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FISICA

Time-Dependent and Three-Dimensional Phenomena in Free-Electron Laser Amplifiers within the Integral-Equation Approach

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Riassunto

Introduzione

Il laser ad elettroni liberi (FEL) è un dispositivo operante con elettroni accelerati. Si tratta di una sorgente di radiazione elettromagnetica ad alta qualità in termini di potenza, coerenza, accordabilità, larghezza di banda e lunghezza d'impulso; il suo principio di funzionamento si basa su fenomeni fisici che coinvolgono elettroni in moto nel vuoto in presenza di un campo e.m. (sia statico che variabile nel tempo).

Tra le molte configurazioni possibili per il FEL, consideriamo l'*amplificatore a passaggio singolo*: il fascio elettronico attraversa un "ondulatore" (una struttura che induce un campo magnetostatico periodico nella regione spaziale attraversata dagli elettroni) per una volta sola, ed all'ingresso dell'ondulatore viene iniettato un "seme" (un fascio ottico generato con un laser convenzionale) che poi viaggia sovrapposto al fascio elettronico. Nel loro cammino gli elettroni oscillano trasversalmente per effetto della forza di Lorentz, e così si accoppiano alla componente trasversale del campo elettrico presente nel fascio ottico; ne risulta uno scambio di energia che, in opportune condizioni, amplifica il fascio ottico (mantenendone inalterata la lunghezza d'onda). All'uscita dall'ondulatore il fascio elettronico viene scartato ed il fascio ottico viene convogliato alla stazione sperimentale.

Originariamente inteso come dispositivo basato sulla meccanica quantistica [1], il FEL è stato poi descritto con successo in ambito puramente classico [2]. L'approccio presentato nel rif. [2] sviluppa un modello monodimensionale, trascurando di conseguenza gli effetti delle disomogeneità lungo le direzioni trasversali; esse sono: variazioni del campo magnetostatico dell'ondulatore, variazioni della densità elettronica, emittanza del fascio elettronico e fenomeni e.m. tridimensionali (diffrazione e guida ottica, che danno luogo ad un profilo trasversale di campo non uniforme). Inoltre, viene trattato sostanzialmente solo il regime stazionario (le caratteristiche di fascio elettronico e fascio ottico sono costanti nel tempo, lungo tutto l'ondulatore); di conseguenza, non vengono studiati effetti legati alla propagazione dell'impulso ottico, quali: variazione del guadagno, distorsione dell'impulso e risposta in frequenza.

L'elaborato

Scopo del presente lavoro è uno studio quantitativo dell'impatto sulle prestazioni del FEL di alcuni fenomeni non stazionari e tridimensionali. Il nostro approccio si basa su quanto segue:

- il campo magnetostatico dell'ondulatore è ancora considerato trasversalmente uniforme;
- il fascio elettronico è assunto essere continuo e longitudinalmente uniforme (la distribuzione elettronica all'ingresso dell'ondulatore non cambia nel tempo) e l'emittanza è ancora trascurata, ma le variazioni trasversali della densità elettronica sono prese in considerazione;
- infine, vengono studiate le conseguenze dell'iniezione di un impulso ottico e dell'azione combinata di diffrazione e guida ottica.

Nel corso degli ultimi 25 anni sono stati fatti molti sforzi al fine di sviluppare una teoria tridimensionale per il FEL. Nel rif. [3] la teoria monodimensionale è estesa al caso di un fascio elettronico uniforme limitato ad un cilindro; esistono allora soluzioni per il campo ottico che presentano un profilo trasversale indipendente dalla posizione longitudinale lungo l'ondulatore: sono i cosiddetti *modi guidati*. L'esistenza di modi guidati indica la presenza di fenomeni di *guida ottica*, i quali contrastano la naturale tendenza del campo a disperdersi per effetto della diffrazione e guidano il fascio ottico lungo la direzione longitudinale; il fenomeno è simile a quello che si incontra nelle guide d'onda metalliche o dielettriche (le fibre ottiche), ampiamente utilizzate nei sistemi di telecomunicazione.

Sempre partendo dalla teoria del rif. [2], proponiamo un'estensione tridimensionale nella quale il profilo trasversale di campo è approssimato da un profilo prefissato; quest'ultimo è il profilo trasversale del modo fondamentale di una guida dielettrica (virtuale). Come nel rif. [3], consideriamo il regime di *campo debole*: il campo ottico è di piccola intensità e le equazioni che descrivono il sistema possono essere opportunamente linearizzate. Un approccio simile al nostro, ma basato sulla fisica del plasma, si trova nei rif.ti [4, 5].

La nostra teoria si basa sull'*Equazione Integrale del FEL*. Si tratta di un'equazione integro-differenziale che descrive l'evoluzione del campo ottico in modo auto-consistente, senza coinvolgere le variabili dinamiche del fascio elettronico; originariamente ottenuta in regime stazionario da Colson et al. [2], è stata poi estesa al caso non stazionario da Gallardo et al. [6].

La prima parte dell'elaborato esordisce con una nuova derivazione dell'Equazione Integrale nel caso stazionario, basata su un'impostazione diversa da quella presentata nel rif. [2] e su altre tecniche matematiche; tale derivazione è stata pubblicata sulla rivista internazionale NIM-A [7]. Segue l'estensione al caso non stazionario, con applicazione allo studio della risposta in frequenza. È di particolare rilievo, nel contesto dell'analisi in frequenza, la derivazione della formula che descrive il fenomeno noto come *frequency pulling*; questa formula è stata dedotta su basi simulative in [8] e dimostrata analiticamente da noi. Presentiamo anche i risultati di una serie di simulazioni monodimensionali svolte dal dott. Simone Spampinati con il codice PERSEO [9], i quali risultano essere in ottimo accordo con le previsioni della nostra teoria.

La seconda parte dell'elaborato è dedicata all'estensione dell'Equazione Integrale al caso tridimensionale, sulla base del modello a guida dielettrica virtuale. Applichiamo questo approccio al caso in cui la densità elettronica abbia profilo trasversale gaussiano; il campo ottico risultante presenta profilo trasversale quasi gaussiano. Scegliendo opportunamente l'indice di rifrazione, il modo fondamentale della guida ha profilo trasversale gaussiano, e fornisce una buona base per descrivere il fascio ottico del FEL. La tecnica è detta *approssimazione a singolo modo gaussiano.* Per appurare l'accuratezza di questo metodo, abbiamo valutato l'evoluzione della potenza ottica in un caso specifico, e confrontato poi il risultato con quello ottenuto da simulazioni effettuate con GENESIS 1.3 [10]. L'accordo è più che soddisfacente.

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Abstract

Introduction

A free-electron laser (FEL) is a device working on an electron accelerator. It is intended to be a source of electromagnetic radiation with high quality in terms of power, coherence, tunability, bandwidth and pulse length; its working principle is based on the physics of electrons moving in vacuum in the presence of an e.m. field (both static and varying in time).

Among the various possible configurations for an FEL, we consider the *single-pass amplifier*: the electron beam passes through an "undulator" (a structure which induces a periodic magnetostatic field into the region where electrons move) just one time, and at the entrance to the undulator a "seed" (an optical beam generated by a conventional laser) is injected which then co-propagates with the electron beam. Due to Lorentz force, electrons wiggle on the transverse plane, thereby coupling to the transverse component of the optical-beam electric field; an energy exchange follows which, under proper conditions, amplifies the optical beam (yet, the wavelength is unaffected). At the exit from the undulator the electron beam is damped and the optical beam is focused to the experimental station.

Originally intended as a quantum-mechanical device [1], the FEL has later been described successfully in a purely classical framework [2]. The approach presented in ref. [2] develops a one-dimensional model, therefore neglecting any effect due to non-homogeneities along transverse directions; these are: variations of the undulator magnetostatic field, variations of the electron-beam density, electron-beam emittance and three-dimensional e.m. phenomena (diffraction and optical guiding, which result in a non-uniform transverse profile for the field). Also, only steady-state regime is substantially considered (electron-beam and optical-beam properties do not vary in time, all along the undulator); as a consequence, effects due to optical-pulse propagation (such as gain degradation, pulse distortion and frequency response) are neglected.

The work

Aim of the present work is a quantitative study of the impact on FEL performance of some time-dependent and three-dimensional phenomena. Our approach is based on what follows:

- the undulator magnetostatic field is still considered as transversally uniform;

- the electron beam is assumed to be continuous and longitudinally uniform (as time goes by, the electron distribution at the entrance to the undulator remains the same) and emittance is still neglected, but transverse variations of electron density are taken into account;

- lastly, consequences of optical-pulse injection and of the combined action of diffraction and optical guiding are analysed.

Over the last 25 years, lots of efforts have been done to develop a three-dimensional FEL theory. In ref. [3], one-dimensional theory is extended to the case of a uniform electron beam limited to a cylinder; solutions exist for the optical field which exhibit a transverse profile not depending on longitudinal position along the undulator: they are the so-called *guided modes.* The existence of guided modes reveals underlying *optical-guiding* phenomena, which counteract the natural diffraction of the field and guide the optical beam along the longitudinal direction; the phenomenon shares many similarities with propagation in waveguides, either metallic or dielectric (i.e., fiber optics), which are widely used in telecommunication systems.

Based on the theory of ref. [2], we propose a three-dimensional extension in which the field transverse profile is approximated by a profile chosen a priori; the latter is the transverse profile of the fundamental mode in a (virtual) dielectric waveguide. As in ref. [3], we consider the *weak-field* regime: the optical field is of weak intensity and the equations describing the system can be linearized properly. A similar approach, yet based on plasma physics, is found in refs. [4, 5].

Our theory is based on the *FEL Integral Equation.* This is an integro-differential equation which describes the optical-field evolution self-consistently, without involving electron-beam dynamic variables; initially derived on steady state by Colson et al. [2], this equation has been later extended to a time-dependent situation by Gallardo et al. [6].

The first part of the work starts by a novel derivation for the Integral Equation on steady state, based on different setting and mathematical techniques with respect to ref. [2]; this novel derivation has been published on the international review NIM-A [7]. A time-dependent extension follows, which is applied to the analysis of frequency response. Within the analysis in the frequency domain, a relevant achievement is the derivation for the formula describing the phenomenon known as *frequency pulling*; this formula, deduced on simulative bases in ref. [8], is here proved analytically for the first time. We also present results from a series of one-dimensional simulations performed by Dr. Simone Spampinati by means of the code PERSEO [9]; these results exhibit an excellent agreement to predictions from our theory.

The second part of the work is devoted to a three-dimensional extension for the Integral Equation, based on the virtual-dielectric-waveguide model. We apply this approach to an electron density having a gaussian transverse profile; the resulting optical field exhibits a quasi-gaussian transverse profile. By choosing properly the index of refraction, the fundamental waveguide mode has a gaussian transverse profile, and yields a fine basis to describe the FEL optical beam. The technique is called *single-gaussian-mode approximation*. In order to analyse the accuracy of this method, we evaluate optical-power evolution in a specific case, and then compare our result to what is found from GENESIS 1.3 [10] simulations. The agreement is more than satisfactory.

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Contents

Chapter 1

Introduction

1.1 The free-electron laser

A Free-Electron Laser (FEL) is a source of coherent light based on the interaction between an ultra-relativistic electron beam and a co-propagating electro-magnetic (e.m.) field, the so-called "optical beam". The former, provided by a particle accelerator, passes through a periodic magnetostatic field (generated by a periodic magnet, called "undulator") which induces a periodic wiggling on the electrons; the latter may be described as an e.m. wave which propagates along the electron drift direction and exhibits an electric field oriented along the electron wiggling direction. This field couples to the transverse component of the electron speed, exchanging energy with the electrons.

Under proper conditions, the field grows from a weak intensity to a strong one (up to many gigawatts); therefore, the FEL evolves from a *weak-field regime* to a *strong-field* one. Growth ends eventually when the optical field is so strong that it modulates the electron distribution up to a point where electrons take back their energy, thereby damping the field. This condition is called *saturation*.

Originally intended as a quantum-mechanical device [17], the FEL has later been described succesfully in a purely classical framework [14]. The treatment reported in reference [14] still provides the easiest theoretical approach to FELs. Within such a theory, the evolution of the optical beam may be deduced by means of various techniques [23, 24, 25, 26]; among them, the most compact one has been proposed by Colson et al. in reference [20]: in the weak-field regime, an integro-differential equation is derived which only involves the optical field as an unknown and describes fully its evolution. This equation is called "FEL Integral Equation" (see also reference [14]).

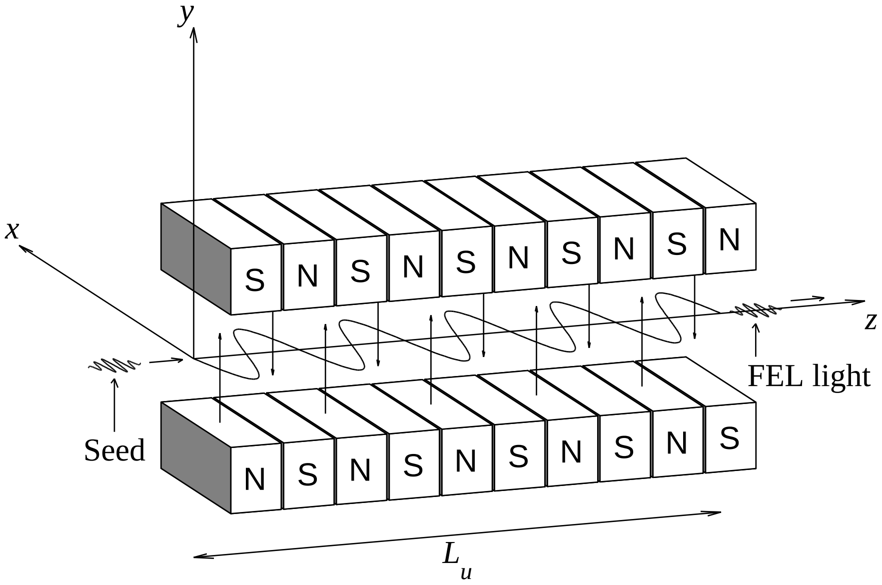

Figure 1.1: FEL setup. Small boxes are the dipoles which compose the undulator; N, S are their north and south poles; small arrows represent the undulator magnetic field; the sinusoidal line is the electron wiggling path; seed and FEL light are represented by the electric field.

The present work is mainly concerned with a theoretical speculation on the FEL phenomenon on the basis of the Integral Equation: we show how this equation allows understanding a recently discovered time-dependent phenomenon called "frequency pulling" and propose a technique for extending the equation to a well-known three-dimensional phenomenon called "optical guiding". Our analysis leads to formulas of practical interest, whose predictions are compared to numerical simulations.

1.2 FEL setup

In order to present clearly the fundamentals of FEL theory, let us describe in some detail the setup of the system. We refer to figure 1.1.

We deal with the so-called "single-pass amplifier" [14, 15, 23, 24, 25, 26, 5, 6, 7, 8]: the electron and optical beams are assumed to go through the undulator only once and the start-up to the interaction is provided by an external optical signal, often called the "seed", which is injected into the undulator together with the electron beam. We consider a setup with a single undulator, where the wavelengths of seed and FEL light coincide.

The undulator magnetostatic field is linearly polarized: it has a constant direction (which we choose as y axis) and oscillates sinusoidally as a function of z. The z axis is the undulator axis, i.e. the drift direction of the electron beam; the undulator begins at $z = 0$ and is long L_u (from a few meters up to 100 m)[1].

Many other FEL setups exist which find interest in theory and experiment. Very often, the single-pass amplifier we deal with is analysed in the case of circular polarization: the undulator field has a constant strength but rotates in the transverse plane as z moves. Here the electron motion is much easier to study than in the case of linear polarization, so the physics of the FEL interaction is understood in a more straightforward way [14]. Within single-pass FELs, an important modification to the amplifier setup is the self-amplified spontaneous emission (SASE) FEL, where the start-up to the interaction is given by the spontaneous emission of the electrons; this removes the need for a seed and allows lasing at x rays, where coherent seeds are not available, but reduces the temporal coherence of FEL light [16, 8]. Other ways for reaching short wavelengths are the coherent armonic generation (CHG) and the high-gain harmonic generation (HGHG) FELs, where the use of (at least) two undulators allows lasing at a higher harmonic with respect to the seed[2]; FEL light is then fully coherent, but the minimum allowed wavelength is limited to the soft-x-ray region due to electron shot noise [16]. The storage-ring FEL and the FERMI project at the Elettra laboratory in Trieste, Italy are, respectively, a CHG and an HGHG FEL [11, 1]. Lastly, we mention the FEL oscillators; in this case, electrons are provided by a storage ring and get through the undulator many times, while light is stored in an optical cavity enclosing the undulator [14]. This allows operation at low gain, but optical cavities are not available at x rays. The storage-ring FEL at Elettra operated as an oscillator until 2006 [10].

1.3 Introduction to FEL theory

This section is devoted to introducing the phenomena which constitute the object of the work. After a short review of steady-state one-dimensional theory, we introduce the time-dependent and three-dimensional phenomena which are studied in the following chapters.

[1] These are the order of magnitude of the undulator length in VISA [5] and LCLS [8].

[2] In the first undulator, the "modulator", the electron beam interacts with the seed and bunches at the seed wavelength; then, in the second undulator, the "radiator", the bunches behave as macro-particles spontaneously emitting at the chosen harmonic.

1.3.1 Steady-state one-dimensional theory

FEL physics can be analysed by considering the electron beam as a collection of point particles, whose motion under the influence of Lorentz force is ruled by relativistic mechanics, and the optical beam as an e.m. field, ruled by Maxwell equations [14, 15, 18, 19]. Other techniques have been proposed, by looking at the electron beam as if it were a continuous distribution of matter and analysing its evolution by means of Vlasov equation [16]; these techniques make use of much powerful mathematical tools, which allow extension to more realistic setups, but pay the price of being more complicated.

In reference [14], the phenomenon is described on the basis of a one-dimensional model which relies on the following hypotheses. The undulator field is transversally uniform; the electron-beam density and energy distribution as well; emittance (which is related to the angular spread of the electrons) is neglected; lastly, also the optical beam is transversally uniform. Within these assumptions, the problem is fully defined once the electron density and energy distribution at the entrance to the undulator are assigned, and proper initial conditions are set by giving the electric field at $z = 0$ as a function of time.

The situation is further simplified by assuming the electron beam to be continuous and time-invariant (its density and energy distribution at $z = 0$ are constant); lastly, the seed is monochromatic (it is a sinusoidal function of time). In these conditions, it is reasonable to assume that all relevant "macroscopic" quantities[3] are time-invariant throughout the undulator: this is called *steady-state* regime.

1.3.2 Time-dependent phenomena

One of the most recently tested FEL configurations is the amplifier seeded by a VUV signal obtained from high-order harmonic generation (HHG) in a gas cell [6, 7]. Optical pulses from an HHG source are very short (a few fs) [12]: their length is comparable to the slippage length[4] which is tipically found in new-generation FELs (some μm) [7, 8, 2, 3, 4]. On the other hand, electron bunches from an FEL linac are much longer (around 1 ps). Within this framework, the electron beam can still be considered as continuous, while the optical-beam length must be accounted for by introducing a time

[3] We consider as macroscopic a quantity which describes a system property in a region of space being large with respect to the optical wavelength: such is any average over electrons in one or more optical wavelengths, as well as the field envelope.

[4] "Slippage" is the relative motion of electrons with respect to light (electrons are slower than light).

dependence on the field envelope. As a straightforward consequence of the interplay between slippage and optical pulse, the FEL interaction is affected by a gain reduction, a modification of the pulse propagation speed and a pulse distortion [30].

When looked at in the frequency domain, these time-dependent phenomena appear as a filtering between the spectra of seed and FEL light: as far as the system is in the weak-field regime, the optical pulse undergoes a linear transformation and its frequency components evolve independent on each other, being only amplified or attenuated (and phase rotated) (see chapter 5). This filtering may induce a shift on the central frequency of the light, by an amount depending on FEL parameters. Now, an HHG seed is hardly tuned but exhibits a broad band (due to its short pulses): based on these speculations, much research is currently going on in order to establish whether it is possible to tune an ultrashort-seeded FEL by moving its resonance frequency[5] and find a simple formula for the frequency of FEL light [32, 33, 34, 35].

1.3.3 Three-dimensional phenomena

In a single-pass FEL amplifier, a long undulator is necessary in order to reach the high-gain regime and obtain a significant gain. As light moves through the undulator, its interaction with electrons may be reduced by diffractive spreading. Yet, under certain circumstances the transverse inhomogeneity of the electron beam may lead to a phenomenon known as *optical guiding*: after a transient in the first part of the undulator, the optical beam is confined near the electron beam and amplified with a stable transverse profile, up to the onset of saturation [38].

Guided propagation is characteristic in waveguides [13]. A *waveguide* is a cylindrical material structure which is homogeneous along the longitudinal axis and inhomogeneous in the transverse plane; examples are the metallic waveguides used in telecommunications and fiber optics. In a waveguide, any (monochromatic) e.m. field can be expressed as a superposition of separate-variable functions, i.e. functions which come out as a product between a function of the transverse position and a function of the longitudinal coordinate, called *modes* of the waveguide. There is a numerable infinity of modes. Each of them has its own transverse and longitudinal profile; if the waveguide is lossless, a mode is either *propagating* (its z-dependence is purely oscillatory) or *evanescent* (its z-dependence is purely (real) exponential). Whether

[5]The resonance frequency is the seed frequency which, on steady state, yields the best electron-light coupling; it can be moved by adjusting the undulator gap or the electron energy.

the mode is propagating or evanescent, it depends on the working frequency: each mode has its own *cutoff frequency*, and propagates if the working frequency is greater than the cutoff.

A similar analysis can be performed in an FEL with a continuous electron beam: the e.m. field can be expressed as a superposition of separate-variable functions which are called *guided modes* [38]. These modes differ from propagating modes of a waveguide in that they present both an oscillatory and a growing z profile: the growth is due to the interaction with the electron beam, which exchanges energy with the field providing a gain. Each mode has its own growth rate, so that one of them will eventually dominate over the others, thereby giving rise to the optical-guiding phenomenon described above.[6]

The guided modes may be determined by solving a non-linear functional eigenvalue problem, which requires a numerical approach [39]. When the transverse distribution of the electron beam is gaussian, the fundamental mode[7] is found to be approximately gaussian in the transverse plane [39]; usually this is the dominant mode, so much research is devoted to finding a simple way for deriving its transverse and longitudinal properties. The most famous results in this sense have been obtained by Xie et al. [39]: based on a variational technique, these authors have derived a gaussian approximation for the mode and a fitting formula for its longitudinal growth. More recently, Hemsing et al. have proposed a virtual-dielectric-waveguide model for describing optical guiding in an FEL [41, 42]; under the condition of considering a continuous, parallel and cold beam and of being on steady state, this technique is capable of estimating the main properties for all modes.

1.4 Original contributions

Having thus introduced the phenomena we deal with, now we present our main original contributions: namely, an analytical theory for the frequency-pulling phenomenon and an extension of FEL equations to a guided mode.

[6]This description only refers to growing modes, which are the relevant part of the optical beam. As it is shown in [38], a complete expansion involves a numerable infinity of growing modes, a numerable infinity of decaying modes and a continuum of purely oscillatory modes.

[7]I.e. the mode whose transverse profile has the fewest oscillations.

1.4.1 The frequency-pulling phenomenon

In a conventional laser, light is generated within an optical cavity. An active medium fills partially the cavity and amplifies the light by a phenomenon known as "stimulated emission". The gain provided by the active medium depends on the radiation frequency; on its part, the cavity behaves as a resonator, holding a small range of frequencies around its resonance and damping the others[8]. The two phenomena are in competition and the central frequency of the laser radiation depends on both. Now, the frequency range over which the cavity resonates is always much smaller than the one over which the active medium provides a significant gain, so the central frequency of the radiation is close to the cavity resonance: the active medium "pulls" slightly the working frequency towards its maximum gain. The phenomenon is known as "frequency pulling" [36].

In an FEL amplifier, the seed pulse carries a whole range of frequencies. The gain due to the FEL interaction exhibits a maximum near the resonance frequency, and the range over which it remains significantly high is comparable to the seed bandwidth. As a consequence, when the central frequency of the seed and the frequency of maximum gain do not coincide the spectrum of the optical pulse is significantly modified by the FEL interaction; in particular, the central frequency is "pulled" towards the frequency of maximum gain.

Based on the analogy with the above-described phenomenon in conventional lasers, recently this frequency shift in FELs has been referred to as *frequency pulling* [33]. In reference [33], a simple formula for the central frequency of FEL light has been proposed; this formula has been deduced on empirical bases by adapting an analogous formula valid for conventional lasers, and checked for the FERMI HGHG FEL [1] by GINGER [45] simulations. Later on, an experiment at the Elettra storage-ring FEL has confirmed the validity of this description [34] and an experiment at Brookhaven National Laboratory has demonstrated a wideband tunability for an FEL amplifier [35].

In chapter 5 we propose a fully analytical theory for frequency pulling in FEL amplifiers, leading to the formula proposed in reference [33] with slight modifications. This theory is based on a time-dependent version of the FEL Integral Equation. The theory is then applied to the FERMI case in direct-seeding configuration[9] and compared to PERSEO simulations [43], showing an excellent agreement.

[8]Here we only deal with the fundamental cavity mode.

