A SEMILINEAR SECOND ORDER ELLIPTIC SYSTEM (*)

by PAUL EGBERTS (in Delft) (**)

SOMMARIO. - *In questa nota si considera un'equazione del tipo*

$$\begin{cases} Lu + \beta(u) \ni f(x,u) & \text{in } \Omega; \\ u = 0 & \text{su } \partial\Omega \end{cases},$$

dove $\Omega \subset \mathbf{R}^n$ $(n \geq 1)$ *è un aperto con frontiera regolare,* $L = \mathrm{diag}(L_1, \ldots, L_N)$ $(N \geq 1)$ *è una matrice diagonale di operatori ellittici,* β *è un grafico massimale monotono in* R^N *ed* $f : \Omega \times \mathbf{R}^N \to \mathbf{R}^N$ *è una funzione di tipo Caratheodory soddisfacente ad una condizione di crescita. Per questa equazione si prova un risultato di esistenza.*

SUMMARY. - *In this note we consider an equation of the form*

$$\begin{cases} Lu + \beta(u) \ni f(x,u) & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega \end{cases},$$

where $\Omega \subset \mathbf{R}^n$ $(n \geq 1)$ *is an open set with smooth boundary* $\partial\Omega$, $L = \mathrm{diag}(L_1, \ldots, L_N)$ $(N \geq 1)$ *is a diagonal matrix of second order elliptic operators,* β *is an m-accretive graph in* $\mathbf{R}^N$ *and* $f : \Omega \times \mathbf{R}^N \to \mathbf{R}^N$ *is a given Caratheodory function satisfying some growth condition. We prove an existence result for this system.*

1. Introduction.

Let $\Omega \subset \mathbf{R}^n$ be a bounded open subset with a smooth boundary and let $\mathbf{R}^N$ be equipped with some innerproduct denoted by $\langle \cdot, \cdot \rangle$ and norm $| \cdot | := \langle \cdot, \cdot \rangle^{\frac{1}{2}}$. In this note we consider a system of the type

$$Lu + \beta(u) \ni f, \tag{1}$$

(*) Pervenuto in Redazione il 27 dicembre 1991.

(**) Indirizzo dell'Autore: Faculty of Mathematics and Informatics – Delft University of Technology – Mekelweg 4, 2628 CD Delft (Netherlands).

where L is an m-accretive operator in $L^2(\Omega;\mathbf{R}^N)$ such that $L - cI$ is accretive for some constant $c > 0$, β an m-accretive (equivalently maximal monotone) graph in $(\mathbf{R}^N, \langle\cdot,\cdot\rangle)$ containing the origin and f a given function in $L^2(\Omega;\mathbf{R}^N)$ or, more generally, f depending on u. As usual we shall identify a graph with its corresponding nonlinear, possibly multivalued operator.

As a consequence of a result due to Brezis and Nirenberg [B-N, Theorem III.6′, Remark III.5], it follows that if the set

$$\{u \in D(L) : \|u\|_1 \leq 1,\ \|Lu\|_1,\ ((Lu,u)) \leq 1\},$$

is relatively compact in $L^1(\Omega;\mathbf{R}^N)$ and β single-valued, continuous, everywhere defined satisfying

$$\forall\, R > 0\ \exists\, C_R > 0 \text{ such that } \langle x, \beta(x)\rangle \geq R|\beta(x)| - C_R\ \forall\, x \in \mathbf{R}^N,$$

then, for $f \in L^2(\Omega;\mathbf{R}^N)$, there exists $u \in L^2(\Omega;\mathbf{R}^N)$, such that $\beta(u) \in L^1(\Omega;\mathbf{R}^N)$, satisfying the system

$$\bar{L}u + \beta(u) = f,$$

where $\bar{L}$ denotes the closure of L in $L^2(\Omega;\mathbf{R}^N) \times L^1(\Omega;\mathbf{R}^N)$.

It is shown that if β is a subdifferential, $\beta = \partial\varphi$, with $\varphi : \mathbf{R}^N \to \mathbf{R}^+$ convex and of class C^1 then β satisfies the above inequality (see [B-N, Remark III.4]).

