \{(x, y) : y > 1\}$, $B = \{(x, y) : y \leq 0\}$ $d(A, B_1(0, 0)) = 1 > 0$ i.e. $A \not\delta B_1(0, 0)$. But $A \cap B_{1+\varepsilon}(0, 0) \neq \emptyset$ for each $\varepsilon > 0$. So B - SP property is not satisfied but is B - TB .

(b) Every Banach space has B - SP , even if it is not B - TB (in infinite dimensional cases).

(15.5) THEOREM. If \mathcal{V} is B - TB or B - SP , then $\tau_W = \tau_{B\delta}$.

Proof. (a) It is enough to show $\tau_{w^+} = \tau_{B\delta^+}$. Let \mathcal{V} be B - TB . Suppose $A = (\tau_{w^+})\text{-lim } A_n$ and $A \in [V(x)^c]^{++}$. Then there exists a $V' \in \mathcal{V}$ such that $V'^3[A] \cap V(x) = \emptyset$. Since $V(x)$ is TB , $V(x) \subset \cup\{V'(x_i) : 1 \leq i \leq m\}$, $x_i \in V(x)$. $A \cap V'^3(x_i) = \emptyset$ implies eventually $A_n \cap V'^2(x_i) = \emptyset$, $1 \leq i \leq m$, i.e. $A_n \in [V(x)^c]^{++}$.

(b) Let \mathcal{V} be B - SP and $A = (\tau_{w^+})\text{-lim } A_n$, $A \in [V(x)^c]^{++}$. Then there is a $V' \in \mathcal{V}$ such that $A \cap V'^2 \circ V(x) = \emptyset$. So eventually $A_n \cap V' \circ V(x) = \emptyset$ i.e. $A_n \in [V(x)^c]^{++}$.

(15.6) REMARKS (a) $\tau_w = \tau_{B\delta}$ does not imply \mathcal{V} is B - TB e.g. an infinite dimensional Banach space.

(b) In a metric space (X, d) consider

(+) For each $0 < \varepsilon < \alpha$, there is a $\delta > \alpha$ such that $B_\delta(x) \subset B_\varepsilon[B_\alpha(x)]$ (See [FLL]).

(J) For each ε, α , $B_\alpha[B_\varepsilon(x)] = B_{\varepsilon+\alpha}(x)$. (This useful condition is a private communication from G. Beer).

It is easy to show that $(J) \Rightarrow (+) \Rightarrow B\text{-}SP \Rightarrow \tau_w = \tau_{B\delta}$.

(c) If A is compact, then $A = (\tau_w)\text{-}\lim A_n$ iff $A = (\tau_{B\delta})\text{-}\lim A_n$.

(15.6) COROLLARY *If \mathcal{V} is $B\text{-}TB$ or $B\text{-}SP$, then $\tau_{B\delta} \subset \tau_B$.*

(15.7) COROLLARY. *Consider the following*

(a) \mathcal{V} is $B\text{-}TB$.

(b) $\tau_w(\mathcal{V}) = \tau_{B\delta}(\mathcal{V})$.

(c) $\tau_{B\delta}(\mathcal{V}) = \tau_{B\delta}(\mathcal{V}')$ for each uniformly equivalent \mathcal{V}' .

(d) $\tau_{B\delta} \subset \tau_B$.

Then (a) \equiv (c) \Rightarrow (b) \Rightarrow (d).

(15.8) COROLLARY. \mathcal{V} is $B\text{-}SP \Rightarrow$ (b) \Rightarrow (d).

16. Weak total boundedness.

(16.1) DEFINITION. (X, \mathcal{V}) is $w\text{-}TB$ (with respect to $\mathcal{F} \subset CL(X)$) iff for each $A \in CL(X)$ (respectively $A \in \mathcal{F} \subset CL(X)$), $V \in \mathcal{V}$, $V[A] \neq X$ implies there exist $V_i \in \mathcal{V}$, $x_i \in X$, $1 \leq i \leq m$, such that if $W = \cup\{V_i(x_i), 1 \leq i \leq m\}$, then $A \not\subset W$, $X - W \subset V[A]$, $X - V[A] \subset W$.