[9]I.e., we consider an FEL amplifier obtained by excluding modulator and dispersion section and directly seeding the radiator by an HHG seed.

1.4.2 FEL equations for a guided mode

The second part of the work is devoted to a simple technique for approximating the fundamental guided mode on steady-state regime. As it is suggested in references [41, 42], we represent the mode by the usual one-dimensional complex envelope, which modulates a three-dimensional carrier given by the fundamental mode of a (virtual) dielectric waveguide.

References [41, 42] provide a full developement for the expansion technique based on the modes of a virtual dielectric waveguide. Yet, this approach proves to be much complicated; the origin of much of its complexity lays in the description of the electron beam, which is based on plasma theory. We develope a more simple technique by attacking the problem on the basis of Colson's theory, which is expected to provide simpler formulas than those obtained from plasma theory. This is done by deriving a novel version of the Pendulum and Field Equations which takes into account transverse profile and wavelength of the guided mode. Under weak-field condition, these equations are then reduced to the Integral Equation for the guided mode, which only involves the field envelope as an unknown and can be solved for the mode longitudinal profile.

We stress upon the fact that our technique is much simpler than the approach presented in references [41, 42]. A further improvement is that it is valid for a warm electron beam[10]. Moreover, it is expected to be easily extended to a time-dependent situation, and the non-linearized Pendulum and Field Equations may be used to simulate numerically the system evolution at the onset of saturation, as far as the transverse profile of the optical field is close to the one of the fundamental guided mode.

1.4.3 Further contributions

During our path towards the Integral Equation we will present two minor original contributions. The first (chapter 2) is concerned with the e.m. wave equation: this equation, which is the most useful tool for analysing any time-varying e.m. field, is usually expressed in terms of the vector potential, either in the Lorentz or in the Coulomb gauge [13]. In FEL theory the Coulomb gauge is universally adopted (this choice is due to the fact that it highlights space-charge contributions); many treatments apply the wave equation to the vector potential and derive the field from the latter (see, e.g., reference [14]), but it is also possible to express directly the wave equation in terms of the electric field by substituting the relation leading from potential to field into the equation for the potential: with this approach any further need for the

[10] I.e., it includes an initial electron-energy spread.

potential is avoided (see, e.g., reference [16]). Note that the equation for the potential is obtained from Maxwell equations by introducing a mathematical artifice (the potential itself) and the equation for the field is derived by eliminating this artifice, so it is clear that the electric-field wave equation is a direct consequence of Maxwell equations: we will show that this is true (i.e., no potential is needed to go from Maxwell equations to the electric-field wave equation).

The second contribution (chapter 4) is a novel procedure for deducing the bunching factor from the Pendulum Equation. The key point in the derivation of the Integral Equation is expressing the bunching factor in terms of the field envelope; this is accomplished by linearizing the Pendulum Equation, integrating and performing an average over the electron initial conditions. The integration has previously been performed by double-integrating the equation (which is a second-order ordinary differential equation) and then expressing the double integral as a single one [15]; this is indeed a mathematically correct way of working, but is also rather complicated and looses the physical meaning of the original equation. We propose an alternate point of view: by leaving mathematical formalisms a little bit aside and working with actual differentials, we will quickly turn the Pendulum Equation into an integral transformation, and our approach is physically transparent (i.e., it keeps the equation physical meaning throughout).

1.4.4 Review contributions

Lastly, the rest of this work (chapter 3) is devoted to a review of Colson's one-dimensional theory [14]; this will provide the basis for the three-dimensional extension and will define our formalism. As far as the latter is concerned, the main difference with respect to what is usually found in literature is that we do not introduce any adimensional quantity, except for the electron phase: on the countrary, we work directly with physical entities, thereby allowing an immediate physical interpretation for the resulting equations.

1.5 Organization of the work

This work is organized as follows. In chapter 2 we derive the equation which rules the propagation of e.m. waves in the presence of an arbitrary current distribution. Next, in part I we deal with one-dimensional FEL theory: in chapter 3 we review the basic equations which rule the FEL interaction, in chapter 4 we propose a novel derivation for the Integral Equation and in chapter 5 we study the frequency-pulling phenomenon. Lastly, in part II we propose a three-dimensional extension for FEL theory suitable for an analysis of the optical-guiding phenomenon: in chapter 6 we extend the FEL equations to a guided mode and in chapter 7 we derive the related extension for the Integral Equation. Chapter 8 reports numerical results of our three-dimensional theory and ends by summarizing positive and negative aspects of this work, as well as some possible improvements.

Chapter 2

The e.m. wave equation

2.1 Introduction

In dealing with any e.m. problem, the first thing to point out is the equation the whole theory will be based on. In classical electrodynamics, the configuration of the e.m. field is described by Maxwell equations [13], but, expecially as far as dynamical problems are concerned, these equations are not very easy to handle directly as they stand. Yet, they can be reduced to relations which involve only one field and stand out as wave equations: this allows a plainer analysis of the situation, since wave equations are separable (both vectorially and with respect to variables) and have been widely studied in literature.

2.2 Lorentz gauge or Coulomb gauge?

The usual method for obtaining e.m. wave equations, customarily used in theoretical and applied electrodynamics (e.g. in antenna theory), gets its start from the introduction of the e.m. potentials, i.e. a scalar field ϕ (the "scalar potential") and a vectorial field $\vec{A}$ (the "vector potential"). We will not report the relationships linking $\phi, \vec{A}$ to $\vec{E}, \vec{H}$: it is enough knowing that these relations allow to obtain, from Maxwell equations, a couple of equations in $\phi, \vec{A}$; moreover, the potentials are not uniquely determined and further simplification can be performed by choosing the right "gauge".

Choosing a gauge means choosing, among all possible $\phi, \vec{A}$ couples leading to the e.m. field, one which allows a simpler link with the sources (charge and current). As mentioned, there is a couple of equations involving potentials and sources; clearly, one will try to simplify them by choosing a gauge in which each of the equations involves only one of the potentials.

Literature reports two ways to accomplish this [13]. The first is the so-called *Lorentz gauge*, which sets a further link between the potentials and obtains wave equations both in ϕ and in $\vec{A}$; this is the choice usually adopted, e.g., in antenna theory. The second way is the so-called *Coulomb gauge*, which imposes the vector potential to be solenoidal (i.e. divergence-free) and is customarily used in magnetostatics, where $\vec{A}$ turns out to satisfy Poisson equation.

A more esoteric usage for the Coulomb gauge is found to be useful in FEL theory. In a time-varying context, by splitting the source current into a "longitudinal" and a "transverse" component the potential equations give rise to a wave equation involving only $\vec{A}$ and the transverse current; the electric field is then expressed as a linear combination of the vector potential (or better its time derivative) and the longitudinal current (or better its time integral), plus an initial value. If the source current is varying mainly along a single direction, as is the case in FELs, its longitudinal and transverse components are easily determined (they are simply the projections along the direction of maximum variation and on the transverse plane) and this approach proves to be very powerful; moreover, the contribution due to the longitudinal current is nothing else than the "space-charge" effect, and at first can be neglected.

A final remark concerning Coulomb gauge in FEL theory. Although early approaches used to deal with the vector potential [14], the most mathematically advanced theories start from a wave equation involving directly the electric field [16], which can be obtained from the equation in $\vec{A}$ by simply substituting the relation $\vec{A} \to \vec{E}$. Since this wave equation in $\vec{E}$ involves only the electric field and the source current, and *not* the potentials, one will expect it to come out as a consequence of Maxwell equations without the need of any potential: and, in fact, this is what we are going to show in next section.

2.3 The wave equation

In this section, we derive the equation which rules the evolution of e.m. waves in the presence of an arbitrary current distribution. First, we recall Maxwell equations; then, we derive the wave equation.

2.3.1 Maxwell equations

In a classical framework, Maxwell equations are generally accepted as a starting point for describing the e.m. field [13]. They are a set of four differential equations, having as source terms a charge distribution and a current distribution and as unknowns, in general, the following four vectorial fields:

- $\vec{E}$ (*electric field*)
- $\vec{H}$ (*magnetic field*)
- $\vec{D}$ (*electric-flux density* or *electric displacement*)
- $\vec{B}$ (*magnetic-flux density* or *magnetic induction*)

all of which are functions of space (denoted by $\vec{r}$) and time (denoted by t).

The first couple of Maxwell equations (the divergence equations) are only needed when dealing with static fields (or better with time-varying fields which include a static component); since we are only concerned with the time-varying part of the e.m. field, we will only consider the second couple (the curl equations):

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{2.1}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} \tag{2.2}$$

where $\vec{J}$ (vectorial function of space and time) is the source-current density.

Lastly, having a set of two equations with four unknowns, we need two more equations in order to have a well-defined problem: these are the so-called *constitutive equations* of the medium, and caracterize the e.m. behaviour of the material filling the region we consider. In FELs, the e.m. field lives in a high-void region, and his co-tenants, the electrons, are taken into account for by including them into the source current (they *are* the source current); so, the FEL e.m. field obeys free-space constitutive equations:

$$\vec{D} = \varepsilon_0 \vec{E}$$

$$\vec{B} = \mu_0 \vec{H}$$

where the constants ε_0, μ_0 are the *permittivity* (or *dielectric constant*) and the *permeability* (or *magnetic permeability*) in free space.

2.3.2 Wave equation

Now, as mentioned, we derive an e.m. wave equation having the field as an unknown directly from Maxwell equations (without introducing any potential). Since electrons exchange energy only with the electric field and not with the magnetic one, we only consider the former and derive a wave equation having $\vec{E}$ as an unknown and the source current as a known term.

Let us start. By taking the curl of equation (2.1) and using the constitutive equations, we get

$$\nabla \times \nabla \times \vec{E} = -\frac{\partial}{\partial t}\nabla \times \vec{B} = -\mu_0 \frac{\partial}{\partial t}\nabla \times \vec{H}$$

where

$$\nabla \times \nabla \times \vec{E} = \nabla \left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E}$$

and, thanks to equation (2.2),

$$-\mu_0 \frac{\partial}{\partial t}\nabla \times \vec{H} = -\mu_0 \frac{\partial^2 \vec{D}}{\partial t^2} - \mu_0 \frac{\partial \vec{J}}{\partial t} = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} - \mu_0 \frac{\partial \vec{J}}{\partial t}$$

where, again, the constitutive equations have been used, and we have introduced the *speed of light*

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

Putting things together, we end up with

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \nabla \left(\nabla \cdot \vec{E}\right) + \mu_0 \frac{\partial \vec{J}}{\partial t} \tag{2.3}$$

Note that the left-hand side of this last equation already has the form it would have in a wave equation.

Now, the source-current function $\vec{J}$ can be splitted in two functions $\vec{J}_{//}$ (*longitudinal current*) and $\vec{J}_{\perp}$ (*transverse current*), the former being curl-free and the latter divergence-free:

$$\vec{J} = \vec{J}_{//} + \vec{J}_{\perp}$$

with

$$\nabla \times \vec{J}_{//} = 0$$

$$\nabla \cdot \vec{J}_{\perp} = 0$$

These two functions are defined uniquely except for a spatially uniform function (i.e., if we add such a function to $\vec{J}_{//}$ and subtract it from $\vec{J}_{\perp}$ the new couple of currents is still a good choice).

With the aid of this splitting, we can write the divergence of equation (2.2) as

$$\nabla \cdot \nabla \times \vec{H} = \frac{\partial}{\partial t}\nabla \cdot \vec{D} + \nabla \cdot \vec{J} = \varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{E} + \nabla \cdot \vec{J}_{//} + \nabla \cdot \vec{J}_{\perp}$$

and, since both $\nabla \cdot \nabla \times \vec{H}$ and $\nabla \cdot \vec{J}_{\perp}$ vanish, we get

$$\frac{\partial}{\partial t}\nabla \cdot \vec{E} = -\frac{1}{\varepsilon_0}\nabla \cdot \vec{J}_{//} \tag{2.4}$$

Equation (2.3) still differs from a wave equation by the first term at its right-hand side, which contains the unknown; by comparing it to formula (2.4), we immediately realize that we can substitute this term by a source term if we take the time derivative of (2.3):

$$\frac{\partial}{\partial t}\left(\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}\right) = \nabla \frac{\partial}{\partial t}\nabla \cdot \vec{E} + \frac{\partial}{\partial t}\left(\mu_0 \frac{\partial \vec{J}}{\partial t}\right) \tag{2.5}$$

By substituting (2.4), we obtain

$$\nabla \frac{\partial}{\partial t}\nabla \cdot \vec{E} = -\frac{1}{\varepsilon_0}\nabla\left(\nabla \cdot \vec{J}_{//}\right) \tag{2.6}$$

Our last trick comes out from the vectorial identity

$$\nabla \times \nabla \times \vec{J}_{//} = \nabla\left(\nabla \cdot \vec{J}_{//}\right) - \nabla^2 \vec{J}_{//}$$

where $\nabla \times \vec{J}_{//} = 0$, so

$$\nabla\left(\nabla \cdot \vec{J}_{//}\right) = \nabla^2 \vec{J}_{//}$$

and formula (2.6) becomes

$$\nabla \frac{\partial}{\partial t}\nabla \cdot \vec{E} = -\frac{1}{\varepsilon_0}\nabla^2 \vec{J}_{//} \tag{2.7}$$

Now, the second term at the right-hand side of equation (2.5) is

$$\frac{\partial}{\partial t}\left(\mu_0 \frac{\partial \vec{J}}{\partial t}\right) = -\frac{1}{\varepsilon_0}\left(-\frac{1}{c^2}\frac{\partial^2 \vec{J}_{//}}{\partial t^2}\right) + \frac{\partial}{\partial t}\left(\mu_0 \frac{\partial \vec{J}_{\perp}}{\partial t}\right) \tag{2.8}$$

Finally, by putting formulas (2.7) and (2.8) into equation (2.5) we get

$$\frac{\partial}{\partial t}\left(\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}\right) = -\frac{1}{\varepsilon_0}\left(\nabla^2\vec{J}_{//} - \frac{1}{c^2}\frac{\partial^2\vec{J}_{//}}{\partial t^2}\right) + \frac{\partial}{\partial t}\left(\mu_0\frac{\partial\vec{J}_\perp}{\partial t}\right)$$

and, after a little sorting,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial\vec{E}}{\partial t} + \frac{1}{\varepsilon_0}\vec{J}_{//}\right) = \frac{\partial}{\partial t}\left(\mu_0\frac{\partial\vec{J}_\perp}{\partial t}\right) \tag{2.9}$$

which, at last, is a wave equation.

Let us play a little bit further with this last equation.

1. *Neglecting space charge.*
   As mentioned, the contribution due to the longitudinal current is the space-charge effect. In order to neglect this phenomenon, we simply substitute $\partial\vec{E}/\partial t + \vec{J}_{//}/\varepsilon_0$ by $\partial\vec{E}/\partial t$ (to include space charge, just read this substitution backwards).[1] The result is

$$\frac{\partial}{\partial t}\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = \frac{\partial}{\partial t}\left(\mu_0\frac{\partial\vec{J}_\perp}{\partial t}\right) \tag{2.10}$$

2. *Neglecting static field.*
   Now, equation (2.10) implies

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = \mu_0\frac{\partial\vec{J}_\perp}{\partial t} \tag{2.11}$$

   except for a time-independent function, which only affects the static component of the electric field. Since we are only concerned with time-varying fields, we work with equation (2.11).

3. *One-dimensional limit.*
   Lastly, if all quantities are spatially varying along a single direction (denoted by z), and in particular $\vec{E} = \vec{E}(z,t)\,, \vec{J}_\perp = \vec{J}_\perp(z,t)$, equation (2.11) turns into

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = \mu_0\frac{\partial\vec{J}_\perp}{\partial t} \tag{2.12}$$

   which is the starting point for a one-dimensional FEL theory.

[1] We do not introduce a further symbol here for the field without space charge, in order to avoid a cumbersome notation.

Part I

One-dimensional theory

Chapter 3

FEL equations

3.1 Introduction

This chapter is devoted to a review of steady-state one-dimensional FEL theory. In section 3.2 we focus on the electron beam and derive the "Pendulum Equation", which rules the electron motion; it allows determining the electron-beam evolution once the optical field is known. Then, in section 3.3 we focus on the optical beam and derive the "Field Equation", which allows determining the optical-beam evolution once the electron distribution is known. By coupling the two equations, we will be able to deduce the FEL evolution.

Our treatment is fundamentally the one presented in references [14, 15, 18, 19]. The main differences lie in the system of units (we use International-System (mskA) units, while [14, 15, 18, 19] use Gaussian (cgs) units) and in the use of dimentional quantities (we describe the system directly by physical quantities, while [14, 15, 18, 19] use dimensionless normalized quantities).

3.2 Electron motion

In this section, we describe the motion of an electron passing through the FEL undulator and interacting with the optical beam. The phenomenon is ruled by relativistic mechanics in the presence of Lorentz force.

3.2.1 Electron wiggling in a linearly-polarized undulator

The main component of an FEL device is an undulator, i.e. a structure which is able to induce a spatially-periodic magnetostatic field into the region where electron motion occurs (see figure 1.1). Usually, such a structure is composed by a set of permanent magnets, placed around the void chamber where electrons run and oriented properly. We take for granted that the field integral along the undulator axis is null, so that incoming electrons do not undergo any net kick as they pass through.

According to magnetostatic-field polarization, undulators may be classified as follows.

- *Linearly-polarized undulators*: the field is oriented along a fixed direction, transverse to the undulator axis, and its amplitude changes periodically along the undulator.

- *Circularly-polarized undulators*: the field has a fixed amplitude and rotates in the transverse plane as a clock's arm.

This is a preliminary distinction, based on ideal behaviours and neglecting the chance of having intermediate situations, such as, e.g., in an *elliptically-polarized undulator*, where the field rotates as in a circularly-polarized undulator but describes an ellipse instead of a circle.

The simplest polarization for an undulator, at least from the technological point of view, is the linear one, and due to this reason (and many others...) most FELs are based on linearly-polarized undulators. As mentioned in section 1.2, we consider only this kind of structure.

Undulator field

Let us introduce a standard system of cartesian coordinates x, y, z: the z axis is the undulator axis, oriented according to the electron motion, and the x, y axes are horizontal and vertical, respectively (the latter being oriented from bottom to top). A linear undulator should ideally induce the following magnetostatic field[1]:

$$\vec{B}_u(z) = B_0 \sin k_u z \, \hat{y} \tag{3.1}$$

where the field is assumed to be oriented vertically, $k_u = 2\pi/\lambda_u$ is the "undulator wavenumber" (λ_u is the *undulator period*, i.e. the field spatial period) and B_0 is a (real) constant giving the field strength. Usually, λ_u is on the order of a few centimeters [9, 6, 7, 8, 3] and B_0 is on the order of 1 T [16]. Clearly, expression 3.1 is valid inside the undulator; the field is assumed to be null elsewhere.

[1]In e.m. theory, the *magnetic field* is $\vec{H}$; $\vec{B}$ is the *magnetic-flux density* or *magnetic induction*. Anyway, here we are concerned only with $\vec{B}$, which we call "magnetic field" for the sake of simplicity.

In reality, a physical undulator cannot induce such a field, due to the fact that it is not unrotational as requested by Maxwell equations. A real field may be expanded as a series of "modes" having an expression similar to (3.1), but with a transverse variation; anyway, we consider a single mode to be dominant and transverse variations to be negligible in the region where the electron motion takes place, so that (3.1) is very close to the real field.[2]

Electron motion: equations

In what follows we describe the motion of an electron driven by the undulator field 3.1. The Lorentz force acting on the electron is[3]

$$\vec{F} = -e\vec{v} \times \vec{B}_u \tag{3.2}$$

where e and $\vec{v}$ are the electron charge (without sign) and velocity. An analysis based on formula (3.2) neglects any variation of the electron energy due to spontaneous emission (a magnetic field does not do any work, nor change the energy); moreover, in an FEL an e.m. field co-propagating with the electrons (the optical beam) is superimposed on the undulator field and gives rise to other forces. Yet, we approximate the optical beam by a plane wave: within an ultra-relativistic approximation for the electron motion (i.e., by assuming the electron to move along the z axis at the speed of light), the net force due to the optical beam is longitudinal and the transverse component of equation (3.2) is correct.

Since the electrons moving in an FEL are ultrarelativistic, in order to determine their motion we need relativistic mechanics: Newton equation is

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{3.3}$$

where $\vec{p}$ is the electron momentum, given by

$$\vec{p} = \gamma m \vec{v}$$

with γ, m and $\vec{v}$ denoting, respectively, the electron relativistic factor, (rest) mass and velocity. The latter is furtherly expressed in terms of its normalized counterpart $\vec{\beta}$ by

$$\vec{v} = c\vec{\beta}$$

[2] A consequence of transverse variation is the "natural focusing": electrons injected off-axis or at non-ideal transverse speed are focused towards the undulator axis. Within our simple theory, natural focusing is neglected.

[3] We give for granted that quantities related to the electron are functions of time; all fields (functions of space and time) have to be considered at the instantaneous electron position, so they too become functions of time only.

(c is the speed of light). Shortly,

$$\vec{p} = mc\gamma\vec{\beta}$$

and

$$\frac{d\vec{p}}{dt} = mc\frac{d\left(\gamma\vec{\beta}\right)}{dt}$$

By substituting this last formula into Newton equation (3.3) and using the Lorentz force (3.2), we get

$$\frac{d\left(\gamma\vec{\beta}\right)}{dt} = -\frac{e}{m}\vec{\beta}\left(t\right) \times \vec{B}_u\left(z\left(t\right)\right)$$

($z\left(t\right)$ is the electron longitudinal position at time t) and substitution of the undulator field (3.1) yields

$$\frac{d\left(\gamma\beta_x\right)}{dt} = \frac{eB_0}{m}\beta_z\left(t\right)\sin k_u z\left(t\right) \tag{3.4}$$

$$\frac{d\left(\gamma\beta_y\right)}{dt} = 0 \tag{3.5}$$

$$\frac{d\left(\gamma\beta_z\right)}{dt} = -\frac{eB_0}{m}\beta_x\left(t\right)\sin k_u z\left(t\right) \tag{3.6}$$

Equations (3.4)-(3.6) rule the electron motion due to the undulator field (and nothing else). They state that the electron speed changes in a way depending on the electron longitudinal position; the initial position and speed depend on the way the electron has been injected into the undulator. As mentioned, equations (3.4) and (3.5) are correct, while (3.6) neglects the contribution due to the optical beam: so, we solve (3.4), (3.5) and express the electron motion in terms of γ.

Electron motion: solution

Now we solve for the electron motion. First of all, from equation (3.5) we immediately realize that the vertical motion is simply a drift (i.e., a motion with constant speed).[4] We assume the electron to be injected without any initial vertical motion, so that

$$\beta_y\left(t\right) = 0 \tag{3.7}$$

[4]This is a consequence of neglecting natural focusing (see note 2). In a real linearly-polarized undulator, there is always some vertical focusing, so the following assumption is unnecessary.

Now, equation (3.4) can be solved easily to obtain the x motion in terms of the longitudinal position. So, we determine the electron motion by writing β_x in terms of z (and γ...) from (3.4), deducing β_z from the relation between β and γ and solving for $z(t)$.

First part: solving (3.4). The general solution is an (arbitrary) constant, which represents a drift, plus any particular solution. In order to find the latter, we note that the equation's right-hand side can be written as

$$\frac{eB_0}{m}\beta_z(t)\sin k_u z(t) = -K\frac{d}{dt}\cos k_u z(t)$$

where

$$K = \frac{eB_0}{mck_u}$$

is the *undulator parameter*; as a consequence, a particular solution is clearly given by

$$\gamma(t)\,\beta_x(t) = -K\cos k_u z(t) \tag{3.8}$$

This solution represents a periodic motion (no drift).[5] So, in order to avoid any drift along the x direction the electron has to be injected with such an x motion that the constant in the general solution is forced to zero.[6] We conclude that

$$\beta_x(t) = -\frac{K}{\gamma(t)}\cos k_u z(t) \tag{3.9}$$

Second part: deducing β_z. It is well known that

$$\beta^2 = 1 - \frac{1}{\gamma^2}$$

which implies

$$\beta_z^2 = 1 - \frac{1 + (\gamma\beta_x)^2}{\gamma^2}$$

By substituting (3.8) we get

$$\beta_z(t) = \sqrt{1 - \frac{1}{\gamma^2(t)}\left(1 + K^2\cos^2 k_u z(t)\right)}$$

[5]Formula (3.8) shows the role of the undulator parameter: it rules the amplitude of transverse-speed oscillations (together with γ). Usually, $K \sim 1$ [16].

[6]This is also true in a real linearly-polarized undulator, unless magnetic poles are shaped properly in order to yield some horizontal focusing.

and if $\gamma >> 1, K$ (which is always the case) this formula can be reduced to

$$\beta_z(t) \approx 1 - \frac{1}{2\gamma^2(t)} \left(1 + K^2 \cos^2 k_u z(t)\right) \tag{3.10}$$

This last approximation is useful for what follows.

Third part: solving for $z(t)$. Since β_z is related to dz/dt, relation (3.10) represents a differential equation in $z(t)$; solving the latter is not so easy, due to the fact that γ is time-varying, but the following procedure yields quickly an approximate solution.