We obtain stronger results if the operator L in $L^2(\Omega;\mathbf{R}^N)$ is of the form

$$Lu = (L_1u_1, L_2u_2, \ldots, L_Nu_n), \quad u = (u_1, u_2, \ldots, u_N),$$

where $L_1, L_2, \ldots, L_N$ are N strictly elliptic m-accretive second order differential operators in $L^2(\Omega)$ with smooth coefficients. We show that equation (1), where β is not necessarily a subdifferential, has a unique solution $u \in \{W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)\}^N$ for all $f \in L^2(\Omega;\mathbf{R}^N)$. In fact we show an existence result for equation (1), where f depends on $u = (u_1, u_2, \ldots, u_N)$ using a fixed point theorem (see also [E]).

2. An existence result to a semilinear second order elliptic system.

We define, for $k = 1, \dots, N$, the following elliptic differential operators

$$L_k u = -\sum_{i,j}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}^{(k)} \frac{\partial u}{\partial x_j}\right) + \sum_{i}^{n} \frac{\partial}{\partial x_i}\left(a_i^{(k)} u\right) + a^{(k)} u \,,$$

for $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, where

$$\begin{cases} a_{ij}^{(k)},\ a_i^{(k)} \in C^1(\overline{\Omega}),\ a^{(k)} \in L^\infty(\Omega), \quad i,j = 1,\dots,n; \\ \sum_{i,j}^{n} a_{ij}^{(k)} \xi_i \xi_j \geq \alpha \sum_{i}^{n} \xi_i^2 \text{ on } \Omega,\ \xi = \{\xi_1,\dots,\xi_N) \in \mathbf{R}^N \text{ for some } \alpha > 0; \\ a^{(k)} \geq 0,\ a^{(k)} + \frac{1}{2} \sum_i \frac{\partial a_i^{(k)}}{\partial x_i} + \alpha\lambda_0 \geq \delta \text{ a.e. for some } \delta > 0\,. \end{cases}$$

Here λ_0 denotes the first eigenvalue of $-\Delta$ with Dirichlet boundary condition. Now we define the operator L in $L^2(\Omega; \mathbf{R}^N)$ by

$$\begin{cases} D(L) = \{W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)\}^N \subset L^2(\Omega; \mathbf{R}^N); \\ Lu = (L_1 u_1, L_2 u_2, \dots, L_N u_N) \text{ for } u = (u_1, u_2, \dots, u_N) \in D(L)\,. \end{cases}$$

The function $f = (f_1, f_2, \dots, f_N) : \Omega \times \mathbf{R}^N \to \mathbf{R}^N$ is assumed to satisfy the following two conditions

$$(H)\begin{cases} (i) & f(\cdot, x) \text{ is measurable for all } x \in \mathbf{R}^N \text{ and } f(\omega, \cdot) \text{ is continuous, } \omega \in \Omega \text{ almost everywhere (the Caratheodory condition).} \\ (ii) & |f(\omega, x)| \leq h(\omega) + c|x|^\beta \text{ for some } h \in L^2(\Omega),\ c \in \mathbf{R}^+ \text{ and } 0 \leq \beta < 1, \text{ for all } x \in \mathbf{R}^N,\ \omega \in \Omega \text{ almost everywhere.} \end{cases}$$

The Niemytski operator F is defined by $F(u)(\omega) = f(\omega, u(\omega))$, for $u \in L^2(\Omega; \mathbf{R}^N)$, $\omega \in \Omega$ almost everywhere. It is well-known that the operator F defines a bounded and continuous operator in $L^2(\Omega; \mathbf{R}^N)$ (see for example [BD]).

Let us denote the innerproduct in $L^2(\Omega; \mathbf{R}^N)$ by $((\cdot, \cdot))$ and $\|\cdot\| := ((\cdot, \cdot))^{\frac{1}{2}}$. The standard innerproduct in $L^2(\Omega)$ is denoted by $(\cdot, \cdot)$ and $\|\cdot\|_2 := (\cdot, \cdot)^{\frac{1}{2}}$.