(16.2) REMARKS.

(a) $TB = B\text{-}TB + w\text{-}TB$ (with respect to \mathcal{F} i.e. finite sets).

(b) In metric spaces, $w\text{-}TB$ implies d is bounded. If in addition (J) [(15.6)(c)] is satisfied, then (X, d) is TB .

(16.3) THEOREM. (X, \mathcal{V}) is $w\text{-}TB$ iff $\tau_{B\delta^+} = \tau_{\delta^+}$.

Proof. (Necessity) Suppose \mathcal{V} is $w\text{-}TB$ and $A \in G^{++} \in \tau_{\delta^+}$, $A \in CL(X)$. Use (16.1) to get $V_i, x_i, 1 \leq i \leq m, W$. Then $A \in \cap\{[V_i(x_i)c]^{++}, 1 \leq i \leq m\} \in \tau_{B\delta^+}$ and $\cap\{[V_i(x_i)c]^{++}, 1 \leq i \leq m\} \subset G^{++}$. Hence $\tau_{B\delta^+} = \tau_{\delta^+}$, $\tau_{B\delta} = \tau_{\delta}$.

(Sufficiency) Suppose $V[A] \neq X$. $A \in [V[A]^0]^{++} \in \tau_{\delta^+} = \tau_{B\delta^+}$ and so there exist $V_i, x_i, 1 \leq i \leq m$, such that $A \in \cap\{[V_i(x_i)^c]^{++}, 1 \leq i \leq m\} \subset [V[A]^0]^{++}$. Setting $W = \cup\{V_i(x_i) | 1 \leq i \leq m\}$ we find that (16.1) is satisfied.

(16.4) COROLLARY. \mathcal{V} is TB implies $\tau_w = \tau_{B\delta} = \tau_{\delta}$.

(16.5) REMARKS. If (X, d) is a metric space, then $\tau_{\sigma}(d)$ denotes the bounded proximal hypertopology ([BL]). Clearly $\tau_w \subset \tau_{B\delta} \subset \tau_{\sigma}(d) \subset \tau_{\delta}$.

$\tau_w = \tau_{\sigma}(d)$ iff (X, d) is $B-TB$.

$\tau_{B\delta} = \tau_{\delta}$ iff (X, d) is $w-TB$.

Hence $\tau_w = \tau_{\delta}$ iff (X, d) is TB ([BLLN]).

(16.6) EXAMPLES. $w-TB$ w.r.t. finite sets but neither $B-TB$ nor TB .

$X = \{2e_1\} \cup \{(1 + 1/n)e_n : n \geq 2\} = \{a_n : n \in \mathbb{N}\} \subset 1_{\infty}$.

We note that $B_{1+1/2}(a_2) = X - \{a_1\}$ is not TB i.e. X is not $B-TB$. For each $n \in \mathbb{N}, r > 0, B_r(a_n) \neq X$, implies $X - B_r(a_n)$ is finite and so $w-TB$.

17. Total boundedness.

(17.1) THEOREM, Consider the following

- (a) (X, \mathcal{V}) is TB .
- (b) $\tau_{\mathcal{H}} = \tau_w$.
- (c) $\tau_w = \tau_{\delta}$.
- (d) $\tau_{\mathcal{H}} \subset \tau_{\mathcal{V}}, (\tau_{\mathcal{H}^-} = \tau_{\mathcal{V}^-})$.
- (e) $\tau_{\mathcal{H}} \subset \tau_B$.
- (f) $\tau_{\delta} \subset \tau_B$.
- (g) $\tau_{\mathcal{H}} = \tau_{\delta}$.
- (h) $\tau_w = \tau_{B\delta}$.
- (i) $\tau_{B\delta} \subset \tau_B$.
- (j) $\tau_{B\delta} \subset \tau_{\delta}$.
- (k) $\tau_{B\delta} = \tau_{\mathcal{H}}$.

Then (a) \equiv (b) \equiv (d) \equiv (e) \equiv (g) \equiv (k) \Rightarrow (c) \Rightarrow (f) \Rightarrow (i)
 \Rightarrow (h)
 \Rightarrow (j) .