First of all, by expanding

$$\cos^2(\,\cdot\,) = \frac{1}{2} + \frac{1}{2}\cos 2(\,\cdot\,)$$

we write (3.10) as

$$\beta_z(t) = 1 - \frac{1}{2\gamma^2(t)} \left(1 + \frac{K^2}{2}\right) - \frac{K^2}{4\gamma^2(t)} \cos 2k_u z(t)$$

This formula shows clearly that the electron speed is given by the superposition of two motions, the former (described by the first and second terms) being very slow[7] and the latter (described by the third term) being much faster due to the ultrarelativistic longitudinal drift of the electron.

In order to describe separately the two motions, we express the electron position as a sum of two terms:

$$z(t) = \bar{z}(t) + \delta z(t) \tag{3.11}$$

The former, $\bar{z}(t)$, is related to the slow motion by imposing the corresponding speed to be described by the slow part of β_z, and the latter, $\delta z(t)$, is related to the fast motion by imposing the corresponding speed to be described by the fast part of β_z.[8] Shortly,

$$\frac{1}{c}\frac{d\bar{z}}{dt} = 1 - \frac{1}{2\gamma^2(t)} \left(1 + \frac{K^2}{2}\right) \tag{3.12}$$

$$\frac{1}{c}\frac{d(\delta z)}{dt} = -\frac{K^2}{4\gamma^2(t)} \cos 2k_u z(t) \tag{3.13}$$

[7] It is assumed that γ changes very slowly. This is true if the field is not too strong (see section 3.2.2), which is always the case in practical FELs.

[8] This defines $\bar{z}$ and δz except for an additive constant.

Obviously, in this manner the speed associated to $z(t)$ is correctly given by β_z. We fix the remaining arbitrary constant by imposing δz to be null when the electron enters the undulator; relation (3.11) imposes $\bar{z}$ as well to be null at this instant.

In conclusion, the slow motion is basically a drift, with $\bar{z}(t)$ increasing according to law (3.12). For convenience, we posticipate the analysis of the fast motion to page 32.

3.2.2 Electron-light interaction

In the previous section we have expressed the motion of the electron in terms of its energy. Now we derive a formula describing the interaction between our electron and the optical beam, which is responsible for electron-energy variations.

The energy-variation formula

Recapitulating, an exact description for the electron dynamics would use Newton law in its relativistic form, with the force acting on the electron expressed according to Lorentz law for a charged particle in an e.m. field. The latter is given by the superposition of the undulator magnetostatic field and the optical beam, which is a time-varying e.m. field; anyway, magnetic forces are perpendicular with respect to the electron trajectory and do not perform any work: as a consequence, an exact description of the electron-energy variation involves only the electric field, which is bound to the optical beam.

So, we now deal only with the electric force, which is given by

$$\vec{F} = -e\vec{E}$$

As time goes by, this force exerts a work on the electron, thereby changing its kinetic energy; in a dt-lasting time interval this work is

$$d\mathcal{W} = \vec{F} \cdot d\vec{\ell}$$

where $d\vec{\ell} = \vec{v}dt$ is the path covered by the electron during the time interval. The electron kinetic energy $\mathcal{E}$ undergoes a variation $d\mathcal{E} = d\mathcal{W}$, and putting things together we conclude that its time derivative is

$$\frac{d\mathcal{E}}{dt} = -e\vec{E} \cdot \vec{v}$$

Now, from relativistic mechanics we know that

$$\mathcal{E} = \gamma mc^2$$

which implies

$$\frac{d\gamma}{dt} = \frac{1}{mc^2}\frac{d\mathcal{E}}{dt} = -\frac{e}{mc^2}\vec{E}\cdot\vec{v}$$

and by introducing the usual normalized velocity $\vec{\beta} = \vec{v}/c$ we end up with the fundamental formula

$$\frac{d\gamma}{dt} = -\frac{e}{mc}\vec{\beta}\,(t)\cdot\vec{E}\,(\vec{r}\,(t)\,,t) \tag{3.14}$$

($\vec{r}\,(t)$ is the electron position at time t), which yields the electron-energy variation due to the electron-field interaction.

The complex envelope

There is a way to express a time-varying e.m. field which turns out to be very useful in FEL theory. Let us consider an electric field[9] $\vec{E}$ in free space: it obeys the wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0$$

whose simplest solution is a *uniform plane monochromatic wave*[10] propagating along a fixed direction; if we choose this direction to be the z axis, such a wave can be expressed as

$$\vec{E}\,(\vec{r},t) = \Re\left\{Ce^{i(\omega t - kz)}\hat{e}\right\} \tag{3.15}$$

Symbols have the following meaning:

- $\omega = 2\pi f$, where f is the chosen frequency;
- k (*wavenumber*) is related to the *wavelength* λ (which is the spatial period) by $k = 2\pi/\lambda$, and has to satisfy the *dispersion relation* $k = \omega/c$;
- $\hat{e}$ is a complex unit vector. It rules the field polarization (linear, circular or elliptical);
- C is a complex constant giving the field amplitude and initial phase.

[9] Since its magnetic counterpart is not involved in (3.14), we do not deal with it.
[10] *Monochromatic*: t variation is sinusoidal;
plane: wavefronts (constant-phase surfaces) are planes;
uniform: amplitude is constant on any chosen wavefront.

From now on we will denote the electric field by means of the symbol $\underset{\sim}{\vec{E}}$: the '$\sim$' under the '$\vec{E}$' indicates the presence of a fast temporal oscillation due to the factor $e^{i\omega t}$ (the need for such a symbol will be clear in a while).

The optical beam propagating within our FEL is expected to be somehow a "wave" moving along the z axis (which we have previously chosen to be the undulator axis – see figure 1.1). We furtherly assume what follows:

- the field is transversally uniform. This leads to a one-dimensional theory;
- the field is linearly polarized along the x axis. This allows the best coupling with the electron beam (electrons wiggle along the x axis);
- the field is stationary, i.e. it is a periodic function of time. This corresponds to steady state.

We are not concerned with harmonics, so we only deal with a monochromatic field (time variation is not simply periodic, but even sinusoidal). Within all these assumptions, we express our electric field as a uniform plane monochromatic wave propagating along the z axis and linearly polarized along the x axis, with the introduction of a further z dependence into the complex amplitude: shortly,

$$\underset{\sim}{\vec{E}} = \underset{\sim}{E_x}\,\hat{x} \tag{3.16}$$

where

$$\underset{\sim}{E_x}(z,t) = \Re\left\{E(z)\,e^{i(\omega t - kz)}\right\} \tag{3.17}$$

The complex function $E(z)$ modulates the carrier $\cos(\omega t - kz)$ in both amplitude and phase, thereby determining the field envelope and phase advance: it is the electric-field *complex envelope*. We consider values for the optical wavelength λ from about 1 μm (near infrared) down to about 100 nm (vacuum ultraviolet)[11].

At this point, by substituting expressions (3.16), (3.17) into formula (3.14) we find

$$\frac{d\gamma}{dt} = -\frac{e}{mc}\beta_x(t)\cdot\Re\left\{E(z(t))\,e^{i(\omega t - kz(t))}\right\} \tag{3.18}$$

[11] These are the order of magnitude of the wavelength for light from a Nd:YAG conventional laser [5] and for the fifth harmonic of light from a Ti:Sa conventional laser, which can be obtained by HHG [6, 7].

which gives the electron-energy variation in terms of time, electron longitudinal position, electron transverse speed and field envelope at the electron position.

Now we eliminate as many unknowns as we can; we will be able to derive a differential equation ruling the electron longitudinal motion provided the field envelope is known along the length of the undulator. The first thing to do is expressing the electron transverse speed in terms of its longitudinal position; this is yielded by formula (3.9), substitution of which into (3.18) gives

$$\frac{d\gamma}{dt} = \frac{eK}{mc}\frac{1}{\gamma(t)} \cdot \Re\left\{E\left(z\left(t\right)\right)\cos k_u z\left(t\right) \cdot e^{i(\omega t - kz(t))}\right\} \tag{3.19}$$

We shall show that the right-hand side of this formula contains a $\bar{z}$-depending factor which oscillates periodically, performing a complete oscillation as $\bar{z}$ covers half an undulator period. As time goes by $\bar{z}$ increases very quickly, so this factor is a fast-oscillating function of time, whose effect on the electron energy is only a negligible ripple. Thus, we can substitute it by its mean value in $\bar{z}$ over half an undulator period: we are going to do this in a while, after a short digression on electron phase and resonance condition.

Electron phase and resonance condition

The role played by electron-speed and field phases is most easily described in FELs based on a circularly-polarized undulator. In such a case, electrons perform a circular motion in the transverse plane and the field is circularly polarized; the angle between electron transverse speed and x axis is $k_u z(t) + \pi$ (it depends on the electron instantaneous longitudinal position) and the angle between electric force (which is opposed to the electric field) and x axis is $\omega t - kz + \pi$ (it depends on longitudinal position and time). The angle between electron speed and electric force (on which the energy transfer depends) is then

$$\zeta(t) = (k + k_u)\, z(t) - \omega t$$

and is called *electron phase*.

Now, as we have seen, the linearly-polarized undulator we are dealing with induces on the electron a fast longitudinal oscillation ($\delta z(t)$) around its "mean" longitudinal position ($\bar{z}(t)$). This oscillation reduces the electron-field coupling and is responsible for the above-mentioned energy ripple, but has no effect on energy-transfer variations: as a consequence, we define the electron phase as

$$\zeta(t) = (k + k_u)\, \bar{z}(t) - \omega t \tag{3.20}$$

involving $\bar{z}$ but neglecting δz.

An electron is said to be *on resonance* when its phase does not change in time (the term “resonance” refers to the fact that in these conditions the energy transfer rate is constant). For a given undulator, and having fixed the optical wavelength, the quantity determining whether the electron is on resonance or not is its energy; for the sake of completeness, we report the simple calculations leading to the relation between undulator parameter K, optical wavelength λ and electron resonance energy γ_r. First of all we introduce the quantity

$$\bar{\beta}_z(t) = \frac{1}{c}\frac{d\bar{z}}{dt} \tag{3.21}$$

which represents the normalized speed associated to the electron mean position $\bar{z}(t)$; then we derive the electron phase with respect to time:

$$\dot{\zeta}(t) = c(k + k_u)\,\bar{\beta}_z(t) - \omega$$

Now the electron phase is constant if its time derivative is null; by imposing this condition and using the relation $\omega = ck$ we get the linear algebraic equation

$$(k + k_u)\,\bar{\beta}_z(t) - k = 0$$

which can be solved to give the resonance speed $\bar{\beta}_{zr}$ in terms of k:

$$\bar{\beta}_{zr} = \frac{k}{k + k_u} \tag{3.22}$$

Finally, this formula may be turned into a relation between λ and γ_r by means of formula (3.12), which yields

$$\bar{\beta}_{zr} = 1 - \frac{1}{2\gamma_r^2}\left(1 + \frac{K_u^2}{2}\right) \tag{3.23}$$

By comparing (3.22) and (3.23), using $k >> k_u$ and substituting k, k_u in terms of λ, λ_u, we find the *resonance formula*

$$\frac{\lambda}{\lambda_u} = \frac{1}{2\gamma_r^2}\left(1 + \frac{K_u^2}{2}\right) \tag{3.24}$$

Formula (3.24) may be used for expressing either γ_r in terms of K and λ, λ in terms of K and γ_r, or K in terms of γ_r and λ. In a real experiment, λ is chosen and usually K is easier to vary than the electron energy, so we solve for K:

$$K = \sqrt{4\frac{\lambda}{\lambda_u}\gamma^2 - 2} \tag{3.25}$$

This last formula yields the value for K which lets an electron at energy γ resonate with an optical beam at wavelength λ.

One last remark about resonance. In any FEL, electrons move in a small energy range around the resonance energy γ_r (otherwise they are unuseful[12]). As a consequence, in all coefficients involved in the equations we can approximate the electron energy by γ_r:

$$\gamma(t) \approx \gamma_r \tag{3.26}$$

An usual value for the resonance energy is $\gamma_r \sim 100$ (~ 100 MeV) [16].

Averaging fast-oscillating terms

As we have mentioned previously, the right-hand side of formula (3.19) contains $\lambda_u/2$-periodic functions of $\bar{z}$, which are fast-oscillating functions of time and can be $\bar{z}$-averaged over a period. We shortly review the procedure [15], since it is used again in sections 3.3, 6.3 and 6.4.

First of all, by substituting (3.26) into (3.13) we find

$$\frac{1}{c}\frac{d(\delta z)}{dt} = -\frac{K^2}{4\gamma_r^2}\cos 2k_u z(t) \tag{3.27}$$

Since $\gamma_r >> K$ and the sinusoidal function oscillates very quickly, this equation tells us that δz will not depart so much from its initial value (zero), i.e. it can be assumed to be very small. So, it is very easy to deduce an approximate solution: by neglecting δz with respect to $\bar{z}$ in the argument z of the sinusoidal function, we deduce[13]

$$\delta z(t) \approx -\frac{K^2}{8k_u\gamma_r^2}\sin 2k_u\bar{z}(t) \tag{3.28}$$

Now we use $k >> k_u$ and (3.28) to expand

$$\cos k_u z(t) \cdot e^{i(\omega t - kz(t))} \approx \frac{1}{2}e^{-i\zeta(t)} f(\bar{z}(t)) \tag{3.29}$$

where

$$f(z) = e^{i\xi \sin 2k_u z} + e^{i2k_u z} \cdot e^{i\xi \sin 2k_u z} \tag{3.30}$$

[12] If γ is not close to γ_r, $\dot{\zeta}$ is significantly different from zero and ζ covers many 2π intervals as the electron passes through the undulator. Then, periods alternate in which the electron, say, first looses energy and then recovers the energy lost: its overall contribution to gain is null.

[13] Note the further approximation $d\bar{z}/dt \approx c$, which follows from (3.12) thanks to $\gamma >> 1, K$.

and

$$\xi = \frac{1}{4}\frac{K^2}{1+K^2/2} \tag{3.31}$$

Note that f is a $\lambda_u/2$-periodic function of z and $f(\bar{z}(t))$ is a fast-oscillating function of time. Then, we substitute f by its mean value

$$\frac{2}{\lambda_u}\int_0^{\lambda_u/2} f(z)\,dz = [JJ] \tag{3.32}$$

where we have introduced, as usual in FEL theory, the factor

$$[JJ] = J_0(\xi_r) - J_1(\xi_r)$$

and J_0, J_1 are the zero-order and first-order Bessel functions. The final formula is

$$\cos k_u z(t) \cdot e^{i(\omega t - kz(t))} \approx \frac{1}{2}[JJ]\,e^{-i\zeta(t)} \tag{3.33}$$

Lastly, by substituting formula (3.33) into formula (3.19) and using approximation (3.26), we find

$$\frac{d\gamma}{dt} = \frac{eK}{2mc}[JJ]\frac{1}{\gamma_r}\cdot\Re\left\{E(z(t))\,e^{-i\zeta(t)}\right\} \tag{3.34}$$

3.2.3 The Pendulum Equation

It is now time to collect the formulae that describe the evolution of electron dynamical quantities and derive a single equation ruling the electron motion. Clearly, the latter depends on the field-envelope distribution (i.e. $E(z)$); in this context we assume $E(z)$ to be known, and later on we will derive a further equation ruling the field evolution on the basis of electron-beam kinematics.

The main relation giving informations on electron dynamics is formula (3.34). It allows determining the energy evolution (i.e. $\gamma(t)$) once the phase $\zeta(t)$ is known: in fact, formulae (3.20), (3.28) and (3.11) relate position z to phase ζ, thus yielding the quantity to be used as an argument for the field envelope E. Now, formula (3.12) allows determining electron mean position $\bar{z}(t)$ from energy $\gamma(t)$, and formula (3.20) yields the phase $\zeta(t)$ to be used in (3.34): in this way we get a loop which provides all the information we need on electron motion.

One further remark about which electron dynamical quantity has to be considered the most relevant. It is the phase, due to three reasons:

- first, the main quantity involved in the right-hand side of formula (3.34) is ζ;
- second, relating z to ζ is an easy task to do (as we have just explained);
- third, the main quantity involved in the source current is ζ (as we shall see later).

So, now we derive an equation ruling the electron-phase evolution, and this will give us all the information we need about electron motion.

We want to obtain a differential equation having ζ as an unknown. To this aim, we derive the phase definition (3.20) as many times as we need in order to get an expression containing the derivative of γ; then, we substitute formula (3.34). By deriving (3.20) two times and substituting definition (3.21), we get

$$\frac{d\zeta}{dt} = c\left(k + k_u\right)\bar{\beta}_z\left(t\right) - \omega$$

$$\frac{d^2\zeta}{dt^2} = c\left(k + k_u\right)\frac{d\bar{\beta}_z}{dt}$$

Now we need a (simple) expression for $d\bar{\beta}_z/dt$. It is found by deriving formula (3.12) and using approximation (3.26):

$$\frac{d\bar{\beta}_z}{dt} = \frac{1}{\gamma_r^3}\left(1 + \frac{K^2}{2}\right)\frac{d\gamma}{dt} \tag{3.35}$$

Lastly, by substituting this relation into the previous one and using (3.34) we obtain

$$\frac{d^2\zeta}{dt^2} = \Omega^2 \cdot \Re\left\{E\left(z\left(t\right)\right)e^{-i\zeta(t)}\right\} \tag{3.36}$$

where we have defined

$$\Omega^2 = \frac{eK}{2m}\left(1 + \frac{K^2}{2}\right)\left[JJ\right]\frac{k + k_u}{\gamma_r^4} \approx \frac{4\pi e}{m\lambda_u}\frac{K}{1 + K^2/2}\left[JJ\right]\frac{\lambda}{\lambda_u}$$

(the last approximation is found from $k >> k_u$ and (3.24)). In FEL theory equation (3.36) is universally known as *Pendulum Equation*: if the field envelope is equal to unity, this equation is formally identical to the one which rules the motion of a pendulum having ζ as the angle with respect to the horizontal plane and Ω as the (angular) frequency of small oscillations.

We end this section by one last remark on equation (3.36). The field envelope is evaluated at the instantaneous electron position, which depends on the unknown (the electron phase); position and phase may be related quite easily (as mentioned previously), but this is not necessary: in chapter 4, we shall introduce an approximation which allows substituting the position of the electron by the "mean" position of its neighbours, thereby simplifying further computations.

3.3 Light evolution

In this section, we describe the evolution of light due to the interaction with the electron beam wiggling in the FEL undulator. The phenomenon is ruled by classical electromagnetics.

We start from the one-dimensional wave equation (2.12) and reduce it to a simpler equation in the field envelope. First of all, we have to split the source current into its longitudinal and transverse components. We assume the electron distribution to be transversally uniform[14], so the current is only varying spatially along the z direction:

$$\vec{J} = \vec{J}(z,t)$$

This allows a very simple choice for the longitudinal and transverse components: a good couple is obtained simply by projecting $\vec{J}$ on the z axis and on its transverse plane. Now, electrons wiggle along the x direction, so $J_y = 0$. In conclusion, we choose

$$\vec{J}_{//} = J_z \hat{z} \tag{3.37}$$

$$\vec{J}_{\perp} = J_x \hat{x} \tag{3.38}$$

This choice may be proved immediately to satisfy the required conditions: $\nabla \times \vec{J}_{//} = 0, \nabla \cdot \vec{J}_{\perp} = 0$.

Within assumptions (3.16) for the field and (3.38) for the transverse current, equation (2.12) is equivalent to the scalar equation

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \underset{\sim}{E_x} = \mu_0 \frac{\partial J_x}{\partial t} \tag{3.39}$$

Further simplification is possible by means of a few approximations.

[14]This is consistent with the assumption of transversally uniform optical field (see section 3.2.2).

3.3.1 The slowly-varying-envelope approximation

Formula (3.17) allows expressing the electric field in terms of its complex envelope. Now we apply this formalism to equation (3.39).

Consider the left-hand side. We expand the d'Alembert operator as

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) = \left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial z} - \frac{1}{c}\frac{\partial}{\partial t}\right)$$

so that

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\underset{\sim}{E_x} = \left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial \underset{\sim}{E_x}}{\partial z} - \frac{1}{c}\frac{\partial \underset{\sim}{E_x}}{\partial t}\right) \tag{3.40}$$

where, from (3.17),

$$\frac{\partial \underset{\sim}{E_x}}{\partial z} = \Re\left\{\left(\frac{dE}{dz} - ikE\left(z\right)\right)e^{i(\omega t - kz)}\right\}$$

and

$$\frac{\partial \underset{\sim}{E_x}}{\partial t} = \Re\left\{i\omega E\left(z\right)e^{i(\omega t - kz)}\right\}$$

Now we apply the *slowly-varying-envelope approximation* (SVEA), according to which the field variation (z derivative) due to the envelope is negligible with respect to the one due to the carrier. In formula,

$$\left|\frac{dE}{dz}\right| << k\left|E\left(z\right)\right| \tag{3.41}$$

In other words, the relative variation of the envelope over an optical wavelength is negligible. Within this approximation, we have

$$\frac{dE}{dz} - ikE\left(z\right) \approx -ikE\left(z\right)$$

and

$$\frac{\partial \underset{\sim}{E_x}}{\partial z} = \Re\left\{-ikE\left(z\right)e^{i(\omega t - kz)}\right\}$$

so that

$$\frac{\partial \underset{\sim}{E_x}}{\partial z} - \frac{1}{c}\frac{\partial \underset{\sim}{E_x}}{\partial t} = -2k\cdot\Re\left\{iE\left(z\right)e^{i(\omega t - kz)}\right\} \tag{3.42}$$

where the relation $\omega/c = k$ has been used.

By substituting (3.42) into (3.40), we find

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) \underset{\sim}{E_x} = -2k \cdot \Re\left\{i\frac{dE}{dz}e^{i(\omega t - kz)} + iE(z)\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)e^{i(\omega t - kz)}\right\}$$

Lastly, the function $\exp\left[i\left(\omega t - kz\right)\right]$ represents a forward travelling wave, i.e. it satisfies the relation

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)e^{i(\omega t - kz)} = 0$$

and we conclude that, within the SVEA, the left-hand side of the wave equation (3.39) reduces to

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) \underset{\sim}{E_x} = -2k \cdot \Re\left\{i\frac{dE}{dz}e^{i(\omega t - kz)}\right\} \tag{3.43}$$

3.3.2 The Field Equation

As one can see, formula (3.43) allows expressing equation (3.39) in terms of the field envelope. Yet, to be consistent with the SVEA, we must also substitute the right-hand side of equation (3.39) by its slowly-varying component. Also, we express the current in terms of electron phases, which allows coupling the resulting equation to the Pendulum Equation (3.36).

Reducing the wave equation

First, we reduce equation (3.39) to a first-order ordinary differential equation in E. By substituting (3.43) into (3.39), we find

$$-2k \cdot \Re\left\{i\frac{dE}{dz}e^{i(\omega t - kz)}\right\} = \mu_0\frac{\partial J_x}{\partial t} \tag{3.44}$$

As mentioned, we must reduce the right-hand side to its slowly-varying component. To this aim, we expand the real-part operator:

$$\frac{dE}{dz}e^{i(\omega t - kz)} - \frac{dE^*}{dz}e^{-i(\omega t - kz)} = -\frac{\eta_0}{i\omega}\frac{\partial J_x}{\partial t} \tag{3.45}$$

We have here introduced the *intrinsic impedance of free space* [13]

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

Now we multiply equation (3.45) by $\exp\left(-i\left(\omega t - kz\right)\right)$; then, we fix space z and time t and average both sides over a λ-long spatial interval centred on

z. Within such an interval, the z derivative of E is approximately constant (due to the SVEA (3.41)), so the first term at the left-hand side does not change, while the second vanishes. The result is

$$\frac{dE}{dz} = -\frac{\eta_0}{i\omega}\left\langle \frac{\partial J_x}{\partial t} e^{-i(\omega t - k\,\cdot\,)} \right\rangle \tag{3.46}$$

where brackets denote the spatial averaging.[15] Before performing the remaining average, we extract the time derivative by means of formula

$$\frac{\partial}{\partial t}\left(J_x e^{-i(\omega t - kz)}\right) = \frac{\partial J_x}{\partial t} e^{-i(\omega t - kz)} - i\omega J_x e^{-i(\omega t - kz)}$$

which allows to re-state equation (3.46) as

$$\frac{dE}{dz} = -\eta_0 \left(1 + \frac{1}{i\omega}\frac{\partial}{\partial t}\right)\left\langle J_x e^{-i(\omega t - k\,\cdot\,)} \right\rangle \tag{3.47}$$

Now, at last, it is time to average the current.

Averaging the source current

In FELs, the source current is due to electrons moving inside the undulator. As a consequence, it is not a regular function: it is always vanishing, except where electrons are (there, it is a Dirac delta). This induces a random field called "shot noise". By averaging the current, we neglect shot noise; yet, in an FEL amplifier shot noise is largely overcome by the seed, so it is unrelevant[16].

An exact expression for the source current is

$$J_x(\vec{r}, t) = -ec \sum_j \beta_x^j(t)\,\delta\left(\vec{r} - \vec{r}_j(t)\right) \tag{3.48}$$

where $\vec{r}_j$ is the position of the j-th electron, β_x^j is its transverse speed and δ denotes the three-dimensional Dirac delta. The summation is extended over

[15] The dot at the exponent in the right-hand side of (3.46) is used to identify the variable over which the average is performed. In this way, the averaging variable is distinguished from z, which is the center of the averaging interval.

