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Some properties of De Giorgi classes

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Dedicated to Giovanni Alessandrini for his 60th Birthday

Abstract. *The De Giorgi classes $[DG]_p(E;\gamma)$, defined in (1)$_\pm$ below encompass, solutions of quasilinear elliptic equations with measurable coefficients as well as minima and Q-minima of variational integrals. For these classes we present some new results (§ 2 and § 3.1), and some known facts scattered in the literature (§ 3–§ 5), and formulate some open issues (§ 6).*

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1. Introduction

Let E be open subset of $\mathbb{R}^N$ and for $y \in \mathbb{R}^N$, let $K_\rho(y)$ denote a cube of edge 2ρ centered at y. The De Giorgi classes $[DG]^\pm_p(E;\gamma)$ in E are the collection of functions $u \in W^{1,p}_{loc}(E)$, for some $p > 1$, satisfying

$$\int_{K_\rho(y)} |D(u-k)_\pm|^p dx \le \frac{\gamma}{(R-\rho)^p} \int_{K_R(y)} |(u-k)_\pm|^p dx \qquad (1)_\pm$$

for all cubes $K_\rho(y) \subset K_R(y) \subset E$, and all $k \in \mathbb{R}$, for a given positive constant γ. We further define

$$[DG]_p(E;\gamma) = [DG]^+_p(E;\gamma) \cap [DG]^-_p(E;\gamma). \qquad (2)$$

A celebrated theorem of De Giorgi [2] states that functions $u \in [DG]_p(E;\gamma)$ are locally bounded and locally Hölder continuous in E. Moreover, non-negative functions $u \in [DG]_p(E;\gamma)$ satisfy the Harnack inequality [7].

Local sub(super)-solutions, in $W^{1,p}_{\rm loc}(E)$, of quasi-linear elliptic equations in divergence form belong to $[DG]^{+(-)}_p(E;\gamma)$ ([12]), with γ proportional to the ratio of upper and lower modulus of ellipticity. Local minima and/or Q-minima of variational integrals with p-growth with respect to $|Du|$ belong to these classes ([10]). Thus the $[DG]_p$-classes include local solutions of elliptic equations with merely bounded and measurable coefficients, only subject to

some upper and lower ellipticity condition. They also include local minima or Q-minima of rather general functionals, even if not admitting a Euler equation.

The interest in the De Giorgi classes stems from the large class of, seemingly unrelated functions they encompass, and from properties, such as local Hölder continuity ([2]), and the Harnack inequality ([7]), typically regarded as properties of solutions of elliptic partial differential equations ([12, 14]).

The purpose of this note is to present some new results on De Giorgi classes (§ 2 and § 3.1), as well as collecting some known facts scattered in the literature (§ 3–§ 5), and formulate some open issues (§ 6) to serve as a basis for further investigations.

2. De Giorgi Classes and Sub(Super)-Harmonic Functions

The generalized De Giorgi classes $[GDG]^{\pm}_p(E;\gamma)$, are the collection of functions $u \in W^{1,p}_{\rm loc}(E)$, for some $p > 1$, satisfying

$$\int_{K_\rho(y)} |D(u-k)_\pm|^p dx \le \frac{\gamma}{(R-\rho)^p}\left(\frac{R}{R-\rho}\right)^{Np}\int_{K_R(y)} |(u-k)_\pm|^p dx \qquad (3)_\pm$$

for all cubes $K_\rho(y) \subset K_R(y) \subset E$, and all $k \in \mathbb{R}$, for a given positive constant γ. Convex, monotone, non-decreasing functions of sub-harmonic functions are sub-harmonic. Similarly, concave, non-decreasing, functions of super-harmonic functions are super-harmonic. Similar statements hold for weak, sub(super)-solutions of linear elliptic equations with measurable coefficients ([14]). The next lemma establishes analogous properties for functions $u \in [DG]^{\pm}_p(E;\gamma)$. Given any such class, we refer to the set of parameters $\{p, \gamma, N\}$ as the *data* and say that a constant $C = C(\text{data})$ depends only on the *data* if it can be quantitatively determined a-priori only in terms of the indicated set of parameters.

LEMMA 2.1. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be convex and non-decreasing, and let $u \in [DG]^+_p(E;\gamma)$. There exists a positive constant $\overline{\gamma}$ depending only on the data, and independent of u, such that $\varphi(u) \in [GDG]^+_p(E;\overline{\gamma})$.*

Likewise let $\psi : \mathbb{R} \to \mathbb{R}$ be concave and non-decreasing, and let $u \in [DG]^-_p(E;\gamma)$. There exist a positive constant $\overline{\gamma}$ depending only on the data, and independent of u, such that $\psi(u) \in [GDG]^-_p(E;\overline{\gamma})$.