Proof. (a) \equiv (d) ([Mi]). (a) \equiv (g) ([NW]). (b) \Rightarrow (c) \Rightarrow (f), (b) \Rightarrow (e) \Rightarrow (d) are trivial and so are the rest of the implications. It is sufficient to prove (a) \Rightarrow (b) which follows from (16.4) $\tau_w = \tau_\delta$ and by (g) $\tau_\delta = \tau_{\mathcal{H}}$.

(17.2) REMARKS. (a) If (X, d) is a metric space then (a) \equiv (c) (see (16.5)).
 (b) If (X, \mathcal{V}) is TB , then $\tau_F \subset \tau_w = \tau_{B\delta} = \tau_\delta = \tau_{\mathcal{H}} \subset \tau_B \subset \tau_V \subset \tau_{LF}$.

18. Pseudocompactness.

We recall that the following are equivalent: (a) (X, τ_0) is pseudocompact, (b) $C(X) = C^*(X)$, (c) each locally finite open family is finite, (d) each compatible uniformity is TB ([GJ] 15 Q).

(18.1) THEOREM, Consider the following

- (a) (X, τ_0) is pseudocompact.
- (b) $\tau_V = \tau_{LF}$.
- (c) $\tau_\delta = \tau_{LF\delta}$.
- (d) $\tau_w = \tau_{LF\delta}$.
- (e) $\tau_{B\delta} = \tau_{LF\delta}$.
- (f) $\tau_w = \tau_{\mathcal{H}}$.
- (g) $\tau_w = \tau_\delta$.
- (h) $\tau_w = \tau_{B\delta}$.
- (i) $\tau_{B\delta} = \tau_{\mathcal{H}}$.
- (j) $\tau_{B\delta} = \tau_\delta$.
- (k) $\tau_\delta = \tau_{\mathcal{H}}$.
- (l) $\tau_{\mathcal{H}} = \tau_{LF\delta}$.
- (m) $\tau_{LF\delta} \subset \tau_B$.

(n) $\tau_\delta \subset \tau_B$.

(o) $\tau_{B\delta} \subset \tau_B$.

(p) $\tau_{LF\delta} \subset \tau_V$.

(q) $\tau_{\mathcal{H}} \subset \tau_V$.

(r) $\tau_{\mathcal{H}} \subset \tau_B$.

Then (a) \equiv (b) \equiv (c) \equiv (d) \equiv (e) \Rightarrow (m) \Rightarrow (r) \Rightarrow (n) \Rightarrow (o);

$$(a) \Rightarrow (k) \Rightarrow (l); \quad (a) \Rightarrow (f) \Rightarrow (g) \Rightarrow (h);$$

$$(f) \Rightarrow (i) \Rightarrow (j);$$

$$(m) \Rightarrow (p); \quad (r) \Rightarrow (q).$$

Proof. See [NS], [DNS].

19. Ball compactness.

(19.1) DEFINITION. (X, \mathcal{V}) is *B-compact* iff for each $V \in \mathcal{V}$, $x \in X$, $V(x) \neq X$ implies $V(x)$ is compact.

(19.2) REMARKS. (a) Terms “nice closed balls” or “boundedly compact” are also used in the literature ([Be], [FLL]).

(b) *B-compact* implies uniformly locally compact and also *B-TB*.

(19.3) THEOREM, Consider the following

(a) (X, \mathcal{V}) is *B-compact*.

(b) $\tau_F = \tau_B$.

(c) $\tau_F = \tau_{B\delta}$.

(d) $\tau_F = \tau_w$.

(e) $\tau_w = \tau_B$.

Then (a) \equiv (b) \equiv (c) \equiv (d) \Rightarrow (e).

Proof. (a) \Rightarrow (b) and (a) \Rightarrow (c). If $V(x) = X$, then $[V(x)^c]^+ = \emptyset$. If $V(x) \neq X$, then $V(x)$ is compact and so $[V(x)^c]^+ = [V(x)^c]^{++} \in \tau_F$.

(b) \Rightarrow (d) \Rightarrow (e) and (c) \Rightarrow (d) \Rightarrow (e) are trivial.