Explicitly, for any function $f(z, t)$,

$$\langle f \rangle = \langle f(\,\cdot\,, t) \rangle = \frac{1}{\lambda} \int f(z', t)\, dz'$$

where the integration is performed over the averaging interval. The result is a function of z and t.

[16] Shot noise is only relevant as a possible way, alternative to seeding, for starting up the radiation; this is done in self-amplified spontaneous emission (SASE) FELs [16, 8].

the j's of all electrons in the beam. Note that (3.48) varies along all spatial directions; yet, within our assumption of a transversally uniform electron beam, transverse variations are only related to shot noise. In order to be consistent with previous assumptions (considering a transversally uniform current and neglecting shot noise), we extend the average in equation (3.47) to a three-dimensional volume V.[17]

From formulas (3.48) and (3.9) (with approximation (3.26)), we find

$$J_x\left(\vec{r},t\right)e^{-i(\omega t-kz)} = \frac{ecK}{\gamma_r}\sum_j \cos k_u z_j\left(t\right)\cdot e^{-i(\omega t-kz_j(t))}\cdot\delta\left(\vec{r}-\vec{r}_j\left(t\right)\right) \tag{3.49}$$

where, due to the presence of a Dirac delta, we have substituted the z at the exponent by $z_j\left(t\right)$. Now, by conjugating (3.29) we find

$$\cos k_u z_j\left(t\right)\cdot e^{-i(\omega t-kz_j(t))} = \frac{1}{2}e^{i\zeta_j(t)}f^*\left(\bar{z}_j\left(t\right)\right)$$

where ζ_j is the phase of the j-th electron and the function f is defined by (3.30). At the argument of f, the smallness of δz_j (see the discussion about (3.27)) allows approximating $\bar{z}_j \approx z_j$; the result is used in (3.49), which turns into

$$J_x\left(\vec{r},t\right)e^{-i(\omega t-kz)} = \frac{ecK}{2\gamma_r}\left(\sum_j e^{i\zeta_j(t)}\delta\left(\vec{r}-\vec{r}_j\left(t\right)\right)\right)f^*\left(z\right)$$

where we have substituted the $z_j\left(t\right)$ at the argument of f by z (thanks to the Dirac delta). Lastly, f varies over a scale $\lambda_u >> \lambda$, so we can approximate[18]

$$\left\langle J_x e^{-i(\omega t-k\,\cdot)}\right\rangle \approx \frac{ecK}{2\gamma_r}nb\left(z\right)f^*\left(z\right) \tag{3.50}$$

---

[17] E.g., let V be a λ-diameter and λ-long cylinder centred on $\vec{r}$. For any function $f\left(\vec{r},t\right)$, we extend

$$\langle f\rangle = \frac{1}{V}\int_V f\left(\vec{r}',t\right)d^3\vec{r}'$$

(we denote the measure of V simply by V). We assume macroscopic quantities (see note 3 at page 7) to be transversally uniform, so the result is still a function of z and t.

[18] We have

$$\left\langle\left(\sum_j e^{i\zeta_j}\delta\left(\,\cdot-\vec{r}_j\right)\right)f^*\right\rangle \approx \left\langle\sum_j e^{i\zeta_j}\delta\left(\,\cdot-\vec{r}_j\right)\right\rangle f^*$$

and

$$\left\langle\sum_j e^{i\zeta_j}\delta\left(\,\cdot-\vec{r}_j\right)\right\rangle = nb$$

where we have introduced the *electron density*

$$n = \frac{N}{V} \tag{3.51}$$

with N being the number of electrons inside the averaging volume V, and the *bunching factor*

$$b(z) = \frac{1}{N} \sum_j e^{i\zeta_j(t)} \tag{3.52}$$

where the summation is extended over the j's of the N electrons inside V. Note that, since the system is on steady state, the electron density is a constant and the bunching factor is a time invariant. Usually, $n \sim 10^{18}$ m^{-3}.

As we have already mentioned previously, for the FEL to operate properly the electron energy must be close to γ_r (see formula (3.26)). Thus, the phase of any electron changes slowly, and its variation over the time the electron takes to move in z by λ_u is negligible. As a consequence, b varies over a scale much larger than λ_u. Now, as in formula (3.19), f is a $\lambda_u/2$-periodic function of z, and its effect on the field envelope is only a negligible ripple. So, in formula (3.50) we substitute f by its mean value $[JJ]$ (see (3.32)), thereby obtaining

$$\left\langle J_x e^{-i(\omega t - k\cdot)} \right\rangle \approx J_0 b(z) \tag{3.53}$$

where we have defined

$$J_0 = \frac{ecK}{2}[JJ]\frac{n}{\gamma_r} = \frac{ec}{\sqrt{2}} \frac{K}{\sqrt{1+K^2/2}}[JJ]\, n\sqrt{\frac{\lambda}{\lambda_u}}$$

(the last expression is found from (3.24)).

We conclude by substituting (3.53) into (3.47): the result is

$$\frac{dE}{dz} = -\eta_0 J_0 b(z) \tag{3.54}$$

We call equation (3.54) *Field Equation*, due to the fact that it rules the evolution of the optical-beam electric field. By coupling this equation to the Pendulum Equation (3.36) (applied to all electrons), a self-consistent system is obtained which describes fully the FEL evolution.

Chapter 4

Integral Equation

4.1 Introduction

This chapter is devoted to a derivation for the FEL Integral Equation which makes use of a different analytical technique with respect to the one employed in reference [20] and in all later works [21, 15]. Our approach proves to be mathematically simpler and more physically transparent with respect to the original one. This work is presented in reference [22].

Within our FEL setup, the relevant physical quantity is $E(z)$; in what follows, we reduce the coupled Field and Pendulum Equations (3.54) and (3.36), whose unknowns are E and ζ, to a single equation which only involves E as an unknown. We follow the idea presented in reference [20]: in the weak-field regime (see section 4.2.1), equation (3.36) is integrated (for every single electron in the beam) and the result is used for expressing the average (bunching factor) at the right-hand side of equation (3.54) in terms of the field envelope, thus leading to the FEL Integral Equation. Our main contribution consists in a novel technique for integrating the Pendulum Equation (3.36).

4.2 Integrating the Pendulum Equation

Before starting with the computations, let us set up a suitable geometrical framework. With reference to figure 4.1, we fix a spatial position $\vec{r} = (\vec{\rho}, z)$ and a temporal instant t, and consider the Field Equation (3.54) at the chosen $\vec{r}, t$ couple. The bunching factor involves an average performed over the electrons whose position at time t is inside a λ-long volume V centred on $\vec{r}$ (see note 17 at page 39), which is identified in the figure by the thick

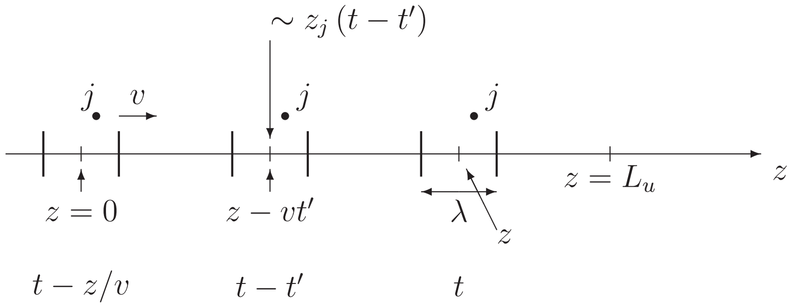

Figure 4.1: the geometrical framework.

lines[1]; so, we have to re-construct the history of these electrons, in order to determine their phase at time t.

In agreement with the approximations underlying equations (3.54) and (3.36), we assume the electron energies to be very close to their mean initial value $\bar{\gamma}_0$ during the whole interaction.[2] As a consequence, the electron longitudinal velocities are very close to the "mean initial velocity" v given by[3] [14, 15]

$$v = c - \frac{c}{2\bar{\gamma}_0^2}\left(1 + \frac{K^2}{2}\right) \tag{4.1}$$

Imagine volume V to be moving from left to right with speed v: as it is shown in figure 4.1, its center has entered the undulator at time $t - z/v$, and at time $t - t'$ ($0 < t' < z/v$) is placed at $z - vt'$.

Now, we choose one of the electrons in V. Since the longitudinal velocity of the electron is very close to v, its position with respect to V remains nearly unchanged during the motion, and as far as the slowly-varying field envelope is concerned the electron can be considered to lie in the middle of the volume[4]. Again, the situation is shown in figure 4.1.

At this point, our task is to write down the phase of the chosen electron in terms of the field envelope. To this aim, we integrate the Pendulum Equation (3.36).

---

[1] In the figure, $\vec{r}$ is intended to be on the z axis ($\vec{\rho} = 0$). Thick lines identify the longitudinal limits of V.

[2] In a high-gain FEL, the maximum efficiency (relative variation of electron energy) is on the order of the "Pierce parameter" ρ [27, 25], and $\rho \sim 10^{-3}$ [16].

[3] The velocity v is defined as the longitudinal velocity an electron would have if its energy were $\bar{\gamma}_0$. Clearly, v is not the average of the initial electron longitudinal velocities; however, it is expected to be very close to the latter, so that the two quantities can be identified.

[4] I.e., we can replace the field envelope at the electron position by its value at the middle of V.

4.2.1 Weak-field approximation

As mentioned, if the initial conditions are set properly the signal generated by a seeded FEL increases exponentially along the undulator, evolving therefore from a weak-field regime to a strong-field one (provided the undulator is not too short). We are concerned with the first part of the phenomenon, when strong amplification occurs.

Let us introduce the symbol $\zeta_f(t)$, denoting the free evolution of the electron phase (i.e. the value $\zeta(t)$ would have in the absence of any optical beam);[5] in other words, ζ_f is the solution to equation (3.36) with $E = 0$, which is simply

$$\ddot{\zeta}(t) = 0 \tag{4.2}$$

The forced evolution of the electron phase (i.e. the actual value of $\zeta(t)$, in the presence of the optical beam), or, in other words, the solution to the general form of equation (3.36), is given by

$$\zeta(t) = \zeta_f(t) + \delta\zeta(t) \tag{4.3}$$

where $\delta\zeta$ is the contribution to the electron phase due to the interaction with the optical beam.

As mentioned, we are concerned with the weak-field regime: so, we expand $\delta\zeta(t)$ to first order in $E(z)$. This is done by approximating $\zeta \approx \zeta_f$ at the right-hand side of the Pendulum Equation (3.36), which thereby reduces to its linearized form

$$\dot{\nu}(t) = \Omega^2 \cdot \Re\left\{E(z(t))\, e^{-i\zeta_f(t)}\right\} \tag{4.4}$$

where, for future convenience, we have introduced the phase velocity $\nu = \dot{\zeta}$.

Equation (4.4) can be integrated in order to get an explicit expression for the electron phase at any time instant. In all earlier derivations this is done by integrating two times and then converting the double integral into a single one. As mentioned, we propose a novel approach which, although requiring a certain care in dealing with differentials, is mathematically simpler and more physically transparent.

[5]Note that our electron emits spontaneous radiation during its travel along the undulator, so that ζ_f is not simply the value ζ would have in the absence of any seeding: it is the phase the electron would have if there were no seed *nor* any spontaneous radiation (i.e., if the electron's energy would stay constant).

4.2.2 Integration

Equation (4.4) states that the optical-beam electric field causes the electron phase velocity to change in time, and can be used to deduce the variation induced in any infinitesimal time interval. We are about to deal with the variation occurring around the time instant $t - t'$, so let us substitute t by $t - t'$. Then, we use the approximation $z(t - t') \approx z - vt'$, which we deduce from figure 4.1. The result is

$$\dot{\nu}(t - t') = \Omega^2 \cdot \Re\left\{E(z - vt')\, e^{-i\zeta_f(t-t')}\right\} \tag{4.5}$$

Now let us move t' to $t' + dt'$ with $dt' < 0$, so that time increases from $t - t'$ to $t - t' - dt'$; the corresponding variation of the phase velocity is

$$d\nu = \dot{\nu}(t - t') \cdot (-dt') = -\Omega^2 \cdot \Re\left\{E(z - vt')\, e^{-i\zeta_f(t-t')}\right\} dt' \tag{4.6}$$

The physical quantity expressed by formula (4.6) can be viewed as an infinitesimal phase velocity imposed on the electron at time $t - t'$. Since ν is the time derivative of ζ, $d\nu$ induces an infinitesimal electron phase at time t, which we denote by $d\zeta$, equal to $d\nu$ times the time elapsed (which is t'). In formula,

$$d\zeta = d\nu \cdot t' = -\Omega^2 \cdot \Re\left\{E(z - vt')\, e^{-i\zeta_f(t-t')}\right\} t'dt' \tag{4.7}$$

The various contributions to the electron phase can be joined together by noting that equation (4.4) depends on E linearly; as a consequence, $\delta\zeta(t)$ is the superposition (integral) of all contributions $d\zeta$, i.e.[6]

$$\delta\zeta(t) = \int_{t'=z/v}^{t'=0} d\zeta = \Omega^2 \cdot \Re\int_0^{z/v} E(z - vt')\, e^{-i\zeta_f(t-t')} t'dt' \tag{4.8}$$

By substituting this formula into expression (4.3), we may write the electron phase at time t in terms of the field envelope explicitly, provided the free evolution ζ_f is determined from equation (4.2) with the proper initial conditions. These are the values of phase and phase velocity at the time instant the electron has entered the undulator, which we denote by ζ_0 and ν_0 respectively.

In conclusion, from equation (4.2) we deduce

$$\zeta_f(t) = \zeta_0 + \nu_0 \cdot \Delta t \approx \zeta_0 + \nu_0 \cdot \frac{z}{v} \tag{4.9}$$

[6] Note that the lower integration limit should be equal to the time Δt elapsed since the electron entered the undulator; here and in what follows, we consider the electron to enter together with the middle point of the moving interval, so that this time is estimated to be z/v.

$$\zeta_f (t - t') = \zeta_0 + \nu_0 \cdot (\Delta t - t') \approx \zeta_0 + \nu_0 \cdot \frac{z}{v} - \nu_0 t' \tag{4.10}$$

(see note 6 for the definition of Δt and an explanation of the approximations) and by substituting formula (4.10) into formula (4.8) we get our main result:

$$\delta\zeta (t) = \Omega^2 \cdot \Re \left\{ e^{-i\zeta_0} e^{-i\nu_0 z/v} \int_0^{z/v} E (z - vt') \, e^{i\nu_0 t'} t' dt' \right\} \tag{4.11}$$

This formula defines an application leading from the function E (considered for all values of its argument between 0 and z) to the function $\delta\zeta$. By substituting the latter and formula (4.9) into expression (4.3), we get the phase at time t of any electron inside volume V (provided ζ_0 and ν_0 are the initial phase and phase velocity for that particular electron).

4.3 Averaging over the electrons

The next step towards the FEL Integral Equation is evaluating the bunching factor (3.52), whose definition involves an average over many electrons. We describe the distribution of electrons through their density in the (ζ, ν) space and assume what follows:

- as mentioned, the system is on *steady state*: this causes the (ζ, ν) distribution over a λ-long spatial interval around $z = 0$ to be t-invariant;
- the electron beam entering the undulator presents no *phase-energy correlation*, i.e. the ν distribution over a λ-long spatial interval around $z = 0$ is ζ-invariant;
- the electron beam entering the undulator presents no *pre-bunching*, i.e. the ζ distribution over a λ-long spatial interval around $z = 0$ is uniform.

The bunching factor involves an exponential with argument ζ, as it was the case for the Pendulum Equation; however, here it is not possible to neglect $\delta\zeta$, since such an approximation would cancel the source term in the Field Equation (3.54). The lowest-order allowed expansion for the exponential is the linear one:

$$e^{i\delta\zeta(t)} \approx 1 + i\delta\zeta (t) \tag{4.12}$$

By writing the phase according to expression (4.3) and then using formulas (4.9) and (4.12), we get[7]

$$b = \left\langle e^{i\zeta} \right\rangle = \left\langle e^{i\zeta_0} e^{i\nu_0 \frac{z}{v}} \right\rangle + i \left\langle e^{i\zeta_0} e^{i\nu_0 \frac{z}{v}} \delta\zeta \right\rangle$$

[7] Here, brackets denote averaging over electrons in V.

Thanks to the absence of phase-energy correlation, the first term can be separated as

$$\left\langle e^{i\zeta_0} e^{i\nu_0 \frac{z}{v}} \right\rangle = \left\langle e^{i\zeta_0} \right\rangle \cdot \left\langle e^{i\nu_0 \frac{z}{v}} \right\rangle$$

where the absence of pre-bunching causes the first factor to vanish, so that

$$\left\langle e^{i\zeta_0} e^{i\nu_0 \frac{z}{v}} \right\rangle = 0$$

Now, we substitute formula (4.11) into the second term. By expressing the real part as the half-sum of its argument and the corresponding complex conjugate, and then using the linearity of the average operator, we get

$$\left\langle e^{i\zeta_0} e^{i\nu_0 \frac{z}{v}} \delta\zeta \right\rangle = \frac{\Omega^2}{2} \int_0^{z/v} E\left(z - vt'\right) \left\langle e^{i\nu_0 t'} \right\rangle t' dt' +$$

$$+ \frac{\Omega^2}{2} \left\langle e^{i2\zeta_0} e^{i2\nu_0 \frac{z}{v}} \int_0^{z/v} E^*\left(z - vt'\right) e^{-i\nu_0 t'} t' dt' \right\rangle$$

As before, the absence of phase-energy correlation allows us to separate

$$\left\langle e^{i2\zeta_0} e^{i2\nu_0 \frac{z}{v}} \int_0^{z/v} E^*\left(z - vt'\right) e^{-i\nu_0 t'} t' dt' \right\rangle =$$

$$= \left\langle e^{i2\zeta_0} \right\rangle \cdot \left\langle e^{i2\nu_0 \frac{z}{v}} \int_0^{z/v} E^*\left(z - vt'\right) e^{-i\nu_0 t'} t' dt' \right\rangle$$

where the first factor vanishes (no pre-bunching), so

$$\left\langle e^{i2\zeta_0} e^{i2\nu_0 \frac{z}{v}} \int_0^{z/v} E^*\left(z - vt'\right) e^{-i\nu_0 t'} t' dt' \right\rangle = 0$$

We conclude that the bunching factor is given by [15, 20]

$$b\left(z\right) = i \frac{\Omega^2}{2} \int_0^{z/v} E\left(z - vt'\right) F\left(t'\right) t' dt' \tag{4.13}$$

where[8]

$$F\left(t'\right) = \left\langle e^{i\nu_0 t'} \right\rangle \tag{4.14}$$

[8]Average (4.14) is performed over a special set of electrons (those involved in the bunching factor (3.52)). This set moves approximately as a rigid body (at speed v), so when it entered the undulator it spread over a λ. Now, the system is on steady state, so the ν distribution at $z = 0$ is t-invariant. In conclusion, the average is performed over any set of electrons in a λ around $z = 0$ at a given time instant, and only depends on t'.

4.4 The Integral Equation

Now it is time to collect our previous results and write down the FEL Integral Equation. By substituting formula (4.13) into the Field Equation (3.54), we get

$$\frac{dE}{dz} = -iA \cdot \int_0^{z/v} E\left(z - vt'\right) F\left(t'\right) t' dt' \tag{4.15}$$

where we have defined

$$A = \frac{\eta_0}{2} J_0 \Omega^2$$

(note that $A > 0$). Equation (4.15) is the *FEL Integral Equation*; its only unknown is the electric-field complex envelope $E\left(z\right)$, and the solution is uniquely determined by its initial value $E\left(0\right)$. At this point we have reached our aim: the original problem, which involves the coupled equations (3.54) and (3.36) in the unknowns E and ζ, is reduced to a single integro-differential equation in the unknown E.

In literature (see, e.g., reference [14]), the Integral Equation is usually expressed in a slightly different form and within an ultra-relativistic approximation for the electron velocity. Now we show how equation (4.15) can be exactly solved in the simple case of a cold beam by means of a procedure originally developed within the ultra-relativistic approximation. Next, we compare our approach to the original one.

Let us assume the beam to be cold, i.e. with no energy spread: all electrons enter the undulator with the same phase velocity ν_0 and the ensemble average at the right-hand side of definition (4.14) can be dropped, thereby reducing the Integral Equation (4.15) to

$$\frac{dE}{dz} = -iA \cdot \int_0^{z/v} E\left(z - vt'\right) e^{i\nu_0 t'} t' dt' \tag{4.16}$$

This integro-differential equation can be solved by means of a mathematical procedure which has been suggested by Dattoli et al. [28, 29, 27].

First of all, we introduce the novel variable of integration $z' = vt'$. Thus, the integral at the right-hand side of equation (4.16) reads

$$\int_0^{z/v} E\left(z - vt'\right) e^{i\nu_0 t'} t' dt' = \frac{1}{v^2} \int_0^{z} E\left(z - z'\right) e^{i(\nu_0/v) z'} z' dz' \tag{4.17}$$

A more compact notation is found with the help of an auxiliary function

$$f\left(z\right) = E\left(z\right) e^{-i(\nu_0/v) z} \tag{4.18}$$

This definition leads in fact to the relation

$$\int_0^z E(z-z') e^{i(\nu_0/v)z'} z' dz' = e^{i(\nu_0/v)z} \int_0^z f(z-z') z' dz' \tag{4.19}$$

and the new integral at the right-hand side will shortly turn out to be easily expressed in an operatorial form. We denote by D the operator which performs the z derivative and adopt the following conventions: first, we define [31]

$$D^{-1} f = \int_0^z f(z') dz' \tag{4.20}$$

Then, we agree that, for any positive integer n, D^n amounts to apply D for n times and D^{-n} amounts to apply D^{-1} for n times; D^0 is the identical operator. Lastly, note that $D^{n_2} D^{n_1} = D^{n_1+n_2}$ for any couple of relative integers n_1, n_2. Within these conventions, it may be proved that

$$\int_0^z f(z-z') z' dz' = D^{-2} f \tag{4.21}$$

Now, by inverting definition f we find

$$E(z) = f(z) e^{i(\nu_0/v)z} \tag{4.22}$$

and

$$\frac{dE}{dz} = e^{i(\nu_0/v)z} \left(Df + i\frac{\nu_0}{v} f \right) \tag{4.23}$$

At this point, we substitute (4.23) and (4.17), (4.19), (4.21) into equation (4.16), and simplify the complex exponentials. This leads to the operatorial equation

$$Df + i\frac{\nu_0}{v} f + i\frac{A}{v^2} D^{-2} f = 0 \tag{4.24}$$

which is fully equivalent to the Integral Equation (4.16).

Now we apply D^2 to (4.24), which is thereby turned into

$$D^3 f + i\frac{\nu_0}{v} D^2 f + i\frac{A}{v^2} f = 0 \tag{4.25}$$

This, at last, is a differential equation, and can be solved by standard techniques. However, some care is needed in setting the initial conditions. As mentioned, the solution to the Integral Equation (4.16) is uniquely determined by its initial value

$$E_0 = E(0)$$

The same holds for the operatorial equation (4.24), where

$$f(0) = E_0$$

Conversely, the solution to the differential equation (4.25) is determined by three initial conditions: namely, the initial values of f, f' and f'' (f', f'' are the first and second derivative of f). Yet, $f'(0)$, $f''(0)$ are easily found from equation (4.24): the latter yields

$$Df = -i\frac{\nu_0}{v}f - i\frac{A}{v^2}D^{-2}f \tag{4.26}$$

$$D^2 f = -i\frac{\nu_0}{v}Df - i\frac{A}{v^2}D^{-1}f \tag{4.27}$$

and $f'(0)$, $f''(0)$ are related to $f(0)$ by imposing $z = 0$ in (4.26), (4.27) and using $(D^{-2}f)(0), (D^{-1}f)(0) = 0$ (see definition (4.20)). The result is

$$f(0) = E_0 \tag{4.28}$$

$$f'(0) = -i\frac{\nu_0}{v}E_0 \tag{4.29}$$

$$f''(0) = -\frac{\nu_0^2}{v^2}E_0 \tag{4.30}$$

The differential equation (4.25), with the initial conditions (4.28), (4.29) and (4.30), can be solved for f. Let λ_1, λ_2, λ_3 be the roots to the characteristic polynomial

$$p(X) = X^3 + i\frac{\nu_0}{v}X^2 + i\frac{A}{v^2} \tag{4.31}$$

Then, the solution is

$$f(z) = c_1 e^{\lambda_1 z} + c_2 e^{\lambda_2 z} + c_3 e^{\lambda_3 z} \tag{4.32}$$

where the complex coefficients c_1, c_2, c_3 are determined by imposing (4.32) to satisfy (4.28), (4.29) and (4.30). At this point, the field envelope E is given by formula (4.22).