Proof. By De Giorgi's theorem ([2, 12]), there exists a constant $C = C(\text{data})$, such that for any $u \in [DG]^{\pm}_p(E;\gamma)$, there holds

$$\|(u-k)_\pm\|_{\infty,K_\rho(y)} \le \frac{C}{(R-\rho)^N}\int_{K_R(y)} (u-k)_\pm dx \qquad (4)$$

for every pair of cubes $K_\rho(y) \subset K_R(y) \subset E$ and all $k \in \mathbb{R}$. It suffices to prove the first statement for $\varphi \in C^2(\mathbb{R})$, and verify that $\varphi(u)$ satisfies $(3)_+$ for cubes $K_\rho \subset K_R$ centered at the origin of $\mathbb{R}^N$. For any such φ and all $h \le k$

$$\big(\varphi(u)-\varphi(h)\big)_+ - \varphi'(h)(u-h)_+ = \int_{\mathbb{R}^+} (u-k)_+ \chi_{[k>h]} \varphi''(k)dk \tag{5}$$

From this, a.e. in E

$$\Big| D\big[\big(\varphi(u)-\varphi(h)\big)_+ - \varphi'(h)(u-h)_+\big]\Big|^p \le \left(\int_{\mathbb{R}} |D(u-k)_+| \chi_{[k>h]}\varphi''(k)dk\right)^p.$$

Integrate over K_ρ, take the p root of both sides, and majorize the resulting term on the right-hand first by the continuous version of Minkowski inequality, then by applying the definition $(1)_+$ of the $[DG]_p^+(E;\gamma)$-classes, and finally by using (4). This gives

$$\begin{aligned}
&\Big\| D\big[\big(\varphi(u)-\varphi(h)\big)_+ - \varphi'(h)(u-h)_+\big]\Big\|_{p,K_\rho} \\
&\qquad \le \int_{\mathbb{R}} \|D(u-k)_+\|_{p,K_\rho} \chi_{[k>h]}\varphi''(k)dk \\
&\qquad \le \frac{C}{R-\rho}\int_{\mathbb{R}} \|(u-k)_+\|_{p,K_{\frac{R+\rho}{2}}} \chi_{[k>h]}\varphi''(k)dk \\
&\qquad \le \frac{CR^{\frac{N}{p}}}{R-\rho}\int_{\mathbb{R}} \|(u-k)_+\|_{\infty,K_{\frac{R+\rho}{2}}} \chi_{[k>h]}\varphi''(k)dk \\
&\qquad \le \frac{CR^{\frac{N}{p}}}{(R-\rho)^{N+1}}\int_{\mathbb{R}} \left(\int_{K_R} (u-k)_+ dx\right) \chi_{[k>h]}\varphi''(k)dk \\
&\qquad = \frac{CR^{\frac{N}{p}}}{(R-\rho)^{N+1}}\int_{K_R} \left(\int_{\mathbb{R}} (u-k)_+ \chi_{[k>h]}\varphi''(k)dk\right) dx \\
&\qquad = \frac{CR^{\frac{N}{p}}}{(R-\rho)^{N+1}}\int_{K_R} \big[\big(\varphi(u)-\varphi(h)\big)_+ - \varphi'(h)(u-h)_+\big] dx \\
&\qquad \le \frac{C}{R-\rho}\left(\frac{R}{R-\rho}\right)^N \big\|\big(\varphi(u)-\varphi(h)\big)_+ - \varphi'(h)(u-h)_+\big\|_{p,K_R}.
\end{aligned}$$

In these calculations, we have denoted by $C = C(p,N,\gamma)$ a generic constant depending only upon the data, and that might be different from line to line. In the last two steps we have interchanged the order of integration with the help of Fubini's Theorem and have applied Hölder's inequality. By the convexity and monotonicity of φ,

$$\big(\varphi(u)-\varphi(h)\big)_+ \ge \varphi'(h)(u-h)_+ \ge 0. \tag{6}$$

Therefore,

$$\big\|D\big(\varphi(u)-\varphi(h)\big)_+\big\|_{p,K_\rho}\le\frac{C}{R-\rho}\left(\frac{R}{R-\rho}\right)^N\big\|\big(\varphi(u)-\varphi(h)\big)_+\big\|_{p,K_R}+\big\|\varphi'(h)D(u-h)_+\big\|_{p,K_\rho}$$

Upon applying the definition of $(1)_+$ of $[DG]^+_p(E;\gamma)$, and then (6), the last term on the right-hand side is majorized by

$$\frac{C}{R-\rho}\big\|(\varphi(u)-\varphi(h))_+\big\|_{p,K_R}.$$

Combining these estimates yields

$$\int_{K_\rho(y)}\big|D\big(\varphi(u)-k\big)_+\big|^p dx\le\frac{\overline\gamma}{(R-\rho)^p}\left(\frac{R}{R-\rho}\right)^N\int_{K_R(y)}\big(\varphi(u)-k\big)_+^p dx \quad (7)$$

for all $k\in\mathbb{R}$ and all $K_\rho(y)\subset K_R(y)\subset E$, for a constant $\overline\gamma=\overline\gamma(\text{data})$. □

If $u\in[DG]^-_p(E;\gamma)$ and φ is convex, there is no guarantee, in general, that $\varphi(u)\in[GDG]^+_p(E;\overline\gamma)$ for some $\overline\gamma=\overline\gamma(p,N,\gamma)$. The next lemma provides some sufficient conditions on φ for this to occur.

LEMMA 2.2. *Let $\varphi:(a,+\infty)\to\mathbb{R}$, for some $a<\infty$ be convex, non-increasing, and such that*

$$\lim_{t\to+\infty}\varphi(t)=\lim_{t\to+\infty}t\varphi'(t)=0, \quad (8)$$

and let $u\in[DG]^-_p(E;\gamma)$, with range in $(a,+\infty)$. There exists a positive constant $\overline\gamma$ depending only on the data, such that $\varphi(u)\in[GDG]^+_p(E;\overline\gamma)$.