(d) \Rightarrow (a) if \mathcal{V} is not B -compact, then there exists $V(x) \neq X$ and not compact. There is a net (x_n) in $V(x)$ with no cluster points. Hence for each compact $K \subset X$, (x_n) is eventually in K^c . So for each $z \in K^c$, $(x_n) \rightarrow z$ in τ_{F^+} , but if $z \in V(x)^c$, then $(x_n) \not\rightarrow z$ in τ_w .

(19.4) REMARKS. If (X, \mathcal{V}) is B -compact, then $\tau_F = \tau_w = \tau_{B\delta} = \tau_B$.

20. Ball equinormal.

(20.1) DEFINITION. (X, \mathcal{V}) is B -equinormal iff for each $x \in X, V \in \mathcal{V}$, $[V(x)^c]^+ = [V(x)^c]^{++}$.

(20.2) THEOREM. Consider the following

- (a) (X, \mathcal{V}) is B -equinormal.
- (b) $\tau_{B\delta} = \tau_B$.
- (c) $\tau_B \subset \tau_\delta$.
- (d) $\tau_B \subset \tau_{\mathcal{H}}$.
- (e) $\tau_B \subset \tau_{LF\delta}$.

Then (a) \equiv (b) \equiv (c) \equiv (d) \Rightarrow (e).

Proof. Clearly (a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e).

(d) \Rightarrow (a). If \mathcal{V} is not B -equinormal, there exist $A \in CL(X), V \in \mathcal{V}$, $x \in X$ such that $A \cap V(x) = \emptyset$ but $A\delta V(x)$. Thus for each $U \in \mathcal{V}$, there is an $x_U \in U[A] \cap V(x)$. Set $A_U = A \cup \{x_U\}$ for each $U \in \mathcal{V}$, then the net $\{A_U : U \in \mathcal{V}\} \rightarrow A$ in $\tau_{\mathcal{H}}$ but not in τ_B . Thus $\tau_B \not\subset \tau_{\mathcal{H}}$.

(20.3) COROLLARY. (a) $\tau_w = \tau_B$ implies \mathcal{V} is B -equinormal.

(b) If \mathcal{V} is B -equinormal and B - TB or B - SP , then $\tau_w = \tau_B = \tau_{B\delta}$.

(c) We note that B -compact implies B -equinormal.

(d) \mathbb{R}^n shows that B -compact + B -equinormal does not imply compact.

21. Equinormal.

(21.1) DEFINITION (X, \mathcal{V}) is *equinormal* iff $\delta = \delta_0$.

(21.2) REMARKS. (a) An equinormal space is normal.

(b) A metric space is equinormal iff it is Atsuji i.e. every continuous function is uniformly continuous. Thus an equinormal metric space is complete and if it is also TB , then it is compact.

(c) W ([GJ] 5.12) is equinormal and pseudocompact (hence TB) but is not compact. It is B -compact.

(21.3) THEOREM Consider the following

(a) (X, \mathcal{V}) is *equinormal*.

(b) $\tau_\delta = \tau_V$.

(c) $\tau_V \subset \tau_{\mathcal{H}}(\tau_{V^+} = \tau_{\mathcal{H}^+})$.

(e) $\tau_V \subset \tau_{LF\delta}$.

(f) $\tau_{B\delta} = \tau_B$.

(g) $\tau_B \subset \tau_\delta$.

(h) $\tau_B \subset \tau_{\mathcal{H}}$.

(i) $\tau_B \subset \tau_{LF\delta}$.

Then (a) \equiv (b) \equiv (c) \Rightarrow (e),

(a) \Rightarrow (f) \Rightarrow (g) \Rightarrow (h) \Rightarrow (i).

Proof. Equivalence of (a) and (c) is well known ([Mi]). (a) \Rightarrow (b) \Rightarrow (c) is trivial and so are the rest of implications.

22. Finite ball separation.

(22.1) DEFINITION. (X, \mathcal{V}) has *F-BS* (*finite ball separation property*) iff for each $A \in CL(X)$, $G \in \tau_0$, $A \subset G$ implies $X - G \subset \cup\{V_i(x_i) \mid 1 \leq i \leq m\}$, $V_i \in \mathcal{V}$, $x_i \in X - G$ and $A \cap V_i(x_i) = \emptyset$.

