ON QUOTIENTS OF HOPF FIBRATIONS (*)

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Sommario. - *In questo articolo dimostriamo l'impossibilità di ottenere una sottofibrazione in cerchi del fibrato di Hopf su S^8.*

Summary. - *In this paper we prove the impossibility of obtaining a circle subfibration of the Hopf fibration over S^8.*

Consider the following Hopf fibrations:

(i) $S^1 \hookrightarrow S^{15} \to \mathbb{CP}^7$,

(ii) $S^3 \hookrightarrow S^{15} \to \mathbb{HP}^3$,

(iii) $S^7 \hookrightarrow S^{15} \to S^8$,

where $\mathbb{C}$ and $\mathbb{H}$ are the complex and quaternion division algebras respectively, and $\mathbb{P}^r$ denotes projective r-space.

The twistor fibration of $\mathbb{HP}^3$ is obtained via a quotient of 2 Hopf maps (items (i) and (ii) above):

$$\begin{array}{ccccc} & & \mathbb{C}^8 - \{0\} & = & \mathbb{H}^4 - \{0\} \\ & & \text{Hopf}_{\mathbb{C}} \downarrow & & \downarrow \text{Hopf}_{\mathbb{H}} \\ \mathbb{CP}^1 & \longrightarrow & \mathbb{CP}^7 & \xrightarrow{\pi} & \mathbb{HP}^3 . \end{array}$$

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One could ask whether other quotients exist, for example,

(a) the quotient of (iii) by S^1 or

(b) the quotient of (iii) by S^3.

However, the lack of associativity of the octonions precludes the possibility of such quotients. Nevertheless, one could ask whether there exist fibrations of the form $\mathbb{CP}^3 \hookrightarrow \mathbb{CP}^7 \to S^8$ or $\mathbb{HP}^1 \hookrightarrow \mathbb{HP}^3 \to S^8$.

Schultz proved that the homotopy analog of b) is not possible.

THEOREM (SCHULTZ [Sc]). *There does not exist a Hurewicz fibration $F \hookrightarrow E \to B$ fibre homotopy equivalent to $\mathbb{HP}^1 \hookrightarrow \mathbb{HP}^3 \to S^8$.*

It was shown in [LV] that there are no PL-bundles of the form $\mathbb{CP}^3_h \hookrightarrow \mathbb{CP}^7_h \to S^8$ where $\mathbb{CP}^k_h$ denotes a PL-manifold homotopy equivalent to $\mathbb{CP}^k$. It was stated at the end of [LV] that the homotopy analog of a) does not exist. In [U], Ucci showed that there exists no Hurewicz fibration of the form $\mathbb{CP}^3 \hookrightarrow \mathbb{CP}^7 \to S^8$. However, as stated, this was not the strongest possible result. Let $\mathfrak{H}\mathbb{CP}^n$, $\mathfrak{H}\mathrm{CaP}^2$ and S^n_h denote spaces homotopy equivalent to complex projective n-space $\mathbb{CP}^n$, the Cayley plane CaP^2 and S^n respectively. In this paper we adapt the proof of Jack Ucci [U] to show:

THEOREM. *There does not exist a Hurewicz fibration fibre homotopy equivalent to*

$$\mathfrak{H}\mathbb{CP}^3 \hookrightarrow \mathfrak{H}\mathbb{CP}^7 \to S^8_h.$$

An immediate corollary of this is the following:

COROLLARY. *The Hopf fibration $\pi : S^{15} \to S^8$ admits no S^1-subfibration arising from a free continuous S^1-action.*

This corollary generalizes the corresponding corollary of [LV] which considered free PL S^1-actions.

Proof of Corollary. The orbit space of a free continuous S^1-action on S^{2n+1} is a homotopy complex projective space $\mathfrak{H}\mathbb{CP}^n$. Thus, if such an S^1-subfibration were to exist, taking a quotient by the S^1-action would give us a Hurewicz fibration of the form $\mathfrak{H}\mathbb{CP}^3 \hookrightarrow \mathfrak{H}\mathbb{CP}^7 \to S^8$, contradicting the Theorem. $\diamondsuit$