Now we can state the existence result.

THEOREM 1. *Let the operator L, the function f and the real number $\delta > 0$ be as above and let β be an m-accretive graph in $\mathbf{R}^N$ such that* $0 \in D(\beta)$. *Then there exists* $u = (u_1, u_2, \ldots, u_N) \in \{W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)\}^N$ *satisfying equation* (1). *Moreover if the function* $g = (g_1, g_2, \ldots, g_N) : \Omega \times \mathbf{R}^N \to \mathbf{R}^N$ *satisfies (H) and if u, v are solutions of* (1) *with right-hand side g respectively f then* $\|u - v\| \leq \frac{1}{\delta}\|Gu - Fv\|$, *where G is the Niemytski operator induced by g.*

An important tool in the proof of this theorem is the following inequality due to Sobolevskii [SO]:

Let M and N be two second order strictly elliptic operators with bounded measurable coefficients and leading coefficients belonging to $C^1(\overline{\Omega})$. *Then there exist constants* $a > 0$, $b \geq 0$ *such that*

$$(Mu, Nu) \geq a\|u\|^2_{W^{2,2}(\Omega)} - b\|u\|_2^2 \ \textit{for all}\ u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)\,. \tag{2}$$

See also [L-U, page 182], [B-E][(1)] and [SK][(1)] .

Another crucial inequality we use is contained in

LEMMA 2. *Let L_0 be the operator in* $L^2(\Omega; \mathbf{R}^N)$ *defined by*

$$\begin{cases} D(L_0) = \{W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)\}^N; \\ L_0 = (-\Delta, \ldots, -\Delta). \end{cases}$$

Let β be an m-accretive graph in $(\mathbf{R}^N, \langle\cdot,\cdot\rangle)$ *satisfying* $0 \in \beta(0)$. *Then*

$$\int_\Omega \langle L_0 u(\omega), \beta_\lambda(u(\omega))\rangle dx \geq 0, \text{ for all } u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega),\ \lambda > 0\,,$$

where β_λ, $\lambda > 0$, *denotes the Yosida-approximation of β, that is,* $\beta_\lambda = \frac{1}{\lambda}(I - (I + \lambda\beta)^{-1})$.

Proof. We note that this lemma is a special case of [C-E, Theorem 1.1]. However, the inequality can be proven directly by partial integration as we will indicate.

---

(1) The author thanks Patrick Fitzpatrick for pointing out these references.

Observe that $0 = \beta_\lambda(0)$ since $0 \in \beta(0)$ and recall that $\beta_\lambda = (\beta_{\lambda,1}, \ldots, \beta_{\lambda,N})$ is Lipschitz continuous. Let $C = (c_{ij})_{i,j=1}^N$ be the positive definite $N \times N$ matrix such that $\langle x, y\rangle = x^T C y$, $x, y \in \mathbf{R}^N$. Since the Yosida-approximation β_λ is m-accretive in $(\mathbf{R}^N, \langle\cdot,\cdot\rangle)$ as well, we have

$$\sum_{i,j=1}^N c_{ij} \sum_{l=1}^N \frac{\partial \beta_{\lambda,i}}{\partial x_l}(u)\xi_l\xi_j, \quad \text{for all } u,\ \xi = (\xi_1, \ldots, \xi_N) \in \mathbf{R}^N .$$