Writing down the field envelope explicitly requires finding the roots to the third-degree polynomial (4.31). This may be done by means, e.g., of Cardano's formulae [27]. Yet, the problem is much simplified if the system is on resonance ($\nu_0 = 0$): in such a case, polynomial (4.31) reduces to

$$p(X) = X^3 + i\frac{A}{v^2}$$

whose roots are

$$\lambda_1 = \frac{\sqrt{3}}{2}\sqrt[3]{\frac{A}{v^2}} - i\frac{1}{2}\sqrt[3]{\frac{A}{v^2}}$$

$$\lambda_2 = -\frac{\sqrt{3}}{2}\sqrt[3]{\frac{A}{v^2}} - i\frac{1}{2}\sqrt[3]{\frac{A}{v^2}}$$

$$\lambda_3 = i\sqrt[3]{\frac{A}{v^2}}$$

By imposing (4.32) to satisfy (4.28), (4.29) and (4.30) with these values for λ_1, λ_2 and λ_3, we get

$$c_1, c_2, c_3 = \frac{E_0}{3}$$

Lastly, if z is large enough the first exponential dominates over the other two (high-gain regime) and

$$E(z) \approx \frac{E_0}{3}\exp\left(\frac{\sqrt{3}}{2}\sqrt[3]{\frac{A}{v^2}}z\right)\exp\left(-i\frac{1}{2}\sqrt[3]{\frac{A}{v^2}}z\right) \tag{4.33}$$

Solution (4.33) is the same that is found in reference [14], under the approximation $v \approx c$. This is seen by introducing Colson's *dimensionless amplitude* $a(\tau)$ and *current* j, which are defined as

$$a(\tau) = \frac{2\pi e L_u N K [JJ]}{mc^2\gamma_r^2}E^*(L_u\tau), \qquad j = \frac{4\pi^2 e^2 L_u^2 N K^2 [JJ]^2}{mc^2\gamma_r^3}n \tag{4.34}$$

where all physical quantities are expressed in Gaussian units and N is the number of undulator periods. By substituting E from (4.33) and using relation

$$\frac{L_u^3 A}{c^2} = \frac{j}{2} \tag{4.35}$$

we find

$$|a(\tau)| = \frac{a_0}{3}\exp\left(\frac{\sqrt{3}}{2}\sqrt[3]{\frac{j}{2}}\tau\right), \qquad \phi(\tau) = \frac{1}{2}\sqrt[3]{\frac{j}{2}}\tau \tag{4.36}$$

($a = |a|\exp(i\phi)$, $a_0 = a(0)$), as reported in reference [14].

Chapter 5

The frequency-pulling phenomenon

5.1 Introduction

In this chapter, we propose a fully analytical theory for frequency pulling in FEL amplifiers (see section 1.4.1). It is based on a time-dependent version of FEL equations; we take these equations from the literature, without proving them. Section 5.2 is a short account on time-dependent FEL theory; it leads to the time-dependent Integral Equation, which is the starting point for our analysis. In section 5.3, we present an interpretation for the FEL amplifier in terms of a dynamic system; such a picture is useful to study the behaviour in the frequency domain. In section 5.4 we derive the "frequency response" of the system, and in section 5.5 we deduce the response to a gaussian seed. This leads to the simple frequency-pulling formula, which yields a prediction for the central frequency of FEL light. Lastly, section 5.6 is devoted to a check for the accuracy of our theory: we consider the FERMI FEL on direct-seeding configuration and compare analytical predictions to PERSEO [43] simulations.

5.2 Time-dependent FEL equations

We consider the FEL setup described in section 1.2. The analysis is based on a generalization of the one-dimensional, steady-state theory presented in chapters 3 and 4. We still consider a continuous electron beam;[1] with respect to steady-state theory, the problem is further simplified by assuming the

[1] I.e., electron parameters at $z = 0$ are t-invariant.

beam to be cold (monoenergetic). At the entrance to the undulator a pulsed seed is injected: thus, the seed is *non* monochromatic, neither periodic.

The relevant physical entity in the optical beam is the electric field, which is responsible for energy exchange between electrons and light: thus, we begin our discussion by assigning the electric field in the seed as a function of time. It is expressed as a carrier[2] at frequency f_s, modulated in both amplitude and phase by a slowly-varying envelope: within the complex formalism, a general formula is[3]

$$\underset{\sim}{\vec{E}_s}(t) = \Re\left\{E_s(t)\, e^{-i\omega_s t}\right\}\hat{x} \tag{5.1}$$

Here, $\vec{E}_{s\sim}(t)$ identifies the physical field, which oscillates quickly in time; the complex-valued function $E_s(t)$ is the electric-field complex envelope; the exponential function represents the carrier; lastly, $\omega_s = 2\pi f_s$.

By generalizing formulas (3.16) and (3.17) of steady-state theory, the electric field in the optical beam is expressed as

$$\underset{\sim}{\vec{E}}(z,t) = \Re\left\{E(z,t)\, e^{i(k_s z - \omega_s t)}\right\}\hat{x}$$

where $k_s = 2\pi/\lambda_s = \omega_s/c$. As in steady-state theory, the problem is reduced to finding the field envelope E at any longitudinal position z (and, in particular, at the undulator end $z = L_u$) once it is known at the entrance to the undulator $z = 0$ (there, $E = E_s$). The novelty is that here E, at any z, is a function of time t.

Now, according to references [14, 15], we "explore" the undulator by following an ideal point which enters at $t = 0$ and moves at speed c; in other words, we introduce the law

$$z = ct \tag{5.2}$$

Within this approach, each longitudinal location is observed at a single time instant; it makes sense for a steady-state system. The moving point spans the undulator in a time interval $T = L_u/c$; we normalize the time variable by defining

$$\tau = \frac{t}{T} \tag{5.3}$$

so that our point enters at $\tau = 0$ and comes out at $\tau = 1$. In terms of the novel variable τ, law (5.2) reads

$$z = L_u \tau \tag{5.4}$$

[2]Here, "carrier" refers to a sinusoidal function of time.

[3]With respect to steady-state theory, the carrier phase is opposite. Consistently, the optical-beam carrier is $\exp(kz - \omega t)$.

In order to observe the various longitudinal locations along the undulator at other time instants, we introduce another ideal point, moving with the former but displaced longitudinally by a length $\tilde{z}$ (positive or negative). In other words, we generalize law (5.4) to

$$z = L_u \tau + \tilde{z} \tag{5.5}$$

This second moving point still spans the undulator in a τ interval equal to unity, but goes in and out before or after the first: in this way, by varying $\tilde{z}$, we can analyse the propagation of an optical pulse.

Recall that the electron beam is slower than light, so it lags behind our ideal moving points. As in steady-state theory, we assume the FEL efficiency to be very low, so we approximate the longitudinal speed of electrons by its value β_{z0} at the entrance to the undulator. Now we consider a site on the electron beam and the ideal point entering the undulator with it, but moving at speed c: when the latter reaches the undulator end, the former has lagged behind by the quantity

$$s = (1 - \beta_{z0}) L_u \tag{5.6}$$

which is known as *slippage length*. The time-dependent Integral Equation reveals its simplest form when the displacement $\tilde{z}$ is normalized with respect to the slippage length, by defining the novel variable

$$\bar{z} = \frac{\tilde{z}}{s} \tag{5.7}$$

At last, we have our independent variables: they are $\bar{z}$ and τ. The physical variables z and t are reconstructed by means of the following transformations:

$$z = s\bar{z} + L_u \tau \tag{5.8}$$

$$t = T\tau \tag{5.9}$$

At this point, the field envelope is normalized conveniently, thereby introducing Colson's dimensionless amplitude $a(\bar{z}, \tau)$ and current j. This is done by means of definitions (4.34); within the present context, in a the factor $E^* (L_u \tau)$ is replaced by $E(s\bar{z} + L_u \tau, T\tau)$, so $a = a(\bar{z}, \tau)$. The Pendulum and Field Equations (3.36), (3.54) are generalized to[4] [14, 15, 18, 19]

$$\frac{d^2\zeta}{d\tau^2} = \Re\left\{ae^{i\zeta}\right\} \tag{5.10}$$

[4]In equation (5.10), $a = a(\bar{z}(\tau), \tau)$ and $\zeta = \zeta(\tau)$; in equation (5.11), $\zeta = \zeta(\tau)$ and brackets denote averaging over electrons in a λ_s around $\bar{z}$ at τ.

$$\frac{\partial a}{\partial \tau} = -j \left\langle e^{-i\zeta} \right\rangle \tag{5.11}$$

For analytical convenience, the initial condition is *not* assigned by giving the field envelope at the entrance to the undulator as a function of time, but by giving a at $\tau = 0$ as a function of $\bar{z}$. Consistently, it is assumed that each site on the optical beam (identified by $\bar{z}$) start interacting with the electron beam at $\tau = 0$ and end at $\tau = 1$.[5]

Within this setting and in the weak-field regime, the evolution of the dimensionless amplitude is ruled by the time-dependent Integral Equation [15, 21, 30]

$$\frac{\partial a}{\partial \tau} = i\pi g_0 \int_0^\tau a\left(\bar{z} + \tau', \tau - \tau'\right) e^{-i\nu_0 \tau'} \tau' d\tau' \tag{5.12}$$

with the initial condition

$$a\left(\bar{z}, 0\right) = a_0\left(\bar{z}\right) \tag{5.13}$$

where a_0 is the spatial distribution of a at $t = 0$ ($z = s\bar{z}$). In equation (5.12), the initial electron phase speed ν_0 is defined as the τ derivative of the electron phase ζ at $\tau = 0$ (i.e., when interaction starts). The *weak-field gain coefficient* g_0 is related to the dimensionless current j and to the so-called *Pierce parameter* ρ [25] by [15, 27]

$$2\pi g_0 = j \tag{5.14}$$

$$\sqrt[3]{\pi g_0} = 4\pi N \rho \tag{5.15}$$

where N is the number of magnetic periods in the undulator. Once initial condition (5.13) be assigned, the Integral Equation (5.12) identifies univocally the function $a\left(\bar{z}, \tau\right)$; in particular, it allows evaluating the field spatial distribution at the end of the interaction, i.e. $a\left(\bar{z}, 1\right)$.

5.3 The FEL amplifier as a linear, time-invariant system

Time-dependent phenomena in FEL amplifiers find an appropriate description within a dynamic-system interpretation. In the simplest meaning of the word, a dynamic system is a physical entity having just an input and an

[5] This is only true for $\bar{z} = 0$: other sites undergo a delay (or advance) in interaction start and end, which causes a negligible inconsistency due to slippage.

output; a signal $x(t)$ enters the system, is processed internally and emerges as a signal $y(t)$ [37]. In general, the system may distort the signal in any way. If the output is linear in the input, i.e.

$$x(t) = ax_1(t) + bx_2(t) \longrightarrow y(t) = ay_1(t) + by_2(t) \tag{5.16}$$

where $x_1(t) \to y_1(t)$, $x_2(t) \to y_2(t)$ and a, b are constants, then the system is said to be *linear*; if a delayed (advanced) input yields an equally delayed (advanced) output, i.e.

$$x(t) = x_0(t-T) \longrightarrow y(t) = y_0(t-T) \tag{5.17}$$

where $x_0(t) \to y_0(t)$ and T is positive (negative), then the system is said to be *time-invariant*. We are concerned with linear and time-invariant (LTI) systems.

An LTI system may be described by means of its *impulse response*, which is the output the system yields when the input is a Dirac delta centered on $t = 0$.[6] The response to any other signal is the convolution between the signal itself and the impulse response. An LTI system can be also represented in the frequency domain: the impulse response is Fourier transformed, thereby obtaining the *frequency response*, and the transform of the response to any signal is the product between the transform of the signal itself and the frequency response. Clearly, if we have a dynamic system and show this last property to hold good, then the system is automatically proved to be LTI. A last remark: in our FEL, whose input/output behaviour is ruled by the Integral Equation (5.12), the independent variable is $\bar{z}$ (not t); we use the terminology "LTI" anyway.

Now we show that the FEL amplifier is an LTI system. Precisely, we prove that $a(\bar{z}, 1)$ is related to $a_0(\bar{z})$ through an LTI system, by showing the Fourier transform of $a(\,\cdot\,, 1)$ to be the product between the transform of a_0 and a function identified univocally by FEL parameters. Since we are transforming in $\bar{z}$, the transformed variable is k; we denote the transforms of a_0 and $a(\,\cdot\,, \tau)$ by $A_0(k)$ and $A(k, \tau)$. Our task is to prove the following

Theorem. A complex function $F(k)$ exists such that

$$A(k, 1) = F(k) \cdot A_0(k) \tag{5.18}$$

for any a_0.

[6] In other words, the impulse response of a system is its Green function.

Proof. By transforming the time-dependent Integral Equation (5.12) with respect to $\bar{z}$, we find [30]

$$\frac{\partial A}{\partial \tau} = i\pi g_0 \int_0^\tau A\left(k, \tau - \tau'\right) e^{-i\nu\tau'} \tau' d\tau' \tag{5.19}$$

where we have defined

$$\nu = \nu_0 - k \tag{5.20}$$

The initial condition corresponding to (5.13) is

$$A\left(k, 0\right) = A_0\left(k\right) \tag{5.21}$$

Here k plays merely the role of a parameter, so equation (5.19) exhibits the same form as the Integral Equation (4.16) for a steady-state system with a cold electron beam. We introduce the auxiliary function

$$f\left(\tau\right) = A\left(k, \tau\right) e^{i\nu\tau}$$

and follow the procedure of section 4.4. In a few steps, equation (5.19) is reduced to the operatorial equation

$$Df - i\nu f - i\pi g_0 D^{-2} f = 0 \tag{5.22}$$

with the initial condition

$$f\left(0\right) = A_0\left(k\right) \tag{5.23}$$

Equation (5.22) with condition (5.23) identifies uniquely the function f; the Fourier-transformed dimensionless amplitude is then given by

$$A\left(k, \tau\right) = f\left(\tau\right) e^{-i\nu\tau}$$

Within the present context, writing down f explicitly is not necessary. It is enough to note that equation (5.22) is linear, so f is proportional to its initial value $A_0\left(k\right)$.[7] As a consequence, a complex constant[8] $F\left(k\right)$ exists such that

$$A\left(k, 1\right) = F\left(k\right) \cdot A_0\left(k\right)$$

which is relation (5.18). $F\left(k\right)$ is given by the solution to equation (5.22) with initial condition set to unity, evaluated at $\tau = 1$ and multiplied by $\exp\left(-i\nu\right)$. □

[7] I.e., f is given by the solution to (5.22) with initial condition set to unity, multiplied by $A_0\left(k\right)$.

[8] Here k is fixed, so this is a constant; yet, it depends on k.

So far, we have been dealing with a representation in the $\bar{z}$ domain; in order to complete the description for the FEL as a dynamic system, we still have to transpose the input/output relation (5.18) into the time domain. Let $a_i(t)$ and $a_o(t)$ be the dimensionless amplitudes of seed and FEL light (at the exit from the undulator), respectively. From a_i we deduce a_0 by noting that before interaction start the optical beam is in free space, so the dimensionless amplitude moves at speed c; in a similar way, from $a(\,\cdot\,, 1)$ we deduce a_o by noting that after the interaction end the optical beam is in free space again. The only relevant detail is that it is convenient to cancel the input/output delay: if there were no interaction, the seed would reach the undulator end with no change but a T delay, so we represent FEL light by centering the time axis on T. Here we are transforming in t, so the transformed variable is ω; we denote the transforms of a_i and a_o by $A_i(\omega)$ and $A_o(\omega)$.

- **Input.** First, we deduce the z distribution of a at $t = 0$. We know that, in $z = 0$, $a = a_i(t)$. As mentioned, a moves at speed c, so its value in any given z at $t = 0$ passes in $z = 0$ at $t = -z/c$. Thus, the z distribution of a at $t = 0$ is given by $a_i(-z/c)$. Now, a_0 is the $\bar{z}$ distribution of a at $\tau = 0$. The space-time couple corresponding to $\bar{z}$ at $\tau = 0$ is $z = s\bar{z}$, $t = 0$ (see (5.8), (5.9)). In conclusion,

$$a_0(\bar{z}) = a_i\left(-\frac{s}{c}\bar{z}\right)$$

By Fourier-transforming this relation with respect to $\bar{z}$, we find

$$A_0(k) = \frac{c}{s} A_i\left(-\frac{c}{s}k\right) \tag{5.24}$$

- **Output.** We know $a(\bar{z}, \tau = 1)$. $\tau = 1$ means $t = T$. a moves at speed c, so its value in $z = L_u$ at any given t is located, at $t = T$, in $z = L_u - c(t - T)$. Since $\tau = 1$, this z corresponds to $\bar{z} = -c(t - T)/s$. Now, as mentioned, we center the time axis on T: thus, $t - T \to t$. In conclusion,

$$a_o(t) = a\left(-\frac{c}{s}t, 1\right)$$

By Fourier-transforming this relation with respect to t, we find

$$A_o(\omega) = \frac{s}{c} A\left(-\frac{s}{c}\omega, 1\right) \tag{5.25}$$

- **Frequency response.** By combining relations (5.18), (5.24) and (5.25), we find

$$A_o(\omega) = H(\omega) \cdot A_i(\omega) \tag{5.26}$$

where

$$H(\omega) = F\left(-\frac{s}{c}\omega\right) \tag{5.27}$$

According to (5.26), the dynamic system relating $a_i(t)$ and $a_o(t)$ is LTI, with frequency response $H(\omega)$ given by formula (5.27).

At this point, the problem is fully outlined: the energy spectrum of FEL light is proportional to

$$|A_o(\omega)|^2 = |H(\omega)|^2 \cdot |A_i(\omega)|^2 \tag{5.28}$$

and by evaluating $|H|^2$ we get the energy-spectrum input/output relation, whence we can deduce the relation between central frequencies.

5.4 Frequency response

If g_0 is big enough (say $g_0 > 1000$), function $|F(k)|^2$ can be approximated by a Gaussian according to [27]

$$|F(k)|^2 \approx G_0 \exp\left(-\frac{\left(\nu - \sqrt{3}\right)^2}{3\sqrt{3}\sqrt[3]{\pi g_0}}\right) \tag{5.29}$$

where

$$G_0 = \frac{1}{9}e^{\sqrt{3}\sqrt[3]{\pi g_0}} = \frac{1}{9}e^{4\sqrt{3}\pi N\rho} \tag{5.30}$$

is the steady-state power gain on resonance.[9]

The spectral energy gain $|H(\omega)|^2$ is obtained by substituting (see formula (5.27))

$$k = -\frac{s}{c}\omega = -\frac{2\pi s}{c}f \tag{5.31}$$

[9] We define the power (energy) gain as the output/input power (energy) ratio. Formula (5.30) is found from (4.36) with (5.14) and (5.15).

i.e., according to definition (5.20),

$$\nu = \frac{2\pi s}{c}\left(f + \frac{c}{2\pi s}\nu_0\right) \tag{5.32}$$

Now we recall formula[10]

$$\nu_0 \approx -2\pi N \frac{\Delta f}{f_r} \tag{5.33}$$

(the approximation is very good), where

$$\Delta f = f_s - f_r \tag{5.34}$$

is the *seed detuning*, f_r being the resonance frequency (see note 10). We also express the slippage length according to[11]

$$s = N\lambda_r \tag{5.35}$$

where $\lambda_r = c/f_r$ is the resonance wavelength. By substituting formulas (5.33) and (5.35) into (5.32), we get

$$\nu = \frac{2\pi N}{f_r}\left(f - \Delta f\right) \tag{5.36}$$

Lastly, we substitute this result into formula (5.29). After a few manipulations, we find

$$\left|H\left(2\pi f\right)\right|^2 \approx G_0 \exp\left(-\frac{\left(f-\mu\right)^2}{2\sigma^2}\right) \tag{5.37}$$

---

[10] From the electron phase (3.20), we find

$$\nu_0 = -L_u k_s \left(1 - \bar{\beta}_z\right) + L_u k_u \bar{\beta}_z$$

(here, $\bar{\beta}_z$ is the electron speed at $\tau = 0$). A paraphrase of formula (3.22) yields

$$\bar{\beta}_z = \frac{k_r}{k_r + k_u}, \qquad 1 - \bar{\beta}_z = \frac{k_u}{k_r + k_u}$$

We denote by f_r, λ_r and k_r the *resonance frequency*, *wavelength* and *wavenumber* (an electron at the initial energy resonates to an optical beam at frequency f_r). By substituting these relations into the former and using $k_r >> k_u$, $k_s = 2\pi f_s/c$, $k_r = 2\pi f_r/c$ and $k_u L_u = 2\pi N$, we find formula (5.33).

[11] On resonance, electrons lag behind light by one optical wavelength at each undulator period. This is seen clearly from the electron phase definition (3.20): if $\dot{\zeta} = 0$, the phase of the optical field $\omega t - k\bar{z}$ at the electron position is equal to the phase of the magnetostatic field $k_u\bar{z}$; as the electron passes through one undulator period, both phases vary by 2π. This means that light has overcome the electron by an optical wavelength.

where

$$\mu = \Delta f - \Delta f_0, \qquad \Delta f_0 = -\frac{\sqrt{3} f_r}{2\pi N} \tag{5.38}$$

$$\sigma^2 = \frac{3\sqrt{3}\rho f_r^2}{2\pi N} \tag{5.39}$$

Expression (5.37) represents a Gaussian having mean value μ and variance σ^2. Once FEL parameters be set, σ^2 and G_0 are constants, and μ only depends on the seed detuning Δf; Δf_0 represents the value for Δf at which $\mu = 0$, i.e. the frequency gain $|H(2\pi f)|^2$ is centred on $f = 0$.

5.5 Frequency pulling

At this point, we inject a gaussian seed $a_i(t)$. Its spectrum $A_i(\omega)$ is a Gaussian, so we can write[12]

$$|A_i(2\pi f)|^2 = \exp\left(-\frac{f^2}{2\sigma_s^2}\right) \tag{5.40}$$

where σ_s is the r.m.s. energy bandwidth of the seed. By combining formulas (5.28), (5.37) and (5.40), we find

$$|A_o(2\pi f)|^2 = G_0 \cdot e^{-2\alpha} \cdot \exp\left(-\frac{(f-M)^2}{2\Sigma^2}\right) \tag{5.41}$$

where

$$M = \frac{\sigma_s^2}{\sigma_s^2 + \sigma^2}\mu \tag{5.42}$$

$$\Sigma^2 = \frac{\sigma_s^2 \sigma^2}{\sigma_s^2 + \sigma^2} \tag{5.43}$$

and

$$\alpha = \frac{1}{4}\frac{\mu^2}{\sigma_s^2 + \sigma^2} \tag{5.44}$$

is the attenuation caused by seed detuning, measured in nepers. Again, expression (5.41) represents a Gaussian having mean value M and variance

[12] Expression (5.40) is correct but for a constant factor, which accounts for seed energy. This is introduced later on.

Σ^2; G_0 only depends on FEL parameters, Σ^2 on FEL parameters and seed length, and M, α on all physical quantities: FEL parameters, seed length and also seed detuning.

Now, recall that we represent the signal at the undulator end by centring the time axis on T. The carrier has a phase velocity equal to c, so it undergoes a phase delay T as it travels throughout the undulator; as a consequence, within our choice of time origin the carrier at the undulator end is simply $\exp(-i\omega_s t)$. So, by representing by $a(t)$ either $a_i(t)$ or $a_o(t)$, seed and FEL light are given by

$$\Re\left\{a(t)\cdot e^{-i\omega_s t}\right\} = \Re\left\{a^*(t)\cdot e^{i\omega_s t}\right\} \tag{5.45}$$

The last expression is useful to deduce the energy spectrum: the factor $\exp(i\omega_s t)$ shifts the Fourier transform of a^* around the seed frequency f_s, thereby yielding the positive frequencies, and the real-part operator mirrors the spectrum around $f = 0$, thereby yielding the negative frequencies. Lastly, the Fourier transform of a^* is $A^*(-\omega)$ ($A(\omega)$ is the Fourier transform of a), so the positive frequencies of the optical energy spectrum are given by $|A|^2$, mirrored around the origin and shifted around the seed frequency.[13]

According to this discussion, the positive frequencies of our seed's energy spectrum are described by the Gaussian

$$e_i(f) = e_0 \exp\left(-\frac{(f-f_s)^2}{2\sigma_s^2}\right) \tag{5.46}$$

(see expression (5.40)) where we have introduced the novel coefficient e_0, which accounts for seed energy. Formula (5.28) is linear, and dimensionless and optical spectra of seed and FEL light are related through the same linear operation; thus, the positive frequencies of FEL light's energy spectrum are described by the Gaussian

$$e_o(f) = G_0 e_0 \cdot e^{-2\alpha} \cdot \exp\left(-\frac{(f-[f_s-M])^2}{2\Sigma^2}\right) \tag{5.47}$$

which is found from formula (5.41) by including coefficient e_0.