Proof of Theorem. Recall that $\mathbb{CP}^7$ is 14-classifying for S^1-bundles. Let ξ denote the bundle $S^1 \hookrightarrow S^{15} \to \mathbb{CP}^7$. Let $\chi : \mathfrak{H}\mathbb{CP}^7 \to \mathbb{CP}^7$ be a homotopy equivalence. Then $\chi^*\xi$ is an S^1-bundle with total space a homotopy 15-sphere, S^{15}_h. We thus have an S^1-action on S^{15}_h with orbit space $\mathfrak{H}\mathbb{CP}^7$:

$$S^1 \hookrightarrow S^{15}_h \xrightarrow{\;g\;} \mathfrak{H}\mathbb{CP}^7.$$

Now suppose that there exists a Hurewicz fibration $\mathfrak{H}\mathbb{CP}^3 \hookrightarrow \mathfrak{H}\mathbb{CP}^7 \to S^8_h$. We then have the following diagram:

$$\begin{array}{ccccc} & & S^1 & & \\ & & \downarrow & & \\ & & S^{15}_h & & \\ & & \downarrow \pi & & \\ \mathfrak{H}\mathbb{CP}^3 & \longrightarrow & \mathfrak{H}\mathbb{CP}^7 & \xrightarrow{\;g\;} & S^8_h\ . \end{array}$$

Let $h : S^{15}_h \to S^8_h$ be defined by the composition $h := g \circ \pi$. This gives us the following diagram:

$$\begin{array}{ccc} \mathfrak{H}\mathbb{CP}^7 & \longrightarrow & \mathfrak{H}\mathbb{CP}^7 \cup_\pi e^{16} \simeq \mathfrak{H}\mathbb{CP}^8 \\ \downarrow g & & \downarrow G := g \cup_\pi \mathrm{id} \\ S^8_h & \longrightarrow & S^8_h \cup_h e^{16} \simeq \mathfrak{H}\mathrm{CaP}^2. \end{array}$$

Let $u \in H^8(\mathfrak{H}\mathrm{CaP}^2; \mathbb{Z})$ denote a generator of the cohomology ring $H^*(\mathfrak{H}\mathrm{CaP}^2; \mathbb{Z})$. Let $v := u^2 \in H^{16}(\mathfrak{H}\mathrm{CaP}^2; \mathbb{Z})$. Observe that $G^*v = x^8$ where $x \in H^2(\mathfrak{H}\mathbb{CP}^8; \mathbb{Z})$ is a generator of $H^*(\mathfrak{H}\mathbb{CP}^8; \mathbb{Z})$.

Let p be an odd prime and consider the Steenrod cohomology operation

$$P^i : H^q(Y;\mathbb{Z}_p) \to H^{q+2i(p-1)}(Y;\mathbb{Z}_p) \quad i \geq 0,\, q \geq 0.$$

In particular, we have $P^1 : H^q(Y;\mathbb{Z}_3) \to H^{q+4}(Y;\mathbb{Z}_3)$. We will let $[y]$ denote the reduction mod 3 of y for any $y \in H^q(Y;\mathbb{Z})$. Thus for $x \in H^2(\mathfrak{H}\mathbb{CP}^8;\mathbb{Z})$ as above, we obtain $P^1[x] = [x^3]$. Since the cohomology ring of the Cayley plane is generated by an element of dimension 8, P^1 acts trivially on $H^*(\mathfrak{H}\mathrm{CaP}^2;\mathbb{Z}_3)$. From the commutative diagram

$$\begin{array}{ccc} H^8(\mathfrak{H}\mathrm{CaP}^2;\mathbb{Z}_3) & \xrightarrow{P^1} & H^{12}(\mathfrak{H}\mathrm{CaP}^2;\mathbb{Z}_3) \\ \downarrow G^* & & \downarrow G^* \\ H^8(\mathfrak{H}\mathbb{CP}^8;\mathbb{Z}_3) & \xrightarrow{P^1} & H^{12}(\mathfrak{H}\mathbb{CP}^8;\mathbb{Z}_3) \end{array}$$

we see that $P^1G^*[u] = G^*P^1[u] = 0$. Now, $G^*[u] = [\lambda x^4]$ for some $\lambda \in \mathbb{Z}$ since x is a generator of the cohomology ring of $\mathfrak{H}\mathbb{CP}^8$. Thus,

$$\begin{aligned} 0 &= P^1G^*[u] = P^1([\lambda x^4]) = [\lambda P^1[x^4]] \\ &= [\lambda P^1([x^2]\cdot[x^2])] = [\lambda(2x^2)P^1[x^2]] \quad \text{by the Cartan formula} \\ &= [\lambda(2x^2)P^1([x]\cdot[x])] = [\lambda(2x^2)(2xP^1[x])] \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{by the Cartan formula} \\ &= [4\lambda x^3\cdot x^3] \quad \text{by item (ii) above} \\ &= [\lambda x^6] = [\lambda][x^6], \end{aligned}$$

and hence $\lambda = 3k$ for some integer k. In other words, $G^*u = 3kx^4$. We obtain

$$0 \neq x^8 = G^*v = G^*u^2 = (G^*u)^2 = 9k^2x^8,$$

a contradiction. This proves the theorem. ♢

Remark. Since the Calabi-Hopf-Penrose fibration $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^3 \xrightarrow{g} S^4$ *does* exist, we shall indicate why the preceding argument cannot be extended to this case. We can mimic the previous argument. Let