Then for $u = (u_1, \ldots, u_N) \in D(L_0)$ we obtain

$$\int_\Omega \langle L_0 u(\omega), \beta_\lambda(u(\omega))\rangle dx = \int_\Omega \sum_{i,j=1}^N c_{ij}(-\Delta)u_j(x)\beta_{\lambda,i}(u(x))\,dx$$

$$= -\sum_{k=1}^n \sum_{i,j=1}^N c_{ij} \int_\Omega \frac{\partial^2 u_j}{\partial x_k^2}(x)\beta_{\lambda,i}(u(x))\,dx$$

$$= \sum_{k=1}^n \sum_{i,j=1}^N c_{ij} \int_\Omega \frac{\partial u_j}{\partial x_k}(x) \frac{\partial}{\partial x_k}\beta_{\lambda,i}(u(x))\,dx$$

$$= \sum_{k=1}^n \sum_{i,j=1}^N c_{ij} \int_\Omega \frac{\partial u_j}{\partial x_k}(x) \sum_{l=1}^N \frac{\partial \beta_{\lambda,i}}{\partial x_l}(u(x)) \frac{\partial u_l}{\partial x_k}(x)\,dx$$

$$= \sum_{k=1}^n \int_\Omega \sum_{i,j=1}^N c_{ij} \sum_{l=1}^N \frac{\partial \beta_{\lambda,i}}{\partial x_l}(u(x)) \frac{\partial u_l}{\partial x_k}(x) \frac{\partial u_j}{\partial x_k}(x)\,dx \geq 0 .$$

Proof of Theorem 1. Since Ω is bounded we may assume that $0 \in \beta(0)$, otherwise consider the m-accretive operator $\beta_0 := \beta - x_0$, where $x_0 \in \beta(0)$. It is well-known that the operators L_k, $k = 1, \ldots, N$ with domain $D(L_k) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ are m-accretive in $L^2(\Omega)$, see [B-S]. Thus L is an m-accretive operator in $L^2(\Omega; \mathbf{R}^N)$. Let the operator L_0 be as in Lemma 2 and define the operator B in $L^2(\Omega; \mathbf{R}^N)$ by

$$u \in D(B), v \in Bu \text{ if and only if } u, v \in L^2(\Omega; \mathbf{R}^N)$$

$$\text{and } v(\cdot) \in \beta(u(\cdot)) \text{ a.e.}$$

The operator B is an m-accretive operator in $L^2(\Omega; \mathbf{R}^N)$.

We will show that the operator $(L + B)^{-1}F : L^2(\Omega; \mathbf{R}^N) \to L^2(\Omega; \mathbf{R}^N)$ is compact and then, using a fixed point theorem, the result will follow. We prove first that the operator $L + B$ is surjective. For that, consider the equation

$$\epsilon u_\lambda + L u_\lambda + B_\lambda u_\lambda = h, \quad \epsilon,\ \lambda > 0 ,$$

where $h \in L^2(\Omega; \mathbf{R}^N)$. By a contraction argument one shows that this approximate equation has a unique solution $u_\lambda \in D(L)$ (see [B, page 34]). By taking the innerproduct in $L^2(\Omega; \mathbf{R}^N)$ with $L_0 u_\lambda$ we get

$$\epsilon((u_\lambda, L_0 u_\lambda)) + ((L u_\lambda, L_0 u_\lambda)) + ((B_\lambda u_\lambda, L_0 u_\lambda)) = ((h, L_0 u_\lambda)) .$$

By Lemma 2,

$$((B_\lambda u_\lambda, L_0 u_\lambda)) \geq 0, \quad \text{for all } \lambda > 0 .$$

Hence $((L u_\lambda, L_0 u_\lambda)) \leq ||f||\,||L_0 u_\lambda||$. Using inequality (2) we obtain that there exist constants $a > 0, b \geq 0$ such that

$$((L u_\lambda, L_0 u_\lambda)) \geq a||L_0 u_\lambda||^2 - b||u_\lambda||^2, \quad \text{for all } \lambda > 0 . \tag{3}$$

It follows that $||L u_\lambda||$ remains bounded if $\lambda \downarrow 0$. Hence by [B, Théorème 2.4] , the operator $L + B$ is m-accretive in $L^2(\Omega; \mathbf{R}^N)$. Thus, there exists a unique $u_\epsilon \in D(L) \cap D(B)$ satisfying

$$\epsilon u_\epsilon + L u_\epsilon + B u_\epsilon \ni h \tag{3}$$

for all $\epsilon > 0$. By the assumptions on the coefficients of the operators L_k we have that $\delta||u||_2^2 \leq (L_k u, u)$ for all $u \in D(L_k)$, $k = 1, \ldots, N$ and therefore

$$\delta|u||^2 \leq ((Lu, u)) \text{ for all } u \in D(L) . \tag{4}$$

Using this estimate we get

$$||u_\epsilon|| \leq \frac{1}{\delta}||h||, \quad \text{for all } \epsilon > 0 .$$