Likewise let $\psi:(-\infty,a)\to\mathbb{R}$, for some $a>-\infty$, be concave, non-increasing, and satisfying

$$\lim_{t\to-\infty}\psi(t)=\lim_{t\to-\infty}t\psi'(t)=0, \quad (9)$$

and let $u\in[DG]^+_p(E;\gamma)$, with range in $(-\infty,a)$. There exists a positive constant $\overline\gamma$ depending only on the data, such that $\psi(u)\in[GDG]^-_p(E;\overline\gamma)$.

Proof. It suffices to prove the first statement for $\varphi\in C^2(\mathbb{R})$ over congruent cubes $K_\rho\subset K_R$ centered at the origin. The starting point is the analog of (5), i.e.,

$$\varphi(u)=\int_{\mathbb{R}}(u-k)_-\varphi''(k)dk. \quad (10)$$

Since $u\in[DG]^-_p(E;\gamma)$, by (4) the function u is locally bounded below in E, and without loss of generality we may assume $u\ge0$. Hence, the representation (10)

is well defined by virtue of the assumption (8) on φ. From this, by taking the gradient of both sides, then taking the p-power, and finally integrating over K_ρ gives

$$\int_{K_\rho} |D\varphi(u)|^p dx = \int_{K_\rho} \left| \int_{\mathbb{R}^+} D(u-k)_- \varphi''(k) dk \right|^p dx.$$

The proof now parallels that of Lemma 2.1. Specifically, apply sequentially the continuous version of Minkowski's inequality, the definition $(1)_-$ of the classes $[DG]_p^-(E;\gamma)$, the sup-bound (4), interchange the order of integration, and use Hölder's inequality. This gives

$$\begin{aligned}
\|D\varphi(u)\|_{p,K_\rho} &\le \int_{\mathbb{R}^+} \|D(u-k)_-\|_{p,K_\rho} \varphi''(k) dk \\
&\le \frac{C}{R-\rho} \int_{\mathbb{R}^+} \|(u-k)_-\|_{p,K_{\frac{R+\rho}{2}}} \varphi''(k) dk \\
&\le \frac{CR^{\frac{N}{p}}}{R-\rho} \int_{\mathbb{R}^+} \|(u-k)_-\|_{\infty,K_{\frac{R+\rho}{2}}} \varphi''(k) dk \\
&\le \frac{CR^{\frac{N}{p}}}{(R-\rho)^{N+1}} \int_{\mathbb{R}^+} \int_{K_R} (u-k)_- \varphi''(k) dk \\
&= \frac{CR^{\frac{N}{p}}}{(R-\rho)^{N+1}} \int_{K_R} \varphi(u) dx \\
&= \frac{C}{(R-\rho)} \left(\frac{R}{R-\rho} \right)^N \|\varphi(u)\|_{p,K_R}.
\end{aligned}$$

Now if φ is convex, non-increasing and satisfying (8), the function $(\varphi-\ell)_+$, for all ℓ in the range of φ, shares the same properties. Hence,

$$\int_{K_\rho(y)} \left| D\big(\varphi(u)-\ell\big)_+ \right|^p dx \le \frac{C}{(R-\rho)^p} \left(\frac{R}{R-\rho} \right)^{Np} \int_{K_R(y)} \big(\varphi(u)-\ell\big)_+^p dx$$

for all cubes $K_\rho(y) \subset K_R(y) \subset E$ and all $\ell \in \mathbb{R}$. □

2.1. Some Consequences

The sup-bound in (4) can be given the following sharper form ([7]).

LEMMA 2.3. *Let $u \in [DG]_p^\pm(E;\gamma)$. Then for all $\sigma > 0$ there exists a constant C_σ depending only upon the data and σ, such that*

$$\sup_{K_\rho(y)} (u-k)_\pm \le C_\sigma \left(\frac{R}{R-\rho} \right)^{\frac{N}{\sigma}} \left(\fint_{K_R(y)} (u-k)_\pm^\sigma dx \right)^{\frac{1}{\sigma}}. \tag{11}$$

If $u \in [DG]_p^-(E;\gamma)$ is non-negative, then Lemma 2.2 with $\varphi(u) = u^{-1}$ and $a = 0$, implies that $u^{-1} \in [GDG]_p^+(E;\gamma)$. Therefore Lemma 2.3, with $k = 0$, implies that for all $\tau > 0$,

$$\frac{1}{\inf\limits_{K_\rho(y)} u} \le C_\tau \left(\frac{R}{R-\rho}\right)^{\frac{N}{\tau}} \left(\fint_{K_R(y)} \frac{1}{u^\tau} dx\right)^{\frac{1}{\tau}}. \tag{12}$$

PROPOSITION 2.4. *Let u be a non-negative function in the De Giorgi classes $[DG]_p(E;\gamma)$. Then for any pair of positive numbers σ and τ*

$$\frac{\sup\limits_{K_\rho(y)} u}{\inf\limits_{K_\rho(y)} u} \le C_\sigma C_\tau \left(\frac{R}{R-\rho}\right)^{N(\frac{1}{\sigma}+\frac{1}{\tau})} \left(\fint_{K_R(y)} u^\sigma dx\right)^{\frac{1}{\sigma}} \left(\fint_{K_R(y)} \frac{1}{u^\tau} dx\right)^{\frac{1}{\tau}}. \tag{13}$$

Inequalities of the form (11) are at the basis of Moser's approach to the Harnack inequality for non-negative weak solutions to quasilinear elliptic equations with bounded and measurable coefficients ([14]). The Harnack inequality will follow from (13) if $\ln u \in BMO(E)$. This fact is established by Moser for non-negative weak solutions of elliptic equations. We will establish that for non-negative functions $u \in [DG]_p^-(E;\gamma)$, one has $\ln u \in BMO(E)$ by using the Harnack inequality established in ([7]).

