A GENERALIZATION OF STILES' ORLICZ-PETTIS THEOREM (*)

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SOMMARIO.- *Si usano elementari metodi matriciali per fornire una generalizzazione di un teorema del tipo Orlicz-Pettis, dovuto a Stiles, sulla convergenza di sottoserie in uno spazio metrico lineare dotato di una base di Schauder.*

SUMMARY.- *We use elementary matrix methods to give a generalization of an Orlicz-Pettis Theorem due to Stiles for subseries convergence in a metric linear space with a Schauder basis.*

In [2], B. Basit used an extension of a theorem of Gelfand to give a new proof of an Orlicz-Pettis Theorem in F-spaces due to W. Stiles ([3]). The proofs given by Basit employed Baire category methods. In this note we show that the matrix methods developed in [1] can be employed to give a generalization of Stiles' version of the Orlicz-Pettis Theorem.

We begin by fixing the notation. Let X be a metric linear space with Schauder basis $\{e_k\}$ and let $\{f_k\}$ be the coordinate functionals with respect to $\{e_k\}$. Assume that the coordinate functionals $\{f_k\}$ are continuous; this condition is automatically satisfied if X is an F-space ([4] 11.4.1). For each i let P_i be the continuous projection defined on X by $P_i x = \sum_{k=1}^{i} <f_k, x> e_k$ so that $\lim_i P_i x = x$ for each $x \in X$. Let $\Gamma = \{f_k : k \in N\} \subseteq X$ and denote by $\sigma(X,\Gamma)$ the weak topology on X induced by Γ.

If (E, τ) is a topological vector space, a sequence $\{x_k\}$ in E is said to be a τ - $\mathcal{K}$ sequence if each subsequence of $\{x_k\}$ has a subsequence $\{x_{n_k}\}$ such that the subseries $\sum x_{n_k}$ is τ-convergent to an element $x \in E$ ([1] 3.1). A τ - $\mathcal{K}$ sequence is τ-convergent to 0, but the converse does not hold in general ([1] 3.3). Note that if the series $\sum x_k$ is τ subseries (s.s.)

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convergent in E, then the sequence $\{x_k\}$ is certainly τ - $\mathcal{K}$ convergent.

The classical version of the Orlicz-Pettis Theorem asserts that a series in a normed space which is s.s. convergent in the weak topology is actually s.s. convergent in the norm topology ([1] §7). In general, a result is referred to as an Orlicz-Pettis Theorem if it asserts that a series which is s.s. convergent in some given topology is actually s.s. convergent in a stronger topology. There have been many such results established; Stiles' version of the Orlicz-Pettis Theorem is that if a series $\sum x_k$ in X is $\sigma(X, \Gamma)$ s.s. convergent, then $\sum x_k$ is s.s. convergent in the original metric topology of X ([2], [3]). Using the notion of $\mathcal{K}$ convergence introduced above, we have the following generalization of Stiles' result.

THEOREM 1. *If $\{x_k\}$ is a $\sigma(X, \Gamma)$ - $\mathcal{K}$ sequence in X, then $\{x_k\}$ converges to* 0 *in the original topology of X.*

Proof. For the proof we employ the Basic Matrix Theorem 2.2 of [1]. Consider the matrix $M = [P_i x_j]$. We show that M satisfies conditions (I) and (II) of [1], 2.2. Condition (I), $\lim_i P_i x_j = x_j$, has been noted above. For condition (II), if $\{m_j\}$ is an increasing sequence of positive integers, then since $\{x_j\}$ is a $\sigma(X, \Gamma)$ - $\mathcal{K}$ sequence there is a subsequence $\{n_j\}$ of $\{m_j\}$ such that the subseries $\sum x_{n_j}$ is $\sigma(X, \Gamma)$ convergent to some $x \in X$. Therefore, for each i,

$$\sum_{j=1}^{\infty} P_i x_{n_j} = \sum_{k=1}^{i} <f_k, x> e_k = P_i x ,$$

and $\lim_i P_i x = x$ so that condition (II) is satisfied. By Theorem 2.2 of [1], $\lim_i P_i x_j = x_j$ uniformly for $i \in N$. Thus,

$$\lim_j x_j = \lim_j \lim_i P_i x_j = \lim_i \lim_j P_i x_j = 0$$

since $\{x_j\}$ is $\sigma(X, \Gamma)$ convergent to 0.

We have the following improvement of Theorem 1.

COROLLARY 2. *If $\{x_k\}$ is a $\sigma(X, \Gamma)$ - $\mathcal{K}$ sequence, then $\{x_k\}$ is a $\mathcal{K}$ sequence with respect to the metric topology of X.*

Proof. Let $|\ |$ be the quasi-norm which induces the metric topology of X. By Theorem 1, $|x_k| \to 0$. Pick a subsequence $\{y_k\}$ of $\{x_k\}$ such that $\sum |y_k| < \infty$ and then choose a subsequence $\{z_k\}$ of $\{y_k\}$ such that the series $\sum z_k$ is $\sigma(X, \Gamma)$ convergent to some $x \in X$. Since $\sum |z_k| < \infty$, the series is $|\ |$-Cauchy in X and, therefore, is $|\ |$-convergent to x. Thus, $\{x_k\}$ is $|\ |$ - $\mathcal{K}$ convergent.

Theorem 1 and Corollary 2 are analogues of 2.7 and 2.8 of [1] for normed spaces.

From Theorem 1 we obtain the following version of the Orlicz-Pettis Theorem.

COROLLARY 3. *If $\sum x_k$ is $\sigma(X, \Gamma)$ s.s. convergent, then $\sum x_k$ is s.s. convergent in the original metric topology of X.*

Proof. If $\sum x_k$ is not $|\ |$-s.s. convergent, there exist $\delta > 0$ and a finite disjoint sequence of subsets of positive integers $\{\sigma_j\}$ such that max $\sigma_j <$ min σ_{j+1} and $|\sum_{k \in \sigma_j} x_k| \geq \delta$ for all j. If $z_j = \sum_{k \in \sigma_j} x_k$, the series $\sum z_j$ is $\sigma(X, \Gamma)$ s.s. convergent so by Theorem 1, $|z_j| \to 0$. This contradiction establishes the result.

Note that Corollary 3 gives a slight improvement of Stiles' result in that we do not assume that the space X is complete. The completeness assumption is replaced by the condition that the coordinate functionals are continuous.

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