$\pi : S^7 \to \mathbb{CP}^3$ denote the Hopf fibration, and let $h := g \circ \pi$. We have the commutative diagram

$$\begin{array}{ccc} \mathbb{CP}^3 & \longrightarrow & \mathbb{CP}^3 \cup_\pi e^8 \simeq \mathbb{CP}^4 \\ \Big\downarrow g & & \Big\downarrow G := g \cup_\pi \mathrm{id} \\ S^4 & \longrightarrow & S^4 \cup_h e^8 \simeq \mathbb{HP}^2. \end{array}$$

Let $u \in H^4(\mathbb{HP}^2; \mathbb{Z})$ be a generator of the cohomology ring of $\mathbb{HP}^2$, and let $v = u^2$. Then as before, we have $G^* v = x^4$ where x is a generator of the cohomology ring of $\mathbb{CP}^4$. Using $\mathbb{Z}_2$ coefficients, observe that $P^1 : H^q(Y; \mathbb{Z}_2) \to H^{q+2}(Y; \mathbb{Z}_2)$ and letting $[y]$ denote the mod 2 reduction of a cocycle y, we have $P^1[x] = x^2$. Again, we see that P^1 acts trivially on $H^*(\mathbb{HP}^2; \mathbb{Z}_2)$. From the commutative diagram

$$\begin{array}{ccc} H^4(\mathbb{HP}^2; \mathbb{Z}_2) & \xrightarrow{P^1} & H^6(\mathbb{HP}^2; \mathbb{Z}_2) \\ \Big\downarrow G^* & & \Big\downarrow G^* \\ H^4(\mathbb{CP}^4; \mathbb{Z}_2) & \xrightarrow{P^1} & H^6(\mathbb{CP}^4; \mathbb{Z}_2) \end{array}$$

we have $P^1 G^*[u] = G^* P^1[u] = 0$. Since $G^*[u] = \lambda x^2$ and $P^1[x^2] = [2x^3] \equiv 0$, we get no information and hence cannot obtain a contradiction as before.

A corollary of the Theorem gives us a weak version of Schultz's theorem:

COROLLARY. *There does not exist a Hurewicz fibration of the form $\mathbb{HP}^1 \hookrightarrow \mathbb{HP}^3 \to S^8$ where $\mathbb{HP}^3$ denotes a standard quaternion projective 3-space.*

Proof. First, recall that we have the quaternionic twistor fibration of the quaternion-Kähler manifold $\mathbb{HP}^3$:

$$\mathbb{CP}^1 \longrightarrow \mathbb{CP}^7 \xrightarrow{\pi} \mathbb{HP}^3.$$

Suppose there exists a Hurewicz fibration $f : \mathbb{HP}^3 \to S^8$. Then, by composition, we obtain a Hurewicz fibration

$$f \circ \pi : \mathbb{CP}^7 \to \mathbb{HP}^3 \to S^8,$$

contradicting the theorem. $\diamondsuit$

Note that this argument made use of the quaternionic twistor fibration of $\mathbb{HP}^3$ in order to obtain the map f from $\mathbb{CP}^7$ to $\mathbb{HP}^3$. It is not clear that given a generic homotopy quaternion projective 3-space $\mathfrak{H}\mathbb{HP}^3$, there exists a Hurewicz fibration $g : \mathfrak{H}\mathbb{CP}^7 \to \mathfrak{H}\mathbb{HP}^3$. If such a map does exist, then the above proof could be used to prove Schultz's Theorem.

Remarks.

Recall from [LV] that a complex 4-plane bundle over S^8 with structure group $U(4)$ has Euler class which is a multiple of six times the generator of $H^8(S^8;\mathbb{Z})$. This fact followed from Bott periodicity. A question to ponder over is "what is the relation between the number 3 (from the $\mathbb{Z}_3$ coefficients in the proof of the main theorem) and the number 6".

References

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