(22.2) REMARKS. (a) \mathbf{R}^n does not have the *F-BS* property.

(b) W has *F-BS*, although it is not compact.

(c) \mathcal{V} equinormal and TB implies *F-BS*.

(22.3) THEOREM. Consider the following

(a) \mathcal{V} has F -BS.

(b) $\tau_B = \tau_{\mathcal{V}}$.

(c) $\tau_{\delta} \subset \tau_B$.

(d) $\tau_{B\delta} = \tau_{\delta}$.

(e) \mathcal{V} is w -TB.

Then (a) \equiv (b) \Rightarrow (d) \equiv (e) \Rightarrow (c).

Proof. Equivalence of (a) and (b) is similar to that of (16.3). (d) \equiv (e) \Rightarrow (c) is trivial. (a) \Rightarrow (d). Suppose $A \in G^{++}$, $G \in \tau_0$. Then there exists a $U \in \mathcal{U}$ such that $U^2[A] \ll G$. By F -BS there exist $V_i \in \mathcal{V}$, $x_i \in X$ such that

$$X - \overline{U[A]} \subset \cap \{V_i(x_i), 1 \leq i \leq m\} = W \text{ and } U[A] \cap W = \emptyset.$$

Clearly, $A \in \cap \{[V_i(x_i)^c]^{++}, 1 \leq i \leq m\}$ and if $F \in \cap \{[V_i(x_i)^c]^{++}, 1 \leq i \leq m\}$ then $F \subset \overline{U[A]} \subset U^2[A] \ll G$ i.e. $F \in G^{++}$. Thus $\tau_{B\delta} = \tau_{\delta}$.

(22.4) REMARKS. Here we record for reference various relationships that exist among the properties studied so far.

(a) compact \Rightarrow pseudocompact \Rightarrow TB \Rightarrow B -TB
 \Rightarrow w -TB.

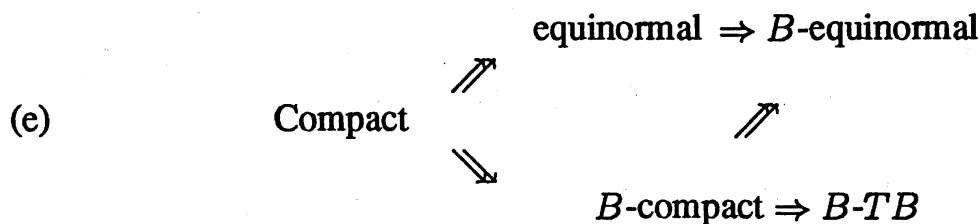
(b) $TB = TB + w$ -TB

(c) compact = TB + B -compact

$$= w$$
-TB + B -compact

$$= F$$
-BS + B -compact

(d) If (X, d) is a metric space, then B -SP + w -TB \Rightarrow TB



(f) F -BS \Rightarrow w -TB.