With formula (5.47) our analytical discussion is substantially complete: the central frequency of FEL light f_{FEL} is given simply by the mean value of Gaussian (5.47), which is $f_s - M$. By substituting M, μ and Δf from

[13] Two remarks are worth. First, A is a Gaussian and spans over all frequencies, so its images at positive and negative optical frequencies interfere; this is neglected. Second, we have not mentioned a few coefficients (the factor $1/2$ introduced by the real-part operator and the conversion factor relating the optical energy to its dimensionless counterpart), which are included into e_0 in (5.46).

definitions (5.42), (5.38) and (5.34), we end up with the *frequency-pulling formula*

$$f_{\mathrm{FEL}} = f_s - \frac{\sigma_s^2}{\sigma_s^2 + \sigma^2} (f_s - f_r) - \frac{\sigma_s^2}{\sigma_s^2 + \sigma^2} \frac{\sqrt{3} f_r}{2\pi N} \tag{5.48}$$

This formula is identical to the one reported in reference [33], except for the last term, which is a constant shift (it only depends on FEL parameters and seed length, not on seed detuning). According to formula (5.48), a seed at the resonance frequency is (frequency-) pulled by an amount equal to the last term, a seed out of resonance is pulled by an amount depending linearly on seed detuning, and pulling only vanishes if

$$f_s = \left(1 - \frac{\sqrt{3}}{2\pi N}\right) f_r \tag{5.49}$$

We conclude by deriving the *energy-gain formula*. The optical-pulse energy, for both seed and FEL light, can be deduced by integrating the energy spectrum; thus, it is clear that the energy of FEL light depends on seed detuning, and exhibits a maximum when frequency pulling vanishes. In fact, according to formula (5.28) the energy spectrum of FEL light is related to the product between Gaussians (5.37) and (5.40); the latter is centred on $f = 0$ and the former on $f = \mu$ (which depends on seed detuning), and the closer these central frequencies are, the greater the product is (and the smaller the frequency pulling): so, the energy of FEL light is maximum when $\mu = 0$ (which corresponds to a vanishing frequency pulling) and decays as $|\mu|$ increases. A quantitative description for this behaviour is derived by integrating formulas (5.46) and (5.47) and comparing the resulting energies; the computation is straightforward, and yields

$$E_o = G e^{-2\alpha} E_i \tag{5.50}$$

where $E_i = \int e_i$, $E_o = \int e_o$ are the energies of seed pulse and FEL-light pulse, and

$$G = \frac{\sigma}{\sqrt{\sigma_s^2 + \sigma^2}} G_0 \tag{5.51}$$

is the energy gain at vanishing frequency pulling.

5.6 Application to FERMI@Elettra

The FERMI project at the Elettra synchrotron-light laboratory in Trieste, Italy is an FEL user facility [1]. Electrons emitted by a photocatode pass through a normal-conducting linear accelerator and operate two single-pass, seeded, HGHG FELs. At present, the first FEL is under commissioning. From now on, we will refer to this as the FERMI FEL.

In the FERMI FEL, electrons and seed are injected into a short undulator, the *modulator*; the system is set on resonance, so the optical beam induces on the electron beam a spatially periodic energy modulation at the seed wavelength. Electrons are then passed through a magnetic chicane, the *dispersion section*, where energy modulation is converted into a density bunching. Lastly, a long undulator, the *radiator*, follows the dispersion section. The radiator is tuned at a harmonic of the seed frequency; the density bunching of incoming electrons exhibits a strong harmonic content, so electrons radiate coherently at the chosen harmonic. Emitted light interacts with electrons and the system eventually reaches the high-gain regime, up to saturation.

The FERMI FEL may operate also as an amplifier. This is done by the so-called *direct seeding*: the modulator is detuned, the dispersion section is turned off and an HHG seed is injected directly into the radiator. We sample our theory for frequency pulling by considering an ideal model for the FERMI FEL on direct-seeding configuration. The system is the FEL amplifier described in section 1.2; the electron beam is cold and the seed is a single short gaussian pulse. System parameters are set to a resonance wavelength of about 60 nm; the resulting weak-field gain coefficient g_0 is equal to about 2000. With such a great value for g_0, our approximations are expected to hold good. Frequency pulling is obtained by seed detuning; theoretical predictions are compared to results from time-dependent PERSEO [43] simulations performed by Dr. Simone Spampinati.

We first consider a seed having a length of 40 fs (power full width at half maximum, FWHM). A variable seed detuning $f_s - f_r$ is introduced by moving the seed central frequency f_s; the frequency-pulling formula (5.48) yields an estimate for the shift induced on FEL-light central frequency, and the energy-gain formula (5.50) allows evaluating the expected output power. Figures 5.1 and 5.2 present a comparison between predictions from these formulas and simulative results. In both plots, the seed detuning, normalized to the resonance frequency f_r, is reported on the horizontal axis. Formula (5.48) is linear, so the theoretical curve in figure 5.1 is a straight line; simulative results show an excellent agreement. On the other hand, formula (5.50) yields a gaussian trend; figure 5.2 shows simulative results to be fitted perfectly by a Gaussian. Theoretical and fitting curve exhibit a similar shape; the evident mismatch between maxima is due to analytical approximations and numerical errors.

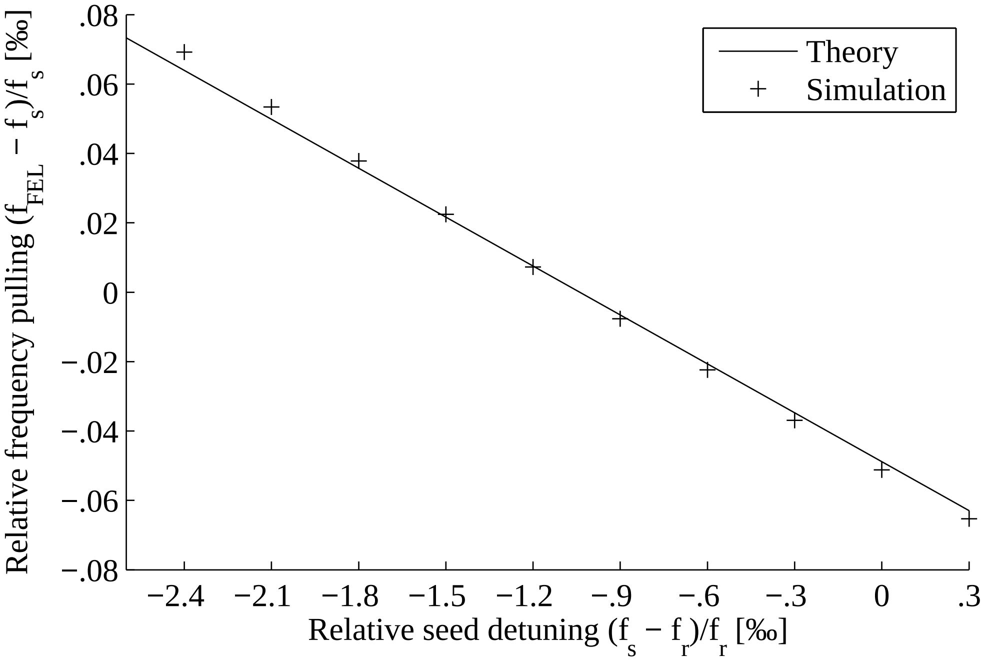

Figure 5.1: Frequency pulling by seed detuning: theory vs. simulations.

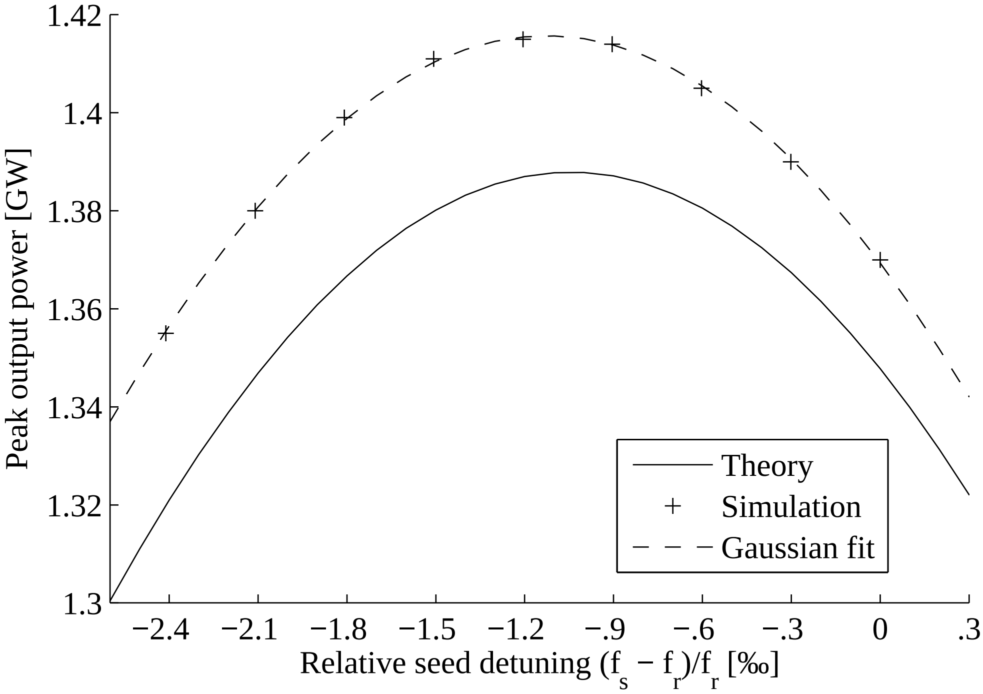

Figure 5.2: Output power as a function of seed detuning.

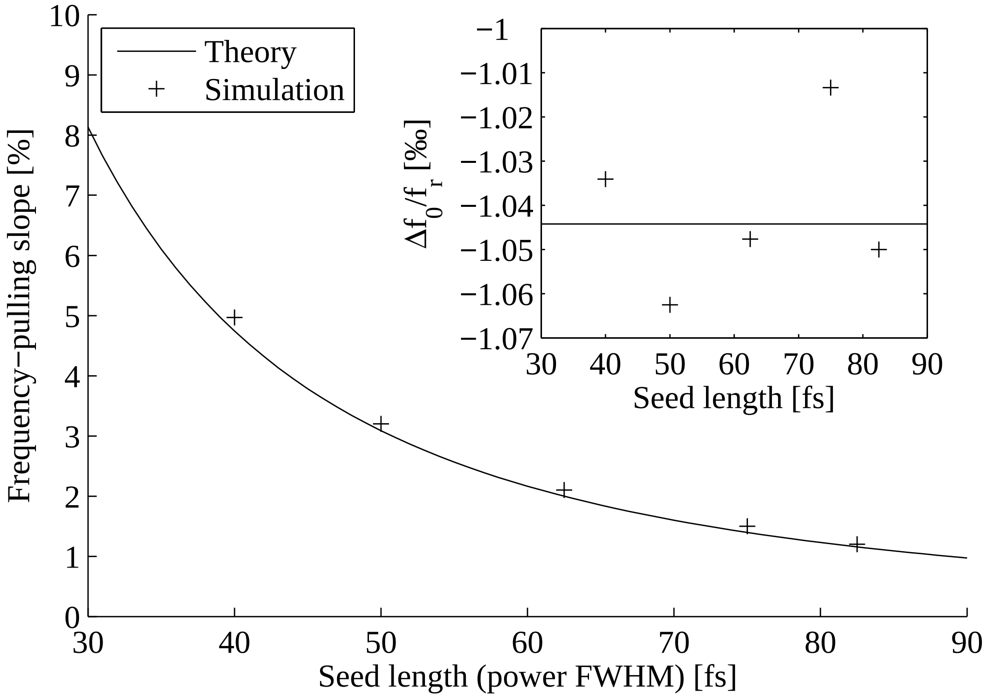

Figure 5.3: Frequency-pulling slope as a function of seed length. Inset: zero point Δf_0.

Now, we consider seeds with different lengths. For each of them, simulative results on frequency pulling are obtained as in figure 5.1; these are then interpolated linearly. In this way, we have a check for slope and zero of the line described by the frequency-pulling formula (5.48). The slope is given by $-\sigma_s^2/\left(\sigma_s^2+\sigma^2\right)$, so it depends on seed length; figure 5.3 reports the theoretical value as a function of seed length, along with simulative results. The agreement is excellent. On the other hand, the zero is given by formula (5.49), so it does *not* depend on seed length; the inset in figure 5.3 shows theoretical value and simulative results. Simulations exhibit a ripple, which is though limited within a few percent around the theoretical value.

Part II

Three-dimensional theory

Chapter 6

FEL equations for a guided mode

6.1 Introduction

In this chapter, we propose an extension to the simple one-dimensional theory described in chapter 3. We loose the hypotesis of a transversally uniform electron beam: thus, the optical beam is guided, as described in section 1.3.3.

A fully analytical solution for guided modes is not feasible. The most advanced numerical approaches to the problem provide accurate solutions, but rely on highly mathematical techniques, which do not allow for a simple physical interpretation [39]. Consequently, the need arises for an easier and more physically transparent approach to the problem. Based on the analogy between optical guiding in FELs and guided propagation in waveguides, attempts have been done to model the system by a "virtual" waveguide. Reference [40] provides a metallic-waveguide model, while references [41, 42] develop a dielectric-waveguide model. In a metallic waveguide, the e.m. field is confined to a limited transverse domain; conversely, a dielectric-waveguide model extends over the whole transverse plane. In an FEL, the transverse domain is unlimited, so dielectric-waveguide models are the most effective.

We consider a single guided mode. It is expressed on the basis of a mode in a (virtual) dielectric waveguide. Then, extensions for the Pendulum and Field Equations (3.36), (3.54) are derived, which describe the evolution of the electron and optical beams. By coupling the extended equations, a simple self-consistent description for the system is obtained, which is valid for a warm electron beam and on strong-field regime (as far as the transverse profile of the optical field is close to the one of the guided mode).

6.2 Modes in a dielectric waveguide

Let us consider a (virtual) *dielectric waveguide*, i.e. a (virtual) space filled by a medium which is homogeneous along the direction identified by the z axis, but potentially inhomogeneous along transverse directions $\vec{\rho}$. We denote the index of refraction by n. Then, the z homogeneity is expressed in mathematical form by the condition

$$n = n(\vec{\rho})$$

In such a medium an electric field $\vec{E}$ may exist which is expressed mathematically by a separate-variable function, i.e.

$$\vec{E}(\vec{r}, t) = \Re\left\{Ce(\vec{\rho})\, e^{i(\omega t - k_z z)}\hat{e}\right\} \tag{6.1}$$

This expression is analogous to formula (3.15) for a plane wave in free space. The field is monochromatic; let f be the frequency, $\omega = 2\pi f$. The real constant k_z is the *longitudinal wavenumber*; the complex scalar function $e(\vec{\rho})$ describes the transverse profile (which is independent on z). Lastly, the complex vector $\hat{e}$ identifies the polarization[1] and the complex constant C gives initial amplitude and phase. Solution (6.1) for the electric field is called a *mode* of the waveguide.

It may be shown (by using Maxwell equations) that k_z and $e(\vec{\rho})$ satisfy (approximately) the following condition [13]:

$$\nabla_\perp^2 e(\vec{\rho}) + \left[n^2(\vec{\rho})\, k^2 - k_z^2\right] e(\vec{\rho}) = 0 \tag{6.2}$$

(k is the wavenumber of a plane wave at frequency f propagating in free space). The valid couples k_z, $e(\vec{\rho})$ are determined from equation (6.2). By re-stating the latter as

$$\left[\nabla_\perp^2 + n^2(\vec{\rho})\, k^2\right] e(\vec{\rho}) = k_z^2 e(\vec{\rho})$$

it is clear that k_z^2 is an eigenvalue for the operator $[\nabla_\perp^2 + n^2(\vec{\rho})\, k^2]$ and $e(\vec{\rho})$ is an associated eigenfunction. Lastly, we introduce the *longitudinal wavelength* λ_z and the *longitudinal index of refraction* n_z by writing

$$k_z = \frac{2\pi}{\lambda_z} = n_z k$$

This formalism is useful for what follows.

[1] In general, the polarization may vary with $\vec{\rho}$ (the situation would imply substituting the product $e(\vec{\rho})\,\hat{e}$ by a single complex vectorial function $\vec{e}(\vec{\rho})$), but such a picture is out of our interest.

6.3 Electron motion

In this section, we extend the analysis presented in section 3.2 to a case in which the optical field is not transversally uniform. We assume the optical beam to be dominated by a guided mode and describe the latter on the basis of the waveguide mode (6.1).

As in one-dimensional theory, first we describe the electron wiggling driven by the undulator field (3.1). Here, the optical beam is no more (approximately) a uniform plane wave: the dominating guided mode exhibits a transverse profile, both in amplitude and phase, so wavefronts are not plane (in general). Even if wavefronts are plane, as in the gaussian approximation we shall develope in section 7.4, the field is not uniform on a given wavefront. As a consequence, the net force due to the optical beam may exhibit a transverse component. Yet, we assume smooth transverse variation for the optical field, so that the force it induces on electrons is mainly longitudinal and transverse dynamics is dominated by the undulator field.

Within this assumption, the analysis of section 3.2.1 still holds good. For any electron in the beam, the longitudinal position is expressed by formula (3.11), in combination with equations (3.12) and (3.13); wiggling still takes place on the horizontal plane, as stated by formula (3.7), and the transverse speed is given by formula (3.9).

6.3.1 Electron-light interaction

As in section 3.2.2, now we derive a formula describing electron-energy variations due to the interaction between electron and optical beam. The fundamental energy-variation formula (3.14) is completely general (i.e. it is valid for any electron in any electric field), so we only have to express the electric field in terms of the waveguide mode (6.1) and state (3.14) in terms of field envelope.

The optical field is still assumed to be x-linearly polarized and monochromatic (which implies the system to be on steady state). Within these assumptions, we express our electric field as a waveguide mode (such as (6.1)) linearly polarized along the x axis, with the introduction of a further z dependence into the complex amplitude:

$$\underset{\sim}{\vec{E}} = \underset{\sim}{E_x}\, \hat{x} \tag{6.3}$$

where

$$\underset{\sim}{E_x}\left(\vec{r}, t\right) = \Re\left\{E\left(z\right) e\left(\vec{\rho}\right) e^{i\left(\omega t - k_z z\right)}\right\} \tag{6.4}$$

As in one-dimensional theory, the complex function $E(z)$ is the electric-field complex envelope.

Now, we substitute expressions (6.3), (6.4) into formula (3.14). We find

$$\frac{d\gamma}{dt} = -\frac{e}{mc}\beta_x(t) \cdot \Re\left\{E(z(t))\, e(\vec{\rho})\, e^{i(\omega t - k_z z(t))}\right\} \tag{6.5}$$

Here, $\vec{\rho}$ is the transverse position of our electron at the entrance to the undulator.[2] Formula (6.5) is the three-dimensional counterpart of (3.18). By expressing electron transverse speed in terms of longitudinal position through (3.9), we find the counterpart of formula (3.19):

$$\frac{d\gamma}{dt} = \frac{eK}{mc}\frac{1}{\gamma(t)} \cdot \Re\left\{E(z(t))\, e(\vec{\rho}) \cos k_u z(t) \cdot e^{i(\omega t - k_z z(t))}\right\} \tag{6.6}$$

Formulae (3.19) and (6.6) exhibit the same longitudinal structure (i.e., they depend on z in the same way). Yet, in (3.19) the exponent involves k, while in (6.6) it involves k_z. To the aim of extending to (6.6) the analysis performed on (3.19), here we define the electron phase as

$$\zeta(t) = (k_z + k_u)\,\bar{z}(t) - \omega t \tag{6.7}$$

Resonance is still defined as the condition of time-independent phase. However, due to the novel definition for ζ, the resonance formula is slightly different from its one-dimensional version (3.24). By deriving (6.7) and using definition (3.21), we find

$$\dot{\zeta}(t) = c\,(k_z + k_u)\,\bar{\beta}_z(t) - \omega$$

Now we impose this expression to be null and use relation $\omega = ck$:

$$(k_z + k_u)\,\bar{\beta}_z(t) - k = 0$$

In conclusion, the novel resonance speed $\bar{\beta}_{zr}$ is related to k and k_z by

$$\bar{\beta}_{zr} = \frac{k}{k_z + k_u} \tag{6.8}$$

For clarity, we keep the symbol γ_r for denoting the one-dimensional resonance energy and introduce the novel symbol Γ, which represents the resonance

[2] According to formula (3.14), the electric field is evaluated at the instantaneous electron position, so the transverse profile $e(\cdot)$ is evaluated at the instantaneous transverse position. Yet, we assume variations of $e(\cdot)$ to be small in the range of transverse electron motion (recall that we assume smooth transverse variation for the optical field), and approximate the instantaneous transverse position of our electron by $\vec{\rho}$.

energy emerging from formula (6.8). An explicit relation between K, λ, n_z and Γ is found by means of formula (3.12), which yields

$$\bar{\beta}_{zr} = 1 - \frac{1}{2\Gamma^2}\left(1 + \frac{K^2}{2}\right) \tag{6.9}$$

By comparing (6.8) and (6.9), using $k_z = n_z k >> k_u$ and substituting k, k_u in terms of λ, λ_u, we find the novel resonance formula[3]

$$\frac{\lambda}{\lambda_u} + (n_z - 1) = \frac{n_z}{2\Gamma^2}\left(1 + \frac{K^2}{2}\right) \tag{6.10}$$

As we shall see in section 8.3, within the range we consider for FEL parameters n_z is so close to unity that $n_z - 1$ is negligible with respect to λ/λ_u, and formula (6.10) does not introduce relevant modifications to what is found from its one-dimensional counterpart (3.24). In particular, the value for K which lets an electron at energy γ resonate with an optical beam at wavelength λ is still given closely by formula (3.25).

As in one-dimensional theory, electrons move in a small energy range around the resonance energy Γ (see note 12 at page 32). Now, as just explained, Γ is close to γ_r. As a consequence, approximation (3.26) is still valid. Thus, the fast electron motion is still given by formula (3.28). By using $k_z = n_z k >> k_u$ and (3.28), we expand

$$\cos k_u z\,(t) \cdot e^{i(\omega t - k_z z(t))} \approx \frac{1}{2} e^{-i\zeta(t)} f\left(\bar{z}\,(t)\right) \tag{6.11}$$

where

$$f\,(z) = e^{i n_z \xi \sin 2k_u z} + e^{i2k_u z} \cdot e^{i n_z \xi \sin 2k_u z} \tag{6.12}$$

and ξ is defined by (3.31). As in one-dimensional theory, f is a $\lambda_u/2$-periodic function of z. We substitute it by its mean value $[JJ]$, with the novel definition

$$[JJ] = J_0\left(n_z \xi\right) - J_1\left(n_z \xi\right) \tag{6.13}$$

---

[3] Relation $k >> k_u$ comes from $\lambda << \lambda_u$, which is always true at the wavelengths we consider. Relation $k_z >> k_u$ comes from the former if $n_z \widetilde{\geq} 1$; n_z depends on the waveguide mode used to represent the optical beam. We do not prove that $n_z \widetilde{\geq} 1$. Yet, the fundamental guided mode in our FEL can be thought of as a superposition of plane waves propagating in vacuum along directions slightly departed from the z axis; the electron beam both amplifies these waves and redirects them towards the z axis. The longitudinal wavelength (distance from a wavefront to the next at the same phase) is then expected to be slightly greater than λ. Such a field is well represented by a waveguide mode having longitudinal wavelength λ_z slightly greater than λ, i.e. $n_z \approx 1$.

The final formula is

$$\cos k_u z\left(t\right) \cdot e^{i\left(\omega t - k_z z(t)\right)} \approx \frac{1}{2}\left[JJ\right] e^{-i\zeta(t)} \tag{6.14}$$

By substituting formula (6.14) into formula (6.6) and using approximation (3.26), we find

$$\frac{d\gamma}{dt} = \frac{eK}{2mc}\left[JJ\right]\frac{1}{\gamma_r} \cdot \Re\left\{E\left(z\left(t\right)\right) e\left(\vec{\rho}\right) e^{-i\zeta(t)}\right\} \tag{6.15}$$

6.3.2 The Pendulum Equation

We conclude this section by deriving a novel version of the Pendulum Equation (3.36), suitable for describing electron motion within a guided mode. By deriving the phase definition (6.7) two times and substituting definition (3.21), we get

$$\frac{d^2\zeta}{dt^2} = c\left(k_z + k_u\right)\frac{d\bar{\beta}_z}{dt}$$

which, combined to expression (3.35) for $d\bar{\beta}_z/dt$ and formula (6.15), yields

$$\frac{d^2\zeta}{dt^2} = \Omega^2 \cdot \Re\left\{E\left(z\left(t\right)\right) e\left(\vec{\rho}\right) e^{-i\zeta(t)}\right\} \tag{6.16}$$

with the novel definition

$$\Omega^2 = \frac{eK}{2m}\left(1 + \frac{K^2}{2}\right)\left[JJ\right]\frac{k_z + k_u}{\gamma_r^4} \approx \frac{4\pi e}{m\lambda_u}\frac{Kn_z}{1 + K^2/2}\left[JJ\right]\frac{\lambda}{\lambda_u}$$

(as in one-dimensional theory, the last approximation is found from $k_z = n_z k >> k_u$ and (3.24)).

Equation (6.16) exhibits the same form as its one-dimensional counterpart (3.36), except for a further dependence on electron transverse position $\vec{\rho}$. Yet, $\vec{\rho}$ does not depend on time (see note 2), so it plays merely the role of a parameter, and equation (6.16) can be handled exactly as the Pendulum Equation (3.36). With respect to one-dimensional theory, the coefficient Ω^2 differs by the presence of n_z inside k_z and $[JJ]$; however, we can approximate $n_z \approx 1$ (as discussed previously), thereby reducing Ω^2 to its one-dimensional value.

6.4 Light evolution

In this section, we extend the analysis presented in section 3.3 to the case considered in section 6.3: the optical field is not transversally uniform; a guided mode dominates the optical beam and is described on the basis of a waveguide mode.

Here, we start from the three-dimensional wave equation (2.11). As in the one-dimensional case, we apply the SVEA and average in order to obtain a simpler equation in the field envelope. First of all, we split the source current into its longitudinal and transverse components. In one-dimensional theory, the current was only varying spatially along the z direction, and the longitudinal and transverse components were easily given by the projections on z and x axes. Now, we are in a fully three-dimensional framework, and the current varies along any spatial direction. Yet, we assume smooth transverse variation for the electron distribution[4], so that transverse variations of the current are slow and the one-dimensional choice is still approximately valid:

$$\vec{J}_{//} \approx J_z \hat{z} \tag{6.17}$$

$$\vec{J}_{\perp} \approx J_x \hat{x} \tag{6.18}$$

Within assumptions (6.3) for the field and (6.18) for the transverse current, equation (2.11) is equivalent to the scalar equation

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \underset{\sim}{E_x} = \mu_0 \frac{\partial J_x}{\partial t} \tag{6.19}$$

Now, as in the one-dimensional case, we simplify this equation by introducing a proper SVEA.