3. De Giorgi Classes, $BMO(E)$ and Logarithmic Estimates

The proof of the following lemma is in [7].

LEMMA 3.1. *Let $u \in [DG]_p^-(E;\gamma)$ be non-negative. There exist positive constants C and σ, depending only upon the data, such that*

$$\fint_{K_\rho(y)} u^\sigma dx \le C \inf_{K_\rho(y)} u^\sigma, \tag{14}$$

for any cube $K_\rho(y)$ such that $K_{2\rho}(y) \subset E$.

Such an inequality, referred to as the weak Harnack inequality, was established by Moser for non-negative super-solutions of elliptic equations with bounded and measurable coefficients ([14]). It is noteworthy that it continues to hold for non-negative functions in $[DG]_p^-(E;\gamma)$, with no further reference to equations.

LEMMA 3.2. *Let $u \in [DG]_p^-(E;\gamma)$ be non-negative. Then $\ln u \in BMO$.*

Proof. By Lemma 3.1

$$\begin{aligned} \fint_{K_\rho(y)} u^\sigma dx \fint_{K_\rho(y)} \frac{1}{u^\sigma} dx &\le \fint_{K_\rho(y)} u^\sigma dx \sup_{K_\rho(y)} \frac{1}{u^\sigma} \\ &= \fint_{K_\rho(y)} u^\sigma dx \frac{1}{\inf\limits_{K_\rho(y)} u^\sigma} \le C \end{aligned} \tag{15}$$

for any cube $K_\rho(y)$ such that $K_{2\rho}(y) \subset E$. Set

$$(\ln u^\sigma)_\rho = \fint_{K_\rho(y)} \ln u^\sigma dx,$$

and estimate

$$\begin{aligned} \fint_{K_\rho(y)} e^{|\ln u^\sigma - (\ln u^\sigma)_\rho|} dx \le e^{-(\ln u^\sigma)_\rho} &\fint_{K_\rho(y)} e^{\ln u^\sigma} dx \\ &+ e^{(\ln u^\sigma)_\rho} \fint_{K_\rho(y)} e^{-\ln u^\sigma} dx. \end{aligned}$$

The second term on the right-hand side is estimated by Jensen's inequality and (15) and yields

$$\begin{aligned} e^{(\ln u^\sigma)_\rho} \fint_{K_\rho(y)} e^{-\ln u^\sigma} dx &\le \fint_{K_\rho(y)} e^{\ln u^\sigma} dx \fint_{K_\rho(y)} \frac{1}{u^\sigma} dx \\ &\le \fint_{K_\rho(y)} u^\sigma dx \fint_{K_\rho(y)} \frac{1}{u^\sigma} dx \le C. \end{aligned}$$

The first term is estimated analogously. Hence, there exists a constant $\bar{C}$, depending only upon the data, such that

$$\fint_{K_\rho(y)} e^{|\ln u^\sigma - (\ln u^\sigma)_\rho|} dx \le \bar{C}$$

for any cube $K_\rho(y)$ such that $K_{2\rho}(y) \subset E$. Thus $\ln u \in BMO(E)$. □

3.1. Logarithmic Estimates Revisited

Let $u \in W^{1,p}_{\mathrm{loc}}(E)$ be a non-negative weak super-solution of an elliptic equation in divergence form, and with only bounded and measurable coefficients. Then there exists a constant C, depending only on p, N, and the modulus of ellipticity of the equation, such that

$$\fint_{K_\rho(y)} |D \ln u|^p dx \le \frac{C}{(R-\rho)^p} \tag{16}$$

for any pair of cubes $K_\rho(y) \subset K_R(y) \subset E$. Such an estimate, established by Moser, permits one to prove that $\ln u \in BMO(E)$, which in turn yields the Harnack inequality. Our approach for functions in the $[DG]_p^-(E;\gamma)$ classes is somewhat different. For non-negative functions in such classes we first establish the weak Harnack estimate (14), and then the latter is used to prove Lemma 3.2. It is not known, whether non-negative functions in $[DG]_p^-(E;\gamma)$ satisfy (16). The next proposition is a partial result in this direction.

PROPOSITION 3.3. *Let $u \in [DG]_p^-(E;\gamma)$ be non-negative and bounded above by some positive constant M. Then*

$$\int_{K_\rho(y)} |D\ln u|^p dx \le \frac{\gamma p}{(R-\rho)^p} \int_{K_R(y)} \ln \frac{M}{u} dx \tag{17}$$

for any pair of cubes $K_\rho(y) \subset K_R(y) \subset E$.

