r-th ORDER LINEAR DIFFERENCE EQUATION OF CHARACTERISTIC J − 1 (*)

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SOMMARIO. - *Viene discusso un metodo computazionale per valutare le caratteristiche di convergenza della soluzione di una equazione lineare alle differenze, di ordine r, nella sua forma canonica.*

Ci si riconduce al computo di talune proprietà empiriche dei coefficienti della soluzione e ad una stima «a priori» della convergenza di ogni forma canonica assegnata.

SUMMARY. - *A computational method of evaluating convergence characteristics of the solution of an r-th order linear difference equation, in its canonical form, is discussed.*

This leads to the computation of certain empirical properties of the coefficients in the solution and an a-priori estimate of the convergence for each given canonical form.

1. Introduction

In two previous papers [1, 2] it was established that given a linear difference equation of order r

$$\triangle y_{n-1}(\lambda) = \lambda \beta_n y_{n+r}(\lambda) \tag{1.1}$$

with $\beta_n \geqslant 0$, $r \geqslant 0$, $n \geqslant 1$ and λ a paremeter, $\exists$ a unique solution given by the series

$$y_n(\lambda) = \sum_{i=o}^{\infty} \alpha_n^{(i)} \lambda^i \tag{1.2}$$

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which converges to an analytic solution and that the coefficients $\alpha_n^{(i)}$ could be computed explicitly [2].

We shall briefly consider here, and in subsequent results, canonical forms of (1.1) and from thence compute empirical properties of the coefficients $\alpha_n^{(i)}$. This ultimately leads us to the positive estimate of the radius of convergence for each canonical form.

An analytic consideration was given [2] to the *r-th* order equation of type $1/n$. In (1.2) of [2] we considered the solution of (1.1) when $\beta_n = \frac{1}{n+s}$.

2. **Characteristic** $J-1$

An *r-th* order difference equation (1.1) is said to be of characteristic $J-1$ if the result holds true of the following:

Lemma J − 1

If $r = s = 0$, then $\alpha_n^{(k)} = 0\ (n)\ \forall\ k \geqslant 0$ and $n \geqslant 1$.

Proof

i. $\alpha_n^{(k)} \leqslant n\ \forall\ k,\ n \geqslant 1$,

and

ii. $\{\alpha_n^{(k)}\}$ is a monotonic increasing sequence of positive numbers, for each given n.

When *i* and *ii* have been proved then $\exists\ \bar{k}$, say ϵ given $\varepsilon > 0$, every $\alpha_n^{(k)}$ with $k > \bar{k}$ belongs to an ε-neighbourhood of n and thus

$$\lim_{k\to\infty} \alpha_n^{(k)} = \sup \alpha_n^{(k)} \leqslant \text{n}.$$

We recall [3] that the $\alpha_n^{(k)}$ satisfy

$$\Delta\, \alpha_{n-1}^{(k+1)} = \frac{\alpha_{n+r}^{(k)}}{n+s},\quad k \geqslant 0,\ n \geqslant 1, \tag{2.1}$$

$$r,\ s \geqslant 0.$$

Thus

$$\alpha_n^{(1)} = \sum_{i_1=1}^{n} \frac{1}{i_1} = \sum_{1}^{n} \delta_{i_1}^{(1)}, \text{ say.}$$

For $j \geqslant 2$,

$$\delta_j^{(1)} < \delta_{j-1}^{(1)},\ \delta_1^{(1)} = 1. \tag{2.2}$$

Now

$$\alpha_n^{(2)} = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \frac{1}{i_1} \cdot \frac{1}{i_2}$$

$$= \sum_{i_2=1}^{n} \frac{\alpha_{i_2}^{(1)}}{i_2} .$$

By virtue of (2.2) we have that

(2.3) $$\frac{1}{j} (\delta_j^{(k)} + \delta_2^{(k)} + ... + \delta_n^{(k)}) \leqslant \delta_1^{(k)} = 1$$

which implies that

(2.4) $$\frac{1}{j} \alpha_j^{(1)} \leqslant \delta_1^{(1)}, \; j \geqslant 1.$$

Therefore

$$\alpha_n^{(2)} = \sum_{i_2=1}^{n} \frac{\alpha_{i_2}^{(1)}}{i_2}$$

$$\leqslant \sum_{i_2=1}^{n} \delta_1^{(1)}$$

$$= n\, \delta_1^{(1)} \equiv n.$$

We note that if $n \geqslant 1$ then

$$\alpha_n^{(k)} \begin{cases} = 1 & \text{for } n = 1, \; \forall\, k \geqslant 1 \\ \leqslant n & \text{for } n > 1, \; k = 1 \end{cases}$$

hold.

By continuing the arguments leading to (2.2), (2.3) and (2.4) it can be shown that

(2.5) $$\frac{1}{j} (\delta_j^{(k)} + \delta_2^{(k)} + ... + \delta_n^{(k)}) \leqslant \delta_1^{(k)} = 1$$

for $j \geqslant 1$.