6.4.1 SVEA for a guided mode

As in section 3.3, we consider first the left-hand side. We write

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \underset{\sim}{E_x} = \nabla_\perp^2 \underset{\sim}{E_x} + \Box_z \underset{\sim}{E_x} \tag{6.20}$$

[4]This is consistent with the assumption of transversally smooth optical field (see section 6.3). In fact, transverse profiles of guided modes depend on electron distribution; in particular, the widths of dominant mode and electron beam are usually comparable [39]. Thus, a transversally smooth electron distribution induces a transversally smooth optical field.

where $\nabla_\perp^2$ is the transverse Laplacian, and we have introduced the *longitudinal d'Alembertian*

$$\Box_z = \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

With respect to the one-dimensional case, the situation is complicated by the presence of the transverse-laplacian term. Yet, by expanding the field according to formula (6.4) and taking into account equation (6.2) we obtain

$$\nabla_\perp^2 \underset{\sim}{E_x} = \Re\left\{E(z)\,e^{i(\omega t - k_z z)}\nabla_\perp^2 e(\vec{\rho})\right\} = \Re\left\{E(z)\,e^{i(\omega t - k_z z)}\left[k_z^2 - n^2(\vec{\rho})\,k^2\right]e(\vec{\rho})\right\}$$

and

$$\Box_z \underset{\sim}{E_x} = \Re\left\{\Box_z\left[E(z)\,e^{i(\omega t - k_z z)}\right]e(\vec{\rho})\right\}$$

By substituting into (6.20), we find

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\underset{\sim}{E_x} = \Re\left\{\left(\Box_z\left[E(z)\,e^{i(\omega t - k_z z)}\right] + k_z^2 E(z)\,e^{i(\omega t - k_z z)}\right)e(\vec{\rho}) + \right.$$
$$\left. - k^2 E(z)\,e^{i(\omega t - k_z z)} n^2(\vec{\rho})\,e(\vec{\rho})\right\} \tag{6.21}$$

which is written in a form suitable for further simplification in a SVEA context, as we shall see in a while.

Now we follow the one-dimensional procedure and decompose the longitudinal d'Alembertian as

$$\Box_z = \left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial z} - \frac{1}{c}\frac{\partial}{\partial t}\right) \tag{6.22}$$

In formula 6.21 this operator is applied to $E(z)\exp(i(\omega t - k_z z))$, so we calculate

$$\frac{\partial}{\partial z}\left[E(z)\,e^{i(\omega t - k_z z)}\right] = \left(\frac{dE}{dz} - ik_z E(z)\right)e^{i(\omega t - k_z z)} \tag{6.23}$$

and

$$\frac{\partial}{\partial t}\left[E(z)\,e^{i(\omega t - k_z z)}\right] = i\omega E(z)\,e^{i(\omega t - k_z z)} \tag{6.24}$$

Within one-dimensional theory, the SVEA states that the field envelope varies much slowlier than the carrier (see formula (3.41). In the present context, E plays the role of an envelope and $\exp(i(\omega t - k_z z))$ the role of a carrier (see formula (6.23)). Thus, we modify slightly the SVEA (3.41), by substituting $k \to k_z$. The novel approximation reads

$$\left|\frac{dE}{dz}\right| << k_z \left|E(z)\right| \tag{6.25}$$

This apparently small change is actually relevant: first of all, it is logical and physically meaningful, and in the second place it induces easier computations. Within approximation (6.25), we have

$$\frac{dE}{dz} - ik_z E(z) \approx -ik_z E(z)$$

and

$$\frac{\partial}{\partial z}\left[E(z)\,e^{i(\omega t - k_z z)}\right] = -ik_z E(z)\,e^{i(\omega t - k_z z)}$$

so that

$$\left(\frac{\partial}{\partial z} - \frac{1}{c}\frac{\partial}{\partial t}\right)\left[E(z)\,e^{i(\omega t - k_z z)}\right] = -i(k + k_z)\,E(z)\,e^{i(\omega t - k_z z)} \tag{6.26}$$

where the relation $\omega/c = k$ has been used.

After substitution of (6.22) and (6.26) into (6.21), the term involving the longitudinal d'Alembertian reduces to

$$\Box_z\left[E(z)\,e^{i(\omega t - k_z z)}\right] =$$

$$= -i(k + k_z)\frac{dE}{dz}e^{i(\omega t - k_z z)} - i(k + k_z)\,E(z)\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)e^{i(\omega t - k_z z)} =$$

$$= -i(k + k_z)\frac{dE}{dz}e^{i(\omega t - k_z z)} + \left(k^2 - k_z^2\right)E(z)\,e^{i(\omega t - k_z z)}$$

due to the relation

$$\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t} = i(k - k_z)$$

valid when this operator is applied to the function $\exp(i(\omega t - k_z z))$. In formula (6.21), the factor weighting the transverse-profile function $e(\vec{\rho})$ is then

$$\Box_z\left[E(z)\,e^{i(\omega t - k_z z)}\right] + k_z^2 E(z)\,e^{i(\omega t - k_z z)} =$$

$$= -i(k + k_z)\frac{dE}{dz}e^{i(\omega t - k_z z)} + k^2 E(z)\,e^{i(\omega t - k_z z)} \tag{6.27}$$

By substituting formula (6.27) into formula (6.21), we conclude that, within SVEA (6.25), the left-hand side of the wave equation (6.19) reduces to

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) \underset{\sim}{E_x} =$$

$$= -k \cdot \Re\left\{i\left(n_z+1\right)\frac{dE}{dz}e^{i(\omega t - k_z z)}e\left(\vec{\rho}\right) + kE\left(z\right)e^{i(\omega t - k_z z)}\left[n^2\left(\vec{\rho}\right) - 1\right]e\left(\vec{\rho}\right)\right\} \tag{6.28}$$

6.4.2 The Field Equation

Formula (6.28) allows expressing equation (6.19) in terms of the field envelope. Now, as in section (3.3), we substitute the source current by its slowly-varying component, expressed in terms of electron phases.

Reducing the wave equation

Here, we reduce equation (6.19) to a first-order ordinary differential equation in E. By substituting (6.28) into (6.19), we find

$$-k\cdot\Re\left\{i\left(n_z+1\right)\frac{dE}{dz}e^{i(\omega t - k_z z)}e\left(\vec{\rho}\right) + kE\left(z\right)e^{i(\omega t - k_z z)}\left[n^2\left(\vec{\rho}\right) - 1\right]e\left(\vec{\rho}\right)\right\} = \mu_0\frac{\partial J_x}{\partial t} \tag{6.29}$$

Then we expand the real-part operator:

$$\frac{dE}{dz}e^{i(\omega t - k_z z)}e\left(\vec{\rho}\right) - i\frac{k}{n_z+1}E\left(z\right)e^{i(\omega t - k_z z)}\left[n^2\left(\vec{\rho}\right) - 1\right]e\left(\vec{\rho}\right) +$$

$$-\frac{dE^*}{dz}e^{-i(\omega t - k_z z)}e^*\left(\vec{\rho}\right) - i\frac{k}{n_z+1}E^*\left(z\right)e^{-i(\omega t - k_z z)}\left[n^2\left(\vec{\rho}\right) - 1\right]e^*\left(\vec{\rho}\right) =$$

$$= -\frac{2\eta_0}{n_z+1}\frac{1}{i\omega}\frac{\partial J_x}{\partial t} \tag{6.30}$$

Now we multiply equation (6.30) by $\exp\left(-i\left(\omega t - k_z z\right)\right)$; then, we fix space $\vec{r}$ and time t and average both sides over a λ_z-long interval centred on z. Within such an interval, the z derivative of E is approximately constant (due to SVEA (6.25)), so the first and second term at the left-hand side do not change, while the third and fourth vanish. The result is

$$\frac{dE}{dz}e\left(\vec{\rho}\right) - i\frac{k}{n_z+1}E\left(z\right)\left[n^2\left(\vec{\rho}\right) - 1\right]e\left(\vec{\rho}\right) = -\frac{2\eta_0}{n_z+1}\frac{1}{i\omega}\left\langle\frac{\partial J_x}{\partial t}e^{-i(\omega t - k_z\,\cdot)}\right\rangle \tag{6.31}$$

where brackets denote the averaging (it is defined as in note 15 at page 38, with substitution $\lambda \to \lambda_z$). Lastly, we extract the time derivative from the average and re-state equation (6.31) as

$$\frac{dE}{dz}e(\vec{\rho}) - i\frac{k}{n_z+1}E(z)\left[n^2(\vec{\rho}) - 1\right]e(\vec{\rho}) = -\frac{2\eta_0}{n_z+1}\left(1+\frac{1}{i\omega}\frac{\partial}{\partial t}\right)\left\langle J_x e^{-i(\omega t - k_z \cdot)}\right\rangle \tag{6.32}$$

Now we average the current.

Averaging the source current

As discussed just before expansion (6.11), approximation (3.26) is still valid, so an exact expression for the source current is still provided by formula (3.48). In a three-dimensional framework, the electron beam is not transversally uniform (in general), so transverse variations of the current are not only related to shot noise. Yet, neglecting the latter requires extension of the average in equation (6.32) to a three-dimensional volume V.[5]

The quantity to average is found from formula (3.49) by substituting $k \to k_z$. Now, by conjugating (6.11) we find

$$\cos k_u z_j(t) \cdot e^{-i(\omega t - k_z z_j(t))} = \frac{1}{2}e^{i\zeta_j(t)} f^*(\bar{z}_j(t))$$

where ζ_j is the phase of the j-th electron, as defined by (6.7), and the function f is defined by (6.12). By approximating $\bar{z}_j \approx z_j$ at the argument of f and substituting into (3.49) (where $k \to k_z$), we find

$$J_x(\vec{r}, t)\, e^{-i(\omega t - k_z z)} = \frac{ecK}{2\gamma_r}\left(\sum_j e^{i\zeta_j(t)}\delta(\vec{r} - \vec{r}_j(t))\right) f^*(z)$$

Lastly, f varies over a scale $\lambda_u >> \lambda_z$, so we approximate (see note 18 at page 39)

$$\left\langle J_x e^{-i(\omega t - k_z \cdot)}\right\rangle \approx \frac{ecK}{2\gamma_r} n_e(\vec{\rho})\, b(\vec{\rho}, z)\, f^*(z) \tag{6.33}$$

[5] In one-dimensional theory, a good choice for V is a λ-diameter and λ-long cylinder centred on $\vec{r}$ (see note 17 at page 39). As explained in note 3, the guided mode is expected to exhibit a longitudinal wavelength greater than λ. The same holds for any other direction; in particular, transverse variations are expected to evolve over a scale greater than λ. Thus, within the present context, a good choice for V is a λ-diameter and λ_z-long cylinder centred on $\vec{r}$.

As a major distinction with respect to one-dimensional theory, here we do not assume macroscopic quantities to be transversally uniform, so any average is a function of z, t *and* transverse position $\vec{\rho}$. This transverse dependence is the origin of optical guiding.

Here, electron density[6] n_e and bunching factor b are defined as in (3.51) and (3.52), with the major distinction that they both depend on transverse position $\vec{\rho}$ (see note 5). Yet, the system is still on steady state, so the electron density is a function of $\vec{\rho}$ and the bunching factor is a function of $\vec{\rho}$ and z:

$$n_e = n_e(\vec{\rho}), \qquad b = b(\vec{\rho}, z)$$

As in one-dimensional theory, electrons move about resonance and b varies in z over a scale much larger than λ_u. So, f only induces a ripple and, in formula (6.33), can be substituted by its mean value $[JJ]$ (as defined by (6.13)), thereby obtaining

$$\left\langle J_x e^{-i(\omega t - k_z \cdot)} \right\rangle \approx J_0 f_e(\vec{\rho})\, b(\vec{\rho}, z) \tag{6.34}$$

where we have defined

$$J_0 = \frac{ecK}{2}[JJ]\frac{n_{e0}}{\gamma_r} = \frac{ec}{\sqrt{2}}\frac{K}{\sqrt{1+K^2/2}}[JJ]\, n_{e0}\sqrt{\frac{\lambda}{\lambda_u}}$$

(as in one-dimensional theory, the last expression is found from (3.24)). Here, we have expressed the electron density as

$$n_e(\vec{\rho}) = n_{e0} f_e(\vec{\rho})$$

where the symbol n_{e0} denotes on-axis density and $f_e(0) = 1$. With respect to one-dimensional theory, the coefficient J_0 differs by $n \to n_{e0}$ and by the presence of n_z inside $[JJ]$ (but $n_z \approx 1$ – see discussion after (6.16)).

Projecting on transverse profile

By substituting formula (6.34) into equation (6.32), we find

$$\frac{dE}{dz}e(\vec{\rho}) - i\frac{k}{n_z+1}E(z)\left[n^2(\vec{\rho}) - 1\right]e(\vec{\rho}) = -\frac{2\eta_0}{n_z+1}J_0 f_e(\vec{\rho})\, b(\vec{\rho}, z) \tag{6.35}$$

Clearly, this equation cannot be satisfied[7]. This inconsistency is a consequence of our approximation for the field transverse profile: we *do not* know this profile, even for a single guided mode, and have *a priori* substituted it by the transverse profile of a waveguide mode. In general, the two profiles do not coincide and our approximate solution does not satisfy FEL equations.

[6]We use the symbol n_e in order to avoid confusion with the waveguide refractive index n.

[7]E.g., if we fix z we find a functional equality that, in general, cannot be satisfied (it should hold for any $\vec{\rho}$).

Yet, a consistent "weak" equation can be found by projecting (6.35) on the functional space generated by $e(\vec{\rho})$, i.e. by performing the functional scalar product between equation (6.35) and function $e(\vec{\rho})$. We define the scalar product between any couple of complex-scalar-valued functions $f(\vec{\rho})$ and $g(\vec{\rho})$ as

$$\langle f|g\rangle = \int\int f(\vec{\rho})\, g^*(\vec{\rho})\, d^2\vec{\rho}$$

(the double integral is extended over the whole transverse plane), and assume $e(\vec{\rho})$ to be normalized:

$$\langle e|e\rangle = 1$$

By projecting (6.35) on $e(\vec{\rho})$, we find[8]

$$\frac{dE}{dz} - i\Lambda \cdot E(z) = -\frac{2\eta_0}{n_z+1} J_0 \langle f_e b(\,\cdot, z)\,|e\rangle \tag{6.36}$$

where we have defined

$$\Lambda = \frac{k}{n_z+1}\left\langle \left[n^2-1\right] e|e\right\rangle$$

This is a novel version of the Field Equation (3.54), suitable for describing light evolution within a guided mode. It differs from its one-dimensional counterpart essentially by the second term at the left-hand side: once the bunching factor is known, the right-hand side is a known term, as it was in (3.54). As in one-dimensional theory, by coupling this equation to the novel Pendulum Equation (6.16) (applied to all electrons), a self-consistent system is obtained which describes fully the FEL evolution.

We conclude this chapter by commenting on initial conditions. They are set by assigning the seed as a function of time; within our assumption of a monochromatic optical field, this is done by assigning the seed complex envelope $E_s(\vec{\rho})$. In formula,

$$\underset{\sim}{\vec{E}}(z=0,\vec{\rho},t) = \Re\left\{E_s(\vec{\rho})\, e^{i\omega t}\right\}\hat{x}$$

By substituting expressions (6.3) and (6.4), we find

$$\Re\left\{E(0)\, e(\vec{\rho})\, e^{i\omega t}\right\}\hat{x} = \Re\left\{E_s(\vec{\rho})\, e^{i\omega t}\right\}\hat{x}$$

which is equivalent to

$$E(0)\, e(\vec{\rho}) = E_s(\vec{\rho}) \tag{6.37}$$

[8] The dot at the first argument of b is used to identify the variable over which the scalar product is performed.

Consistent with the derivation underlying equation (6.36), we project (6.37) on $e(\vec{\rho})$, thereby extracting the initial condition

$$E(0) = \langle E_s | e \rangle \tag{6.38}$$

Under proper assumptions for electron-beam properties at the entrance to the undulator, condition (6.38) will enable us to solve for FEL evolution.

Chapter 7

Integral Equation for a guided mode

7.1 Introduction

In this chapter, we apply the procedure described in chapter 4 to the Pendulum and Field Equations for a guided mode proposed in chapter 6. Under weak-field condition, equations (6.16) and (6.36) are linearized and then combined to obtain an extension to the Integral Equation (4.15), suitable for describing light evolution within a guided mode.

The chapter ends with an appendix on a specific case. Namely, we consider a transversally gaussian electron beam and approximate the optical beam on the basis of a transversally gaussian waveguide mode. This description is valid when the optical beam is dominated by the fundamental guided mode.

7.2 Bunching factor

The reference geometrical framework is substantially the same as in one-dimensional theory: so, we still refer to figure 4.1. We choose a spatial position $\vec{r} = (\vec{\rho}, z)$ and evaluate the bunching factor (3.52) at $\vec{r}$. This involves an average over a λ_z-long volume V centred on $\vec{r}$ (see note 5 at page 5), which is still identified in the figure by the thick lines. The average is on electrons in V at a given temporal instant t, so now we determine their phase.

Due to small FEL efficiency, electron longitudinal velocities are close to the "mean initial velocity" v given by formula (4.1). We apply the moving-volume technique detailed in section (4.2): for any electron in V, we approximate

$$z\left(t-t'\right)\approx z-vt' \tag{7.1}$$

when this quantity appears as an argument for the field envelope.

As in one-dimensional theory, we are concerned with weak-field regime. With reference to the notation developed in section 4.2.1 (see expression (4.3)), we expand $\delta\zeta\left(t\right)$ to first order in $E\left(z\right)$ by approximating $\zeta\approx\zeta_f$ at the right-hand side of the Pendulum Equation (6.16), which thereby reduces to its linearized form

$$\dot{\nu}\left(t\right)=\Omega^2\cdot\Re\left\{E\left(z\left(t\right)\right)e\left(\vec{\rho}\right)e^{-i\zeta_f(t)}\right\} \tag{7.2}$$

Since $\vec{\rho}$ plays merely the role of a parameter, this equation can be integrated by means of the technique developed in section 4.2.2. By substituting t by $t-t'$ and using approximation (7.1), formula (7.2) yields

$$\dot{\nu}\left(t-t'\right)=\Omega^2\cdot\Re\left\{e\left(\vec{\rho}\right)E\left(z-vt'\right)e^{-i\zeta_f(t-t')}\right\} \tag{7.3}$$

Formula (7.3) is identical to formula (4.5), except for the complex coefficient $e\left(\vec{\rho}\right)$; yet, the latter can be thought of as being part of the complex envelope E. Thus, a straightforward application of the technique of section 4.2.2 yields

$$\delta\zeta\left(t\right)=\Omega^2\cdot\Re\left\{e^{-i\zeta_0}e^{-i\nu_0 z/v}e\left(\vec{\rho}\right)\int_0^{z/v}E\left(z-vt'\right)e^{i\nu_0 t'}t'dt'\right\} \tag{7.4}$$

The free evolution ζ_f is still given by formula (4.9).

Now we evaluate the bunching factor (3.52) by averaging over electrons in V. As in one-dimensional theory, we assume the system to be on steady state and the electron beam to be free from phase-energy correlation and pre-bunching. By following the procedure detailed in section 4.3, we find

$$b\left(\vec{\rho},z\right)=i\frac{\Omega^2}{2}\left[\int_0^{z/v}E\left(z-vt'\right)F\left(t'\right)t'dt'\right]e\left(\vec{\rho}\right) \tag{7.5}$$

where the function $F\left(t'\right)$ is defined by (4.14).

7.3 Integral Equation

At this point we collect our previous results and derive an extension for the FEL Integral Equation, suitable for describing the evolution of a guided mode. First of all, from formula (7.5) we find

$$\left\langle f_e b\left(\,\cdot\,,z\right)|e\right\rangle=i\frac{\Omega^2}{2}\left[\int_0^{z/v}E\left(z-vt'\right)F\left(t'\right)t'dt'\right]\left\langle f_e e|e\right\rangle$$

Now, by substituting this formula into the Field Equation (6.36), we get

$$\frac{dE}{dz} = -iA \cdot \int_0^{z/v} E\left(z - vt'\right) F\left(t'\right) t' dt' + i\Lambda \cdot E\left(z\right) \tag{7.6}$$

where we have defined

$$A = \frac{\eta_0}{n_z + 1} J_0 \Omega^2 \left\langle f_e e | e \right\rangle$$

Equation (7.6) extends the FEL Integral Equation (4.15) to a guided mode. As for its one-dimensional counterpart, the only unknown is the electric-field complex envelope $E(z)$, and the solution is uniquely determined by its initial value $E(0)$ (which is set by condition (6.38)).

Now, as in section 4.4, we solve equation (7.6) for a cold beam. By dropping the ensemble average at the right-hand side of definition (4.14), the Integral Equation (7.6) reduces to

$$\frac{dE}{dz} = -iA \cdot \int_0^{z/v} E\left(z - vt'\right) e^{i\nu_0 t'} t' dt' + i\Lambda \cdot E\left(z\right) \tag{7.7}$$

This integro-differential equation only differs from its one-dimensional counterpart (4.16) by a term proportional to the unknown, and can be solved by means of the mathematical procedure detailed in section 4.4. We introduce the auxiliary function (4.18) and express equation (7.7) in operatorial form. A straightforward generalization of one-dimensional procedure leads to equation

$$Df + i\left(\frac{\nu_0}{v} - \Lambda\right) f + i\frac{A}{v^2} D^{-2} f = 0 \tag{7.8}$$

which is fully equivalent to the Integral Equation (7.7). Now we apply D^2 to (7.8), which is thereby turned into

$$D^3 f + i\left(\frac{\nu_0}{v} - \Lambda\right) D^2 f + i\frac{A}{v^2} f = 0 \tag{7.9}$$

This differential equation can be solved by standard techniques. Before doing this, we set initial conditions. Equation (7.8) yields

$$Df = -i\left(\frac{\nu_0}{v} - \Lambda\right) f - i\frac{A}{v^2} D^{-2} f \tag{7.10}$$

$$D^2 f = -i\left(\frac{\nu_0}{v} - \Lambda\right) Df - i\frac{A}{v^2} D^{-1} f \tag{7.11}$$

By imposing $z = 0$ in (4.18), (7.10) and (7.11), we find

$$f\left(0\right) = E_0 \tag{7.12}$$

$$f'(0) = -i\left(\frac{\nu_0}{v} - \Lambda\right)E_0 \tag{7.13}$$

$$f''(0) = -\left(\frac{\nu_0}{v} - \Lambda\right)^2 E_0 \tag{7.14}$$

where $E_0 = E(0)$. Now we solve differential equation (7.9) with initial conditions (7.12), (7.13) and (7.14). Let λ_1, λ_2, λ_3 be the roots to the characteristic polynomial

$$p(X) = X^3 + i\left(\frac{\nu_0}{v} - \Lambda\right)X^2 + i\frac{A}{v^2} \tag{7.15}$$

Then, the solution is

$$f(z) = c_1 e^{\lambda_1 z} + c_2 e^{\lambda_2 z} + c_3 e^{\lambda_3 z} \tag{7.16}$$

where the complex coefficients c_1, c_2, c_3 are determined by imposing (7.16) to satisfy (7.12), (7.13) and (7.14). The field envelope E is given by formula (4.22).

As in one-dimensional theory, we solve explicitly on resonance ($\nu_0 = 0$). Polynomial (7.15) reduces to

$$p(X) = X^3 - i\Lambda X^2 + i\frac{A}{v^2}$$

Roots are given explicitly by Cardano's formulae.[1] Yet, such expressions do not provide any significant physical insight, and numerical computation is more easily performed by dealing directly with the polynomial. So, we assume λ_1, λ_2, λ_3 to be known.[2] Due to the linear structure of initial conditions (7.12), (7.13) and (7.14), the coefficients in formula (7.16) are proportional to E_0. Thus, we solve for the case $E_0 = 1$ and multiply the solution by E_0. By imposing (7.16) to satisfy (7.12), (7.13) and (7.14), we find the linear algebraic system

$$\begin{cases} c_1 + c_2 + c_3 = 1 \\ \lambda_1 c_1 + \lambda_2 c_2 + \lambda_3 c_3 = i\Lambda \\ \lambda_1^2 c_1 + \lambda_2^2 c_2 + \lambda_3^2 c_3 = -\Lambda^2 \end{cases}$$

which is easily solved numerically.[3] Within the case study we consider in chapter 8, the solution exhibits the same behaviour as in one-dimensional

[1] Within the case study we consider in chapter 8, the order of magnitude of A/v^2 and Λ is the same, so the term in X^2 cannot be neglected and roots cannot be approximated by simple formulae.

[2] E.g., they may be estimated via MATLAB computations.

[3] Again, e.g., via MATLAB.

theory: λ_1 yields a growing exponential, λ_2 a decaying one and λ_3 an oscillatory one. Then, if z is large enough the first exponential dominates over the other two (high-gain regime) and

$$E(z) \approx c_1 E_0 e^{az} e^{ibz} \tag{7.17}$$

where a and b are the real and imaginary part of λ_1.