Proof. The arguments being local may assume that $y = \{0\}$. By the definition $(1)_-$ of classes, for all $0 < t < M$,

$$\int_{K_\rho} |D(u-t)_-|^p dx \le \frac{\gamma}{(R-\rho)^p} \int_{K_R} (u-t)_-^p dx.$$

Multiply both sides by t^{-p-1} and integrate over $(0, M)$. The left-hand side is transformed as

$$\begin{aligned}
\int_0^M \frac{dt}{t^{p+1}} \int_{K_\rho} |D(u-t)_-|^p dx &= \int_{K_\rho} \left(\int_0^M |D(u-t)_-|^p \frac{1}{t^{p+1}} dt \right) dx \\
&= \int_{K_\rho} |Du|^p \left(\int_0^M \frac{1}{t^{p+1}} \chi_{[u<t]} dt \right) dx \\
&= \int_{K_\rho} |Du|^p \left(\int_u^M \frac{1}{t^{p+1}} dt \right) dx \\
&= \int_{K_\rho} \left(-\frac{1}{p} \frac{|Du|^p}{M^p} + \frac{1}{p} \frac{|Du|^p}{u^p} \right) dx \\
&= \frac{1}{p} \int_{K_\rho} |D\ln u|^p dx - \frac{1}{pM^p} \int_{K_\rho} |Du|^p dx.
\end{aligned}$$

The integral on the right-hand side is transformed as

$$\begin{aligned}
&\int_0^M \frac{1}{t^{p+1}} \left(\int_{K_R} (u-t)_-^p \, dx \right) dt \\
&\qquad = \int_{K_R} \left(\int_u^M \frac{(t-u)^p}{t^{p+1}} dt \right) dx \\
&\qquad = \int_{K_R} \left[-\frac{1}{p} \frac{(t-u)^p}{t^p} \Big|_u^M + \int_u^M \frac{(t-u)^{p-1}}{t^{p-1}} \frac{dt}{t} \right] dx \\
&\qquad = -\frac{1}{pM^p} \int_{K_R} (M-u)^p dx + \int_{K_R} \left(\int_u^M \left(\frac{t-u}{t} \right)^{p-1} \frac{dt}{t} \right) dx \\
&\qquad \le -\frac{1}{pM^p} \int_{K_R} (M-u)^p dx + \int_{K_R} \ln \frac{M}{u} dx.
\end{aligned}$$

Combining the previous estimates gives

$$\begin{aligned}
\int_{K_\rho} |D \ln u|^p dx \le \frac{1}{M^p} \left(\int_{K_\rho} |Du|^p dx - \frac{\gamma}{(R-\rho)^p} \int_{K_R} (M-u)^p dx \right) \\
+ \frac{\gamma p}{(R-\rho)^p} \int_{K_R} \ln \frac{M}{u} dx.
\end{aligned}$$

Since $u \in [DG]_p^-(E;\gamma)$, the term in round brackets on the right-hand side is non-positive and can be discarded. □

REMARK 3.4. Applying Lemma 2.2 to $\varphi(u) = \ln_+(M/u)$, gives the weaker estimate

$$\int_{K_\rho(y)} |D \ln u|^p dx \le \frac{\overline{\gamma}}{(R-\rho)^p} \int_{K_R(y)} \left(\ln \frac{M}{u} \right)^p dx. \tag{18}$$

4. Higher Integrability of the Gradient of Functions in the De Giorgi Classes

PROPOSITION 4.1. *Let $u \in [DG]_p^{\pm}(E)$. Then there exist constants $C > 1$ and $\sigma > 0$, dependent only upon the data, such that, for any pair of cubes $K_\rho(y) \subset K_R(y) \subset E$, there holds*

$$\left(\fint_{K_\rho(y)} |Du|^{p(1+\sigma)} dx \right)^{\frac{1}{p(1+\sigma)}} \le C \left(\frac{R}{\rho} \right)^{\frac{N}{p}} \left(\frac{R}{R-\rho} \right) \left(\fint_{K_R(y)} |Du|^p dx \right)^{\frac{1}{p}}. \tag{19}$$

Proof. Let u be in the classes $[DG]_p(E;\gamma)$ defined in (2). For any pair of cubes $K_\rho(y) \subset K_R(y) \subset E$, write down $(1)_+$ and $(1)_-$ for the choice

$$k = u_R \stackrel{\text{def}}{=} \fint_{K_R(y)} u dx.$$

Adding the resulting inequalities gives

$$\int_{K_\rho(y)} |Du|^p dx \le \frac{\gamma}{(R-\rho)^p} \int_{K_R(y)} |u - u_R|^p dx.$$

By the Sobolev-Poincaré inequality

$$\fint_{K_R(y)} |u - u_R|^p dx \le C_q\, R^p \left(\fint_{K_R(y)} |Du|^q dx \right)^{\frac{p}{q}}, \qquad \text{for all } \; q \in \left[\frac{Np}{N+p}, p \right]$$

for a constant $C_q = C_q(N,q)$. Hence, for all such q

$$\fint_{K_\rho(y)} |Du|^p dx \le C_q \gamma \left(\frac{R}{R-\rho} \right)^p \left(\frac{R}{\rho} \right)^N \left(\fint_{K_R(y)} |Du|^q dx \right)^{\frac{p}{q}}$$

for all pair of congruent cubes $K_\rho(y) \subset K_R(y) \subset E$. The conclusion follows from this and the local version of Gehring's lemma ([9]), as appearing in [11]. □