Let the inequality i hold for n and k simultaneously. For each fixed n,

$$\alpha_n^{(k+1)} = \sum_{i_{k+1}=1}^{n} \frac{1}{i_{k+1}} \alpha_{i_{k+1}}^{(k)}$$

$$= \sum_{j=1}^{n} \alpha_j^{(k+1)}, \text{ say.}$$

Now

(2.6) $$\delta_1^{(k)} = 1, \ \delta_j^{(k)} < \delta_{j-1}^{(k)} \text{ for } j \geqslant 2.$$

Thus we have

$$\delta_1^{(k+1)} = 1$$

$$\delta_2^{(k+1)} = \frac{1}{2}(\delta_1^{(k)} + \delta_2^{(k)})$$

(2.7) $$\delta_3^{(k+1)} = \frac{1}{3}(\delta_1^{(k)} + \delta_2^{(k)} + \delta_3^{(k)})$$

$$\cdots\cdots\cdots\cdots$$

$$\delta_n^{(k+1)} = \frac{1}{n}(\delta_1^{(k)} + \dots + \delta_n^{(k)}).$$

We see immediately that for $j \geqslant 2$, $k \geqslant 1$,

$$\triangle_k \, \delta_1^{(k-1)} \equiv 0, \qquad \triangle_k \, \delta_i^{(k-1)} > 0,$$

and

$$-\triangle_j \, \delta_{j-1}^{(k+1)} = \frac{1}{j-1}[\delta_1^{(k)} + \dots + \delta_{j-1}^{(k)}]$$

$$-\frac{1}{j}[\delta_1^{(k)} + \dots + \delta_j^{(k)}]$$

$$= \frac{1}{j(j-1)}[\delta_1^{(k)} + \dots + \delta_{j-1}^{(k)} - (j-1)\,\delta_j^{(k)}]$$

which, by virtue of (2.7) becomes

$$\frac{1}{j(j-1)} \sum_{i=2}^{j} \triangle_k \, \delta_i^{(k-1)} > 0.$$

Hence,

(2.8) $$\delta_{j-1}^{(k+1)} > \delta_j^{(k+1)}, \quad j \geqslant 2, \quad k \geqslant 0.$$

Therefore,

$$\alpha_n^{(k+1)} = \sum_{i_{k+1}=1}^{n} \frac{1}{i_{k+1}} \alpha_{i_{k+1}}^{(k)}$$

$$\leqslant \delta_1^{(k+1)} = n, \quad \forall \ k \geqslant 0.$$

Similarly,

$$\alpha_{n+1}^{(k+1)} = \sum_{i_{k+1}=1}^{n} \frac{1}{i_{k+1}} \sum_{1}^{i_{k+1}} \delta_{i_k}^{(k)} + \frac{1}{n+1} \delta_{n+1}^{(k+1)}$$

$$\leqslant n + \frac{1}{n+1} < n + 1.$$

Thus, i of the lemma holds for every $k \geqslant 0$, $n \geqslant 1$.

Comparing the terms of

$$\alpha_n^{(k)} = \sum_{i_k=1}^{n} \delta_{i_k}^{(k)}$$

with those of

$$\alpha_n^{(k+1)} = \sum_{i_{k+1}=1}^{n} \delta_{i_{k+1}}^{(k+1)}$$

and using (2.8), we have that

$$\delta_j^{(k+1)} = \frac{1}{j} (\delta_1^{(k)} + \delta_2^{(k)} + \ldots + \delta_j^{(k)}) > \delta_j^{(k)}$$

so that

$$\alpha_n^{(k)} < \alpha_n^{(k+1)} \ \forall \ k \geqslant 0 \text{ and fixed } n.$$

Thus $\{\alpha_n^{(k)}\}$ is a monotonic increasing sequence and the result follows, viz. that.

$$\{\alpha_n^{(k)}\} \to n \ \forall \ n, \text{ as } k \to \infty. \tag{2.9}$$

Results were computed on the *PDP* 11/34 minicomputer. From the computational returns over several cases (not tabulated here) it could be conjectured that the rate of convergence to n can be approximated by a linear function of n.

Table I shows somevalues and convergence nature of $\alpha_n^{(k)}$.

Table I - Values of $\alpha_n^{(k)}$ for $(r, s) = (0, 0)$

k \ n	10	20	30	40	
0	1	1	1	1	
1	2.928968	3.597740	3.994987	4.278543	4.499205
2	5.064311	7.269947	8.786036	9.963087	10.933991
3	6.856653	11.032841	14.247380	16.920511	19.235979
4	8.130249	14.175092	19.241219	23.690458	27.701386
5	8.942059	16.453923	23.150936	29.283367	34.988578
6	9.422708	17.947980	25.882502	33.374307	40.512495
7	9.693092	18.857912	27.634993	36.102163	44.310167
8	9.839843	19.382809	28.689194	37.796338	46.730337
9	9.917516	19.673718	29.293164	38.792518	48.184143
10	9.957910	19.830277	29.626698	39.354190	49.018235
11	9.978659	19.912746	29.805886	39.660902	49.480094
12	9.989228	19.955520	29.900210	39.824390	49.728985
13	9.994579	19.977460	29.949123	39.909978	49.860384
14	9.997278	19.988626	29.974214	39.954194	49.928698
15	9.998635	19.994277	29.986986	39.976816	49.963816
16	9.999316	19.997126	29.993451	39.988310	49.981720
17	9.999724	19.998559	29.996711	39.994121	49.990794
18	9.999829	19.999278	29.998351	39.997049	49.995375
19	9.999914	19.999639	29.999174	39.998520	49.997680
20	9.999957	19.999819	29.999586	39.999259	49.998837
21	9.999979	19.999910	29.999793	39.999629	49.999418
22	9.999989	19.999955	29.999896	39.999814	49.999709
23	9.999995	19.999977	29.999948	39.999907	49.999854
24	9.999997	19.999989	29.999974	39.999954	49.999927
25	9.999998	19.999994	29.999987	39.999977	49.999964
26	9.999999	19.999997	29.999994	39.999988	49.999982
27	10.000000	19.999999	29.999997	39.999994	49.999991
28	10.000000	20.000000	29.999998	39.999997	49.999995
29	10.000000	20.000000	29.999999	39.999998	49.999998
30	10.000000	20.000000	30.000000	39.999999	49.999999
31	10.000000	20.000000	30.000000	40.000000	49.999999
32	10.000000	20.000000	30.000000	40.000000	50.000000

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