By passing to the limit it follows that there exists a unique $u \in D(L) \cap D(B)$ such that $Lu + Bu \ni h$. Since the function $h \in L^2(\Omega; \mathbf{R}^N)$ is arbitrary, the surjectivity of $L + B$ is proved.

Observe, using (3) and (4), that $||Lu|| \leq c||h||$ for some constant $c > 0$. Therefore the operator

$$(L + B)^{-1} : L^2(\Omega; \mathbf{R}^N) \to D(L)$$

is bounded ($D(L)$ equipped with the graph norm of L). By a well-known embedding theorem, $D(L)$ is compactly embedded in $L^2(\Omega; \mathbf{R}^N)$. Thus we may conclude that the operator

$$(L + B)^{-1}F : L^2(\Omega; \mathbf{R}^N) \to L^2(\Omega; \mathbf{R}^N)$$

is compact. If we can show that the set

$$S = \{v \in L^2(\Omega; \mathbf{R}^N) : v = \sigma(L + B)^{-1}Fv, \sigma \in [0, 1]\}$$

is bounded in $L^2(\Omega; \mathbf{R}^N)$ it follows that $(L + B)^{-1}F$ has a fixed point (see for example [G-T]). Taking the innerproduct in $L^2(\Omega; \mathbf{R}^N)$ of $Lv + Bv \ni \sigma F(v)$ with v and using (4), (H) and that $((Bu, u)) \geq 0$ for all $u \in D(B)$, we obtain that for all $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\delta||v||^2 \leq \sigma((F(v), v)) \leq \sigma\{\epsilon||v||^2 + C_\epsilon\},$$

and the boundedness of the set S is proved. Since $(L + B)^{-1}$ is Lipschitz continuous with constant $\frac{1}{\delta}$ the last statement of the theorem follows.

References

[B] Brezis H., *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, Math Studies 5, North Holland, Amsterdam, 1973.

[B-E] Brezis H. and Evans L.C., *A variational inequality approach to the Bellman-Dirichlet equation for two elliptic operators*, Arch. Rational Mech. Anal., *71*, (1979), 1-13.

[B-N] Brezis H. and Nirenberg L., *Characterizations of the ranges of some nonlinear operators and applications to boundary value problems*, Ann. Sc. Norm. Sup. di Pisa, serie 4, *5* (1978), 225-326.

[B-S] Brezis H. and Strauss W.A., *Semi-linear second order elliptic equations in L^1*, J. Math. Soc. Japan, *25* (1973) 565-590.

[BD] Browder F.E., *Nonlinear functional analysis and nonlinear integral equations of Hammerstein and Urysohn type*, Contributions to Nonlinear Functional Analysis, ed. E.H. Zarantonello. Academic Press, New York-London 1971.

[C-E] Clément Ph. and Egberts P., *On the sum of two maximal monotone operators*, Diff. and Int. Eq. *6* (1990), 1113-1124.

[E] Egberts P., *On the sum of accretive operators*, Ph.D. thesis, Delft, The Netherlands 1992.

[G-T] Gilbarg D. and Trudinger N.S., *Elliptic partial differential equations of second order*, Springer, Berlin, 1983.

[L-U] Ladyzhenskaya O.A. and Ural'tseva N.N., *Linear and quasilinear elliptic equations*, New York, Academic Press, 1968.

[SK] Skrypnik I.V., *Nonlinear elliptic boundary value problems*, Teubner-Texte zur Math. *91*, 1986.

[SO] Sobolevskii P.E., *On equations with operators forming an acute angle*, (Russian) Dokl. Akad. Nauk USSR, *116* (1957) 754-757.