Remark 4.2. Hence, the higher integrability of the gradient of solutions of elliptic equations with measurable coefficients ([15]), and more generally of Q-minima ([10]), continues to hold for function in the De Giorgi classes. If $u \in [DG]_p^{\pm}(E;\gamma)$, the conclusion is in general false, as one can verify starting from sub(super)-harmonic functions. However, essentially the same arguments give the inequality

$$\fint_{K_\rho(y)} |D(u-k)_\pm|^p dx \le C_q \gamma \left(\frac{R}{R-\rho} \right)^p \left(\frac{R}{\rho} \right)^N \left(\fint_{K_R} |Du|^q dx \right)^{\frac{p}{q}}$$

for all $q \in \left[\frac{Np}{N+p}, p \right]$, and

$$\text{all } \; k \ge \fint_{K_R(y)} u dx \quad \text{if } \; u \in [DG]_p^+(E;\gamma),$$
$$\text{all } \; k \le \fint_{K_R(y)} u dx \quad \text{if } \; u \in [DG]_p^-(E;\gamma).$$

5. Measure Theoretical Decay Estimates of Functions in De Giorgi Classes

For a non-negative function $f \in L^1_{\rm loc}(E)$ one estimates the measure of the set $[f > t]$ relative to a cube $K_\rho(y) \subset E$, as $\mu\big([f > t] \cap K_\rho(y)\big) \le t^{-1}\|f\|_{1,K_\rho(y)}$. Estimates of the measure of the set $[f < t]$ relative to $K_\rho(y)$ are not, in general, a consequence of the mere integrability of f. One of De Giorgi's estimates of [2], is that if u is a non-negative function in $[DG]^-_p(E;\gamma)$, then

$$\frac{\big|[u < t] \cap K_\rho(y)\big|}{|K_\rho|} \le \frac{C(N,p,\gamma)}{|\ln t|^{1/p}} \qquad \text{asymptotically as } \ t \to 0, \tag{20}$$

provided $\big|[u > t] \cap K_\rho(y)\big| \ge \frac{1}{2}|K_\rho|$. Here $|\sigma|$ denotes the Lebesgue measure of a measurable set $\sigma \subset \mathbb{R}^N$. The next proposition improves on this estimate.

PROPOSITION 5.1. *Let $u \in [DG]^-_p(E;\gamma)$ be non-negative, and assume that for some $t_o > 0$ and $\alpha \in (0,1)$, there holds*

$$\frac{\big|[u > t_o] \cap K_\rho(y)]\big|}{|K_\rho|} \ge \alpha. \tag{21}$$

There exist positive constants $C, t_*, \sigma = C, t_*, \sigma(N,p,\gamma,t_o,\alpha)$, depending only on the indicated parameters and independent of u, such that

$$\frac{\big|[u < t] \cap K_\rho(y)\big|}{|K_\rho|} \le \frac{C}{|\ln t|^{\sigma|\ln t|^{\frac{1}{2}}}}, \qquad \textit{for } \ t < t_*. \tag{22}$$

Proof. In what follows we denote by C a generic positive constant that can be determined a-priori only in terms of $\{N,p,\gamma,t_o,\alpha\}$ and that it may be different in the same context. The arguments being local to concentric cubes $K_\rho(y) \subset K_{2\rho}(y) \subset E$, may assume $y = \{0\}$ and write $K_\rho(0) = K_\rho$. Let n_o be the smallest positive integer such that $2^{-n_o} \le t_o$, and for $n \ge n_o$ set

$$A_{n,\rho} \stackrel{\rm def}{=} \left[u < \frac{1}{2^n}\right] \cap K_\rho, \qquad \text{for } \ n \ge n_o.$$

The discrete isoperimetric inequality ([3, Chapter I, Lemma 2.2]), reads

$$(\ell - h)\big|[u < h] \cap K_\rho\big| \le C(N)\frac{\rho^{N+1}}{\big|[u > \ell] \cap K_\rho\big|}\int_{[h<u<\ell]\cap K_\rho} |Du|dx$$

for any two levels $0 < h < \ell$. Applying it with

$$\ell = \frac{1}{2^n}, \quad h = \frac{1}{2^{n+1}}, \quad \text{so that} \quad [h < u < \ell] \cap K_\rho = A_{n,\rho} - A_{n+1,\rho},$$

and taking into account (21), yields

$$\frac{1}{2^{n+1}}\big|A_{n+1,\rho}\big| \le \frac{C(N)}{\alpha}\rho^N \int_{A_{n,\rho}-A_{n+1,\rho}} |Du|dx.$$