23. Tables.

COINCIDENCE OF COMPARABLE HYPERTOPOLOGIES

<i>Hyperspace</i>	(X, τ_0, \mathcal{V})	<i>metric case</i>
(1) $\tau_F = \tau_w$	<i>B</i> -compact	<i>B</i> -compact
(2) $\tau_B = \tau_V$	<i>F</i> - <i>BS</i>	<i>F</i> - <i>BS</i>
(3) $\tau_V = \tau_{LF}$	pseudocompact	compact
(4) $\tau_F = \tau_B$	<i>B</i> -compact	<i>B</i> -compact
(5) $\tau_B = \tau_{LF}$	<i>F</i> - <i>BS</i> + pseudocompact	compact
(6) $\tau_F = \tau_V$	compact	compact
(7) $\tau_w = \tau_{LF}$	equinormal + pseudocompact	compact
(8) $\tau_F = \tau_{LF}$	compact	compact
(9) $\tau_{B\delta} = \tau_\delta$	<i>w</i> - <i>TB</i>	<i>w</i> - <i>TB</i>
(10) $\tau_\delta = \tau_V$	equinormal	Atsuji
(11) $\tau_F = \tau_{B\delta}$	<i>B</i> -compact	<i>B</i> -compact
(12) $\tau_{B\delta} = \tau_V$	<i>w</i> - <i>TB</i> +equinormal	Atsuji+ <i>w</i> - <i>TB</i>
(13) $\tau_\delta = \tau_{LF}$	equinormal + pseudocompact	compact
(14) $\tau_\delta = \tau_F$	compact	compact
(15) $\tau_{B\delta} = \tau_{LF}$	equinormal + pseudocompact	compact
(16) $\tau_\delta = \tau_{\mathcal{H}}$	<i>TB</i>	<i>TB</i>
(17) $\tau_{LF\delta} = \tau_{LF}$	equinormal	Atsuji
(18) $\tau_{B\delta} = \tau_{\mathcal{H}}$	<i>TB</i>	<i>TB</i>
(19) $\tau_\delta = \tau_{LF\delta}$	pseudocompact	compact
(20) $\tau_{\mathcal{H}} = \tau_{LF}$	equinormal, \mathcal{V} fine	Atsuji
(21) $\tau_{B\delta} = \tau_{LF\delta}$	pseudocompact	compact
(22) $\tau_\delta = \tau_{LF}$	equinormal + pseudocompact	compact
(23) $\tau_{B\delta} = \tau_{LF}$	equinormal + pseudocompact	compact
(24) $\tau_F = \tau_{\mathcal{H}}$	compact	compact
(25) $\tau_w = \tau_{LF\delta}$	pseudocompact	compact
(26) $\tau_F = \tau_{LF\delta}$	compact	compact

IMPLICATIONS AMONG COMPARABLE HYPERTOPOLOGIES

 \Rightarrow

- | | |
|--|-------------------|
| (1) B -compact | $\tau_B = \tau_w$ |
| (2) B -equinormal + B - TB or B - SP | $\tau_w = \tau_B$ |
| (3) $\tau_w = \tau_B$ | B -equinormal |
| (4) equinormal + TB | $\tau_w = \tau_V$ |
- In the metric case they are equivalent to compact.*

- | | |
|------------------------------|---------------------------|
| (5) B - TB or B - SP | $\tau_w = \tau_{B\delta}$ |
| (6) TB | $\tau_w = \tau_\delta$ |
- In the metric case they are equivalent*

- | | |
|-------------------|--|
| (7) pseudocompact | $\tau_{\mathcal{H}} = \tau_{LF\delta}$ |
|-------------------|--|

EQUIVALENCES AMONG NON COMPARABLE

- | | |
|---------------------------|-------------------------------------|
| (1) B -equinormal | $\tau_{B\delta} = \tau_B$ |
| (2) equinormal | $\tau_V \subset \tau_{\mathcal{H}}$ |
| (3) TB | $\tau_{\mathcal{H}} \subset \tau_V$ |
| (4) equinormal+ TB | $\tau_{\mathcal{H}} = \tau_V$ |
| (5) B -equinormal | $\tau_B \subset \tau_{\mathcal{H}}$ |
| (6) TB | $\tau_{\mathcal{H}} \subset \tau_B$ |
| (7) B -equinormal+ TB | $\tau_B = \tau_{\mathcal{H}}$ |

IMPLICATIONS AMONG NON COMPARABLE

 \Rightarrow

- | | |
|------------------------------------|----------------------------------|
| (1) B - TB or B - SP | $\tau_{B\delta} \subset \tau_B$ |
| (2) B -equinormal | $\tau_B \subset \tau_\delta$ |
| (3) TB | $\tau_\delta \subset \tau_B$ |
| (4) B -equinormal+ TB | $\tau_\delta = \tau_B$ |
| (5) pseudocompact | $\tau_{LF\delta} \subset \tau_V$ |
| (6) equinormal | $\tau_V \subset \tau_{LF\delta}$ |
| (7) equinormal+pseudocompact | $\tau_{LF\delta} = \tau_V$ |
| (8) B -equinormal | $\tau_B \subset \tau_{LF\delta}$ |
| (9) pseudocompact | $\tau_{LF\delta} \subset \tau_B$ |
| (10) B -equinormal+pseudocompact | $\tau_B = \tau_{LF\delta}$ |

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