Majorize the right-hand side by the Hölder inequality, then raise both terms to the power $\frac{p}{p-1}$, and majorize the right-hand side by $(1)_-$ in the definition of the classes $[DG]_p^-(E;\gamma)$. These sequential estimates yield

$$\begin{aligned}\frac{1}{2^{n\frac{p}{p-1}}}|A_{n+1,\rho}|^{\frac{p}{p-1}} &\le C\rho^{\frac{p}{p-1}}\left(\int_{K_\rho}\big|D\big(u-\tfrac{1}{2^n}\big)_-|^p dx\right)^{\frac{1}{p-1}}\big|A_{n,\rho}-A_{n+1,\rho}\big| \\ &\le C\left(\int_{K_\rho}\big(u-\tfrac{1}{2^n}\big)_-^p dx\right)^{\frac{1}{p-1}}\big|A_{n,\rho}-A_{n+1,\rho}\big| \\ &\le \frac{C}{2^{n\frac{p}{p-1}}}\,|A_{n_o,2\rho}|^{\frac{1}{p-1}}\,\big|A_{n,\rho}-A_{n+1,\rho}\big|.\end{aligned}$$

This in turn yields the recursive inequalities

$$\big|A_{n+1,\rho}\big|^{\frac{p}{p-1}} \le C(N,p,\gamma,\alpha)\,\big|A_{n_o,2\rho}\big|^{\frac{1}{p-1}}\,\big|A_{n,\rho}-A_{n+1,\rho}\big|.$$

Let n_* be a positive integer to be chosen. Adding them from n_o to n_*-1 gives

$$\big|A_{n_*,\rho}\big| \le \frac{C(N,p,\gamma,\alpha)}{(n_*-n_o)^{\frac{p-1}{p}}}\,\big|A_{n_o,2\rho}\big|^{\frac{1}{p}}\,\big|A_{n_o,\rho}\big|^{\frac{p-1}{p}}. \tag{23}$$

Return now to the assumption (21) and estimate

$$\frac{\big|[u>t_o]\cap K_{2\rho}(y)\big|}{|K_{2\rho}|} \ge \frac{\big|[u>t_o]\cap K_\rho(y)\big|}{2^N|K_\rho|} \ge \frac{\alpha}{2^N}.$$

Therefore, the same arguments leading to (23) can be repeated over the cube $K_{2\rho}$ and give

$$\big|A_{n_*,2\rho}\big| \le \frac{C(N,p,\gamma,\alpha)}{(n_*-n_o)^{\frac{p-1}{p}}}\,\big|A_{n_o,4\rho}\big|^{\frac{1}{p}}\,\big|A_{n_o,2\rho}\big|^{\frac{p-1}{p}}. \tag{24}$$

While the constant C in (24) differs from the one in (23), we may take them to be equal by taking the largest. The assumption (21) continue to hold with t_o replaced by 2^{-n_*}. Hence, the previous arguments can be repeated and yield the analogues of (23)–(24), i.e.,

$$\begin{aligned}\big|A_{2n_*,\rho}\big| &\le \frac{C(N,p,\gamma,\alpha)}{(n_*-n_o)^{\frac{p-1}{p}}}\,\big|A_{n_*,2\rho}\big|^{\frac{1}{p}}\,\big|A_{n_*,\rho}\big|^{\frac{p-1}{p}} \\ \big|A_{2n_*,2\rho}\big| &\le \frac{C(N,p,\gamma,\alpha)}{(n_*-n_o)^{\frac{p-1}{p}}}\,\big|A_{n_*,4\rho}\big|^{\frac{1}{p}}\,\big|A_{n_*,2\rho}\big|^{\frac{p-1}{p}}\end{aligned}$$

for the same constant C. Combining them gives

$$\left|A_{2n_*,\rho}\right| \le \frac{C^2 4^{2N}}{(n_* - n_o)^{2\frac{p-1}{p}}} |K_\rho|.$$

Iteration of this procedure yields

$$\left|A_{jn_*,\rho}\right| \le \frac{C^j 4^{jN}}{(n_* - n_o)^{j\frac{p-1}{p}}} |K_\rho| \qquad \text{for all } \ j \in \mathbb{N}.$$

Choose n_* so large that $n_* - n_o > \frac{1}{2} n_*$, and then take $j = n_*$. By possibly modifying the various constants, the previous inequality yields

$$\left|A_{j^2,\rho}\right| \le \frac{C^j 4^{jN}}{j^{j\frac{p-1}{p}}} |K_\rho| \qquad \text{for all } \ j \in \mathbb{N}.$$

The constant C being fixed, for each $0 < \varepsilon < \frac{p-1}{p}$ there exists j^* so large that

$$\left|A_{j^2,\rho}\right| \le \frac{1}{j^{j\varepsilon}} |K_\rho| \qquad \text{for all } \ j \ge j^*.$$

Fix now $t \le 2^{-j^{*2}}$ and let j be the largest integer such that $2^{-(j+1)^2} \le t \le 2^{-j^2}$. For such choices

$$\frac{\left|[u < t] \cap K_\rho\right|}{|K_\rho|} \le \frac{\left|A_{j^2,\rho}\right|}{|K_\rho|} \le \frac{C}{|\ln t|^{\frac{\varepsilon}{2}|\ln t|^{\frac{1}{2}}}}.$$

□

The parabolic version of this result has been used in [6].

6. Boundary Behavior of Functions in the De Giorgi Classes

Let $h \in W^{1,p}_{\rm loc}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$. The De Giorgi classes $[DG]^{+(-)}_p(\bar{E}; \gamma, h)$, in the closure of E are the collection of functions $u \in W^{1,p}_{\rm loc}(\bar{E})$, such that $(u - h) \in W^{1,p}_o(E \cap K_R(y))$, for all cubes $K_R(y)$ centered at some $y \in \partial E$, and satisfying

$$\int_{K_\rho(y)\cap E} |D(u-k)_{+(-)}|^p dx \le \frac{\gamma}{(R-\rho)^p} \int_{K_R(y)\cap E} (u-k)^p_{+(-)} dx \qquad (25)$$

for all pairs of congruent cubes $K_\rho(y) \subset K_R(y)$, centered at some $y \in \partial E$ and all levels

$$k \ge \sup_{K_R(y)\cap\partial E} h, \qquad \Big(k \le \inf_{K_R(y)\cap\partial E} h\Big). \qquad (26)$$

We let further

$$[DG]_p(\bar E;\gamma,h)=[DG]_p^+(\bar E;\gamma,h)\cap[DG]_p^-(\bar E;\gamma,h).$$

Functions in $[DG]_p(\bar E;\gamma,h)$ are continuous up to points $y\in\partial E$, provided E satisfies a positive geometric density at y, i.e., there exist ρ_o and $\eta\in(0,1)$, such that (see [12])

$$\big|E^c\cap K_\rho(y)\big|\ge\eta|K_\rho(y)|,\qquad\text{for all }\ \rho\le\rho_o.$$

For $1<p<N$, the p-capacity of the compact set $E^c\cap\bar K_\rho(y)$ is defined by

$$c_p[E^c\cap\bar K_\rho(y)]=\inf_{\substack{\psi\in W_o^{1,p}(\mathbb{R}^N)\cap C(\mathbb{R}^N)\\ E^c\cap\bar K_\rho(y)\subset[\psi\ge1]}}\int_{\mathbb{R}^N}|D\psi|^p dx.\tag{27}$$

For $1<p<N$, the relative p-capacity of $E^c\cap\bar K_\rho(y)$ with respect to $K_\rho(y)$ is

$$\delta_y(\rho)=\frac{c_p[E^c\cap\bar K_\rho(y)]}{\rho^{N-p}},\qquad(1<p<N).\tag{28}$$

If $p=N$, and for $0<\rho<1$, the N-capacity of the compact set $E^c\cap\bar K_\rho(y)$, with respect to the cube $K_{2\rho}(y)$, is defined by

$$c_N[E^c\cap\bar K_\rho(y)]=\inf_{\substack{\psi\in W_o^{1,N}(K_{2\rho}(y))\cap C_o(K_{2\rho}(y))\\ E^c\cap\bar K_\rho(y)\subset[\psi\ge1]}}\int_{K_{2\rho}(y)}|D\psi|^N dx.\tag{29}$$

The relative capacity $\delta_y(\rho)$ can be formally defined by (28), for all $1<p\le N$. For $p=N$, we let $\delta_y(\rho)\equiv c_N[E^c\cap\bar K_\rho(y)]$, as defined by (29). For a positive parameter ϵ denote by $I_{p,\epsilon}(y,\rho)$ the Wiener integral of ∂E at $y\in\partial E$, i.e.,

$$I_{p,\epsilon}(y,\rho)=\int_\rho^1[\delta_y(t)]^{\frac{1}{\epsilon}}\frac{dt}{t}.\tag{30}$$

The celebrated Wiener criterion states that a harmonic function in E is continuous up to $y\in\partial E$ if and only if the Wiener integral $I_{2,1}(y,\rho)$ diverges as $\rho\to0$ ([16]).

It is known that weak solutions of quasilinear equations in divergence form, and with principal part exhibiting a p-growth with respect to $|Du|$, when given continuous boundary data h on ∂E, are continuous up to $y\in\partial E$ if $I_{p,(p-1)}(y,\rho)$ diverges as $\rho\to0$ ([8]). Since such solutions belong to the boundary $[DG]_p(\bar E;\gamma,h)$ classes ([10]), it is natural to ask whether the divergence of the Wiener integral $I_{p,(p-1)}(y,\rho)$, is sufficient to insure the boundary continuity for functions $u\in[DG]_p(\bar E;\gamma,h)$.

The only result we are aware of in this direction is due to Ziemer ([17]). It states that a function $u \in [DG]_p(\bar{E};\gamma,h)$ is continuous up to $y \in \partial E$ if

$$\int_\rho^1 \exp\Big(-\frac{1}{\delta_y(t)^{\frac{1}{p-1}}}\Big)\frac{dt}{t} \;\to\; \infty \quad \text{as } \rho \to 0. \tag{31}$$

Ziemer's proof follows from a standard De Giorgi iteration technique. It has been recently established that local minima of variational integrals when given continuous boundary data h are continuous up to $y \in \partial E$ provided ([5]) $I_{p,\varepsilon}(y,\rho)$ diverges as $\rho \to 0$. Here ε is a number that can be determined a-priori only in terms of the growth properties of the functional. While such minima are in the classes $[DG]_p(\bar{E};\gamma,h)$, the result is not known to hold for functions merely in such classes. Also the optimal parameter $e = (p-1)$ remains elusive. A similar result has been recently obtained with a different approach in [1].

The significance of a Wiener condition for Q-minima, is that the structure of ∂E near a boundary point $y \in \partial E$, for u to be continuous up to y, hinges on minimizing a functional, rather than solving an elliptic p.d.e.

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