7.4 Gaussian approximation

So far, we have been dealing with an electron beam having a transverse profile shaped according to an unspecified function $f_e(\vec{\rho})$. We have expressed the optical beam on the basis of a mode in a dielectric waveguide; the index of refraction in the waveguide is shaped as $n(\vec{\rho})$. Now we sample this technique by considering a transversally gaussian electron beam. The transverse profile of the optical beam is approximately gaussian [39], so we set $n(\vec{\rho})$ in such a way that the fundamental mode exhibit a gaussian transverse profile.

The electron density is given by

$$n_e(\vec{\rho}) = n_{e0} f_e(\vec{\rho}), \qquad f_e(\vec{\rho}) = \exp\left(-\frac{\rho^2}{2\sigma_e^2}\right)$$

where σ_e is the r.m.s. radius of the electron beam (note that f_e only depends on $\rho = |\vec{\rho}|$, so the beam is circularly simmetric). Modes in the waveguide are determined by equation (6.2), which involves transverse profile $e(\vec{\rho})$ and longitudinal index of refraction n_z. We consider a circularly simmetric waveguide, which allows circularly simmetric modes to exist. The fundamental mode is gaussian if

$$n^2(\vec{\rho}) = n_0^2 - \frac{1}{k^2\sigma^4}\rho^2$$

where n_0 is the on-axis refractive index and σ turns out to be the r.m.s. radius of the fundamental mode [41, 42]. In fact, this mode is characterized by

$$e(\vec{\rho}) = \frac{1}{\sqrt{\pi}\sigma} \exp\left(-\frac{\rho^2}{2\sigma^2}\right), \qquad n_z = n_0\sqrt{1 - \frac{2}{k^2 n_0^2 \sigma^2}} \tag{7.18}$$

The on-axis refractive index n_0 and the r.m.s. radius σ are free parameters; they are set properly in order to minimize the discrepancy between virtual-dielectric-waveguide model and physical reality.

Having thus sketched out the situation, now we proceed with computation of the coefficients involved in equation (7.6). Namely:

- the current I, i.e. the electron-beam current passing through any transverse plane. Evaluation of this current is mandatory to derive n_{e0} from I, which is measurable in a physical system;
- the coefficient A;
- the coefficient Λ;
- the initial value E_0.

To this purpose, it is useful keeping in mind that

$$\int\!\!\int e^{-a\rho^2} d^2\vec{\rho} = \frac{\pi}{a} \tag{7.19}$$

(integration extended over the whole transverse plane), a being any fixed number.

1. Current I.
   We have

   $$I = evn_{e0} \int\!\!\int f_e\left(\vec{\rho}\right) d^2\vec{\rho}$$

   where

   $$f_e\left(\vec{\rho}\right) = e^{-a\rho^2}, \qquad a = \frac{1}{2\sigma_e^2}$$

   From formula (7.19), we find

   $$\int\!\!\int f_e\left(\vec{\rho}\right) d^2\vec{\rho} = 2\pi\sigma_e^2$$

   so

   $$I = 2\pi evn_{e0}\sigma_e^2$$

   and

   $$n_{e0} = \frac{I}{2\pi ev\sigma_e^2} \tag{7.20}$$

2. Coefficient A.
   There is a scalar product to perform. By making use of formula (7.19), we get

$$\langle f_e e|e\rangle = \frac{1}{\pi\sigma^2}\int\!\!\int \exp\left(-\left(\frac{1}{2\sigma_e^2}+\frac{1}{\sigma^2}\right)\rho^2\right) d^2\vec{\rho} = \frac{1}{1+\sigma^2/2\sigma_e^2}$$

   Then, the coefficient is

$$A = \frac{\eta_0}{n_z+1}\frac{J_0\Omega^2}{1+\sigma^2/2\sigma_e^2}$$

3. Coefficient Λ.
   Again, there is a scalar products to perform. We have

$$\left\langle \left[n^2-1\right] e|e\right\rangle = \left(n_0^2-1\right)\langle e|e\rangle - \frac{1}{k^2\sigma^4}\left\langle \rho^2 e|e\right\rangle$$

   Now, it is cumbersome but easy to show that[4]

$$\left\langle \rho^2 e|e\right\rangle = \sigma^2\langle e|e\rangle$$

   and, as a consequence,

$$\left\langle \left[n^2-1\right] e|e\right\rangle = \left(n_0^2-1\right) - \frac{1}{k^2\sigma^2}$$

   Then, the coefficient is

$$\Lambda = \frac{k}{n_z+1}\left[\left(n_0^2-1\right)-\frac{1}{k^2\sigma^2}\right]$$

4. Initial value E_0.
   One more scalar product. We consider a transversally gaussian seed: its complex envelope is

$$E_s\left(\vec{\rho}\right) = E_{s0}\exp\left(-\frac{\rho^2}{2\sigma_s^2}\right)$$

[4] The following formula is obtained through an integration by parts according to variable ρ^2.

where E_{s0} is the field on axis and σ_s is the r.m.s. radius. From formulae (6.38) and (7.19), we find[5]

$$E_0 = \langle E_s | e \rangle = \frac{E_{s0}}{\sqrt{\pi}\sigma} \int\int \exp\left(-\frac{1}{2}\left(\frac{1}{\sigma_s^2} + \frac{1}{\sigma^2}\right)\rho^2\right) d^2\vec{\rho} =$$

$$= 2\left(\frac{\sigma_s}{\sigma} + \frac{\sigma}{\sigma_s}\right)^{-1} \cdot \sqrt{\pi}\sigma_s E_{s0} \tag{7.21}$$

We conclude this chapter by proposing a way to estimate the on-axis index of refraction n_0. In one-dimensional theory, the optical field is given by

$$\underset{\sim}{\vec{E}}(z,t) = \Re\left\{E(z)\, e^{i(\omega t - kz)}\right\}\hat{x}$$

where the envelope E has the form $\exp(az) \cdot \exp(ibz)$. Thus, the field is

$$\underset{\sim}{\vec{E}}(z,t) \sim e^{az} \cos\left[\omega t - (k-b)\, z\right]\hat{x}$$

which represents a wave having $k - b$ as a wavenumber. Now, we estimate the longitudinal wavenumber of the guided mode to be approximately the same, and set waveguide parameters such that $k_z \approx k - b$. Around the z axis, the waveguide is approximately homogeneous, so its fundamental mode is approximately a uniform plane wave having $n_0 k$ as a wavenumber. In conclusion, we set

$$n_0 = 1 - \frac{b}{k}$$

[5] In the last formula, we highlight the contributions due to seed power and overlap between seed and guided mode. The seed power is (approximately: actually, the wave impedance differs slightly from η_0)

$$P_s = \frac{\langle E_s | E_s \rangle}{2\eta_0} = \frac{\pi\sigma_s^2 \left|E_{s0}\right|^2}{2\eta_0}$$

and the power in the guided mode is

$$P_0 = \frac{\left|E_0\right|^2}{2\eta_0} = 4\left(\frac{\sigma_s}{\sigma} + \frac{\sigma}{\sigma_s}\right)^{-2} \cdot \frac{\pi\sigma_s^2 \left|E_{s0}\right|^2}{2\eta_0}$$

so

$$\frac{P_0}{P_s} = 4\left(\frac{\sigma_s}{\sigma} + \frac{\sigma}{\sigma_s}\right)^{-2}$$

which is the square of the first factor in formula (7.21).

Since the value of b is well known from one-dimensional theory, this formula provides a straightforward way to estimate n_0, and the only free parameter in the virtual-dielectric-waveguide model is the mode r.m.s. radius σ. At present, the problem of estimating σ is still an open issue.

Chapter 8

Numerical results and conclusions

8.1 Introduction

In this chapter, we apply the technique developed in chapters 6 and 7 to a specific case. The FEL setup extends the one considered in section 5.6 by including transverse variations of electron and optical beams; however, the system is on steady state. The problem is first addressed on simulative basis by means of the code GENESIS 1.3 [44]; this allows setting parameters properly in order to obtain a linear, strong-guiding system, suitable for an analysis based on our theory. System parameters and simulative results on the transverse profile of the optical beam are presented in section 8.2. Next, we build a proper (virtual) waveguide and address the problem by our theory. Section 8.3 presents waveguide parameters and theoretical results on optical-power growth, along with a comparison to simulation.

We end this work by section 8.4, where we summarize positive and negative aspects of our approach to both frequency pulling and optical guiding, and propose further developements.

8.2 Optical guiding in FERMI@Elettra

We consider the same FEL amplifier used in section 5.6 to test the theory of frequency pulling. For the sake of clarity, we report the complete setup.

The FERMI FEL in direct-seeding configuration is modelled by means of the setup described in section 1.2. The real radiator is composed by 6 modules of 44 magnetic periods; we neglect drift sections and model the radiator by a single undulator having 264 periods (this results in an undu-

Parameter	Symbol	Value
Undulator period	λ_u	5.5 cm
Number of undulator periods	N_u	264
Electron-beam r.m.s. radius	σ_e	100 μm
Electron energy[1]	γ	1.2 GeV
Current	I	1.5 kA
Optical wavelength	λ	60 nm
Seed r.m.s. radius	σ_s	300 μm
Seed power	P_s	1 kW

Table 8.1: System parameters.

lator length L_u of about 15 m). Thus, our ideal setup does not allow for a strong-focusing lattice. Magnetic poles exhibit an ideal planar surface, so the magnetostatic field is linearly polarized and does not provide any horizontal focusing; vertical focusing is neglected.

A continuous, parallel, cold electron beam at 1.2 GeV enters the undulator and co-propagates with an optical beam at 60 nm; the latter originates from a monochromatic seed. The system is on steady state. We model the transverse profiles of electron density and seed field by Gaussians, symmetrical with respect to the z axis.

Table 8.1 reports all parameters in our model. In the FERMI FEL, the nominal electron-beam current is equal to 750 A [1]; GENESIS 1.3 simulations show that the resulting gain is not sufficient for the fundamental guided mode to dominate the optical beam significantly over the undulator length. Conversely, our choice $I = 1.5$ kA yields a strong optical guiding. Now, a seed at 60 nm is obtained through HHG; the available seed power is then on the order of 1 MW [6]. However, such a power causes the system to saturate well before the undulator end, thereby compromising optical guiding. Thus, we use the very low power $P_s = 1$ kW, which shifts the onset of saturation well beyond 10 m (2/3 of the undulator length). Lastly, a narrow seed (similar to the electron beam) exhibits strong diffraction in the first undulator meters; then, optical guiding occurs. As a consequence, the optical-beam size undergoes wide oscillations. Our choice $\sigma_s = 300\mu$m yields modest diffraction, along with a reasonable overlap to the electron beam. Not reported in table 8.1 is the undulator parameter K, which is set according to formula (3.25).[2]

[1]Symbol γ denotes electron relativistic factor; the energy is γmc^2.

[2]As explained in section 6.3.1 (see the discussion on formula (6.10)), the value for K which sets the system on resonance is estimated accurately by one-dimensional theory.

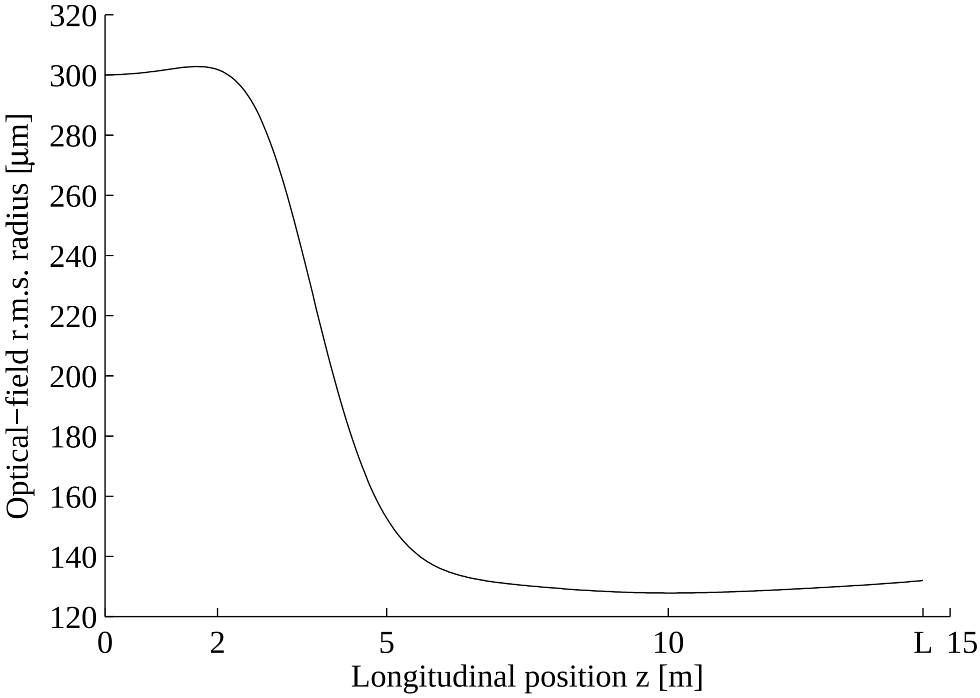

Figure 8.1: Optical-field r.m.s. radius (GENESIS 1.3 simulation).

Before comparing systematically our theory to GENESIS 1.3 simulations, we present some simulative results; they show parameter values in table 8.1 to be an optimal setting for optical guiding, and yield an estimate for the size of the fundamental guided mode. Figure 8.1 shows the r.m.s. radius of the optical field as a function of longitudinal position z inside the undulator. In the first couple of meters, the seed is scarcely affected by the electron beam, which is as yet unbunched; thus, the optical beam exhibits a modest diffraction. Then, optical guiding takes over, and light is confined quickly within the core of the electron beam. A stable configuration is evident over a wide range (a few meters) around $z = 10$ m; there, the optical beam is dominated by the fundamental guided mode. Diffraction is at play again towards the undulator end: a modest saturation lowers the gain (see section 8.3), and optical guiding becomes less effective. An estimate for the r.m.s. radius of the fundamental guided mode is obtained by reading the plot at $z = 10$ m; we find a value of about 130 μm, slightly greater than the r.m.s. radius of the electron beam.

In consideration of these results, beyond $z = 5$ m (1/3 of the undulator length) we expect the optical beam to be well described by a single guided mode. According to reference [39], the transverse profile is expected to be

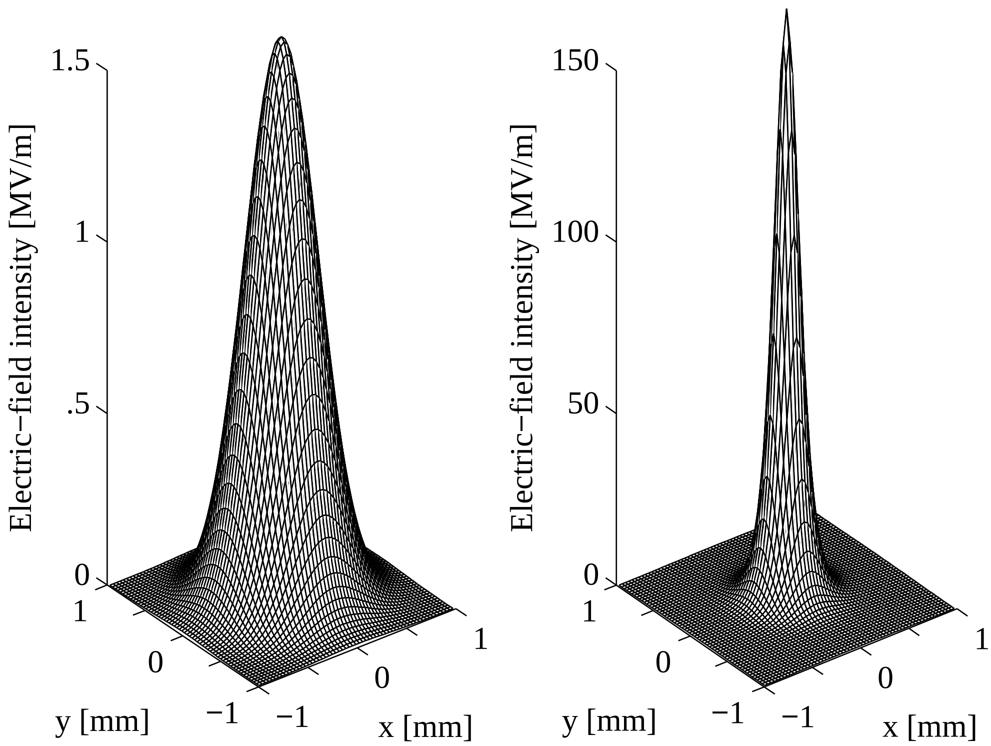

Figure 8.2: Seed (left) and optical-beam ($z = 10$ m, right) transverse profiles.

approximately gaussian; on the basis of previous considerations, we estimate an r.m.s. radius of about 130 μm. Now, we use these results to build up a proper virtual dielectric waveguide, and express the optical beam on the basis of the fundamental waveguide mode, as explained in chapter 6. Before doing this, a look at simulated transverse profile is worth, in order to establish whether a Gaussian is actually a close description. First of all, we give an idea on light guiding due to FEL interaction by comparing the transverse profiles of seed and optical beam. Figure 8.2 presents three-dimensional plots of electric-field modulus[3], taken at $z = 0$ (seed) and $z = 10$ m (dominant guided mode). The guided mode is much more narrow (and intense) than the seed, a clear evidence of optical guiding (along with high gain).

From a qualitative point of view, both plots resemble Gaussians. Actually, such is the seed. The guided mode exhibits a slightly sharper peak; yet, its overall shape encourages a gaussian approximation. A quantitative anal-

[3] The monochromatic field is here represented by a phasor (i.e., the field at a given spatial position is expressed as a complex vector – the phasor – modulating a sinusoidal function of time at the field frequency). Figures report the phasor modulus, which gives the amplitude of field time oscillations and is related to power density.

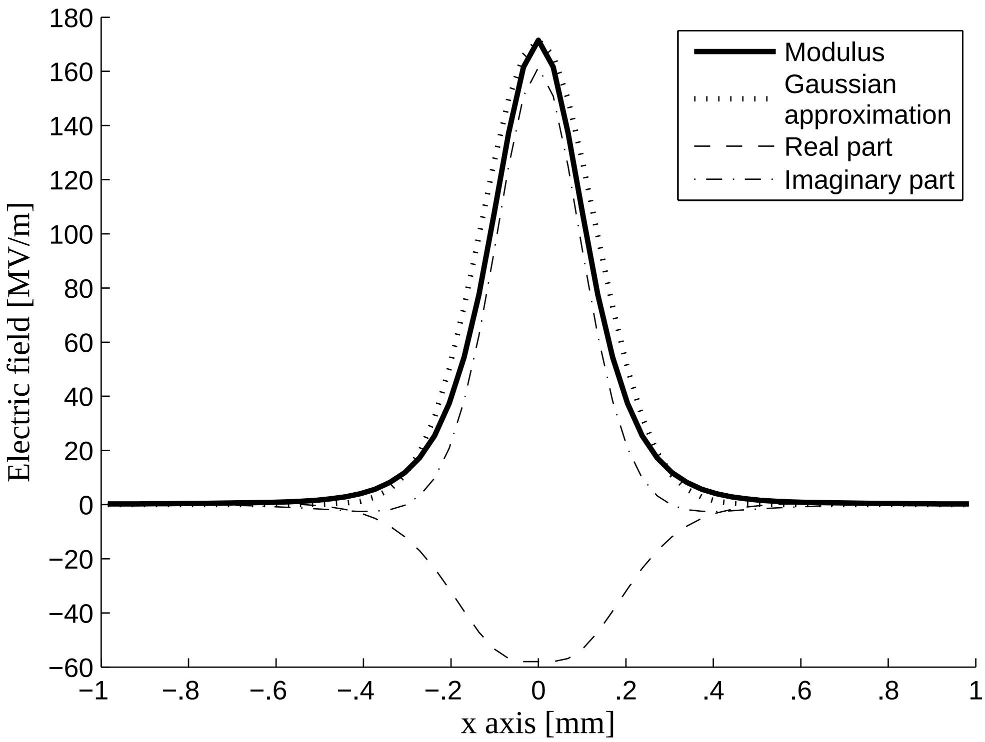

Figure 8.3: Electric-field ($z = 10$ m) modulus, real and imaginary part, and gaussian approximation.

ysis is best performed by means of a two-dimensional plot: axial symmetry allows representing the guided-mode transverse profile just along a single direction. Figure 8.3 shows the electric-field modulus at $z = 10$ m along the x axis; superimposed are the real and imaginary part of the field. A Gaussian with an r.m.s. width of 130 μm and a peak value equal to the one of the field modulus is also reported. Clearly, the Gaussian reproduces closely the field modulus; however, it does not take into account phase variations. In section 8.3, we build up a virtual dielectric waveguide such that its fundamental mode exhibit a gaussian transverse profile with r.m.s. radius of 130 μm (for the field), and express the optical beam on the basis of such a mode. As we shall see, neglecting the phase causes a significant numerical error.

8.3 Single-gaussian-mode approximation

This section is devoted to a comparison between our theory and GENESIS 1.3 simulations. First of all, we relate optical power P to electric field. The former is intended as the power flowing through a given transverse plane; as such, it is a function of z. We define the *differential power gain*

$$g(z) = \frac{1}{P(z)}\frac{dP}{dz}$$

Within one-dimensional theory, the field is, in principle, transversally uniform, so the optical beam carries an infinite power; this model is applied to a real system by assuming light to be confined within a cylinder.[4] Due to transverse uniformity, g can be evaluated as

$$g(z) = \frac{1}{p(z)}\frac{dp}{dz}$$

where p denotes power density.[5] Now, the (time-averaged) power density carried by a (longitudinally propagating) wave is given by the squared modulus of the (transverse) electric field (intended as a phasor – see note 3), halved and divided by the wave impedance, which we approximate by free-space intrinsic impedance η_0. Within our one-dimensional optical beam, the phasor modulus is simply $|E|$ ($E(z)$ is the electric-field complex envelope – see expression (3.17)), so

$$g(z) = \frac{1}{|E(z)|^2}\frac{d|E|^2}{dz} \tag{8.1}$$

Except for the very first part of the undulator, the system is on high-gain regime, so $|E(z)| \sim e^{az}$ (see formula (4.33)). As a consequence, g is independent on z and $P(z) \sim e^{gz}$. We define the *gain length*

$$L_g = \frac{1}{g} \tag{8.2}$$

which represents the undulator length over which power grows by a factor e (Neper number).

Now we approximate the optical beam by a single guided mode with gaussian transverse profile, as explained in section 7.4. The technique is called single-gaussian-mode (SGM) approximation. The field (electric-field phasor) modulus is given by $|E| \cdot |e(\vec{\rho})|$ (see expression (6.4)), so

$$p(z, \vec{\rho}) = \frac{|E(z)|^2}{2\eta_0} \cdot |e(\vec{\rho})|^2$$

[4]We assume full overlap between light and electron beam, so all light undergoes FEL interaction.

[5]The *power density* is defined as the power flowing through a unit area. It corresponds to the longitudinal component of the Poynting vector.

The optical power is obtained by integrating p over the transverse plane. Since $e(\vec{\rho})$ is normalized ($\langle e|e\rangle = 1$), the result is

$$P(z) = \frac{|E(z)|^2}{2\eta_0} \tag{8.3}$$

As a consequence, formula (8.1) is still valid. Now, we still have $|E(z)| \sim e^{az}$ (see formula (7.17)), so g is independent on z, $P(z) \sim e^{gz}$ and definition (8.2) for the gain length is still valid.

At this point we are ready to present some numerical results. Electrons are ultrarelativistic, so we approximate $v \approx c$ (v is the "mean" electron velocity defined by formula (4.1)). The on-axis electron density n_{e0} is found from formula (7.20). Within one-dimensional theory, we set an electron density $n = n_{e0}$. From formula (4.33), we find a gain length

$$L_g\ (1\mathrm{D}) = 73\ \mathrm{cm}$$

Now, we set parameters for the virtual dielectric waveguide. The on-axis index of refraction n_0 is chosen according to the recipe proposed at the end of section 7.4; the result is

$$n_0 - 1 = 3.8 \cdot 10^{-9}$$

The r.m.s. radius of the fundamental waveguide mode is set as

$$\sigma = 130\ \mu\mathrm{m}$$

according to the results of GENESIS 1.3 simulations presented in section 8.2. From formula (7.18), we find a longitudinal index of refraction n_z given by

$$n_z - 1 = -9.2 \cdot 10^{-9}$$

This extremely small value confirms the prediction of note 3 at page 3. Also, ratio $\lambda/\lambda_u = 1.1 \cdot 10^{-6}$ is much greater than $n_z - 1$, which confirms the validity of one-dimensional resonance formula (3.24) (see the discussion on formula (6.10)). In all numerical computations, we approximate $n_z \approx 1$. Lastly, from formula (7.17) we find a gain length

$$L_g\ (\mathrm{SGM}) = 90\ \mathrm{cm}$$

These results are now compared to simulation. We refer to figure 8.4, where the simulated differential gain g is plotted as a function of longitudinal position z inside the undulator. A sequence of various regimes is evident; a detailed analysis confirms what we have deduced from figure 8.1. Namely, in the first couple of meters the electron beam is as yet unbunched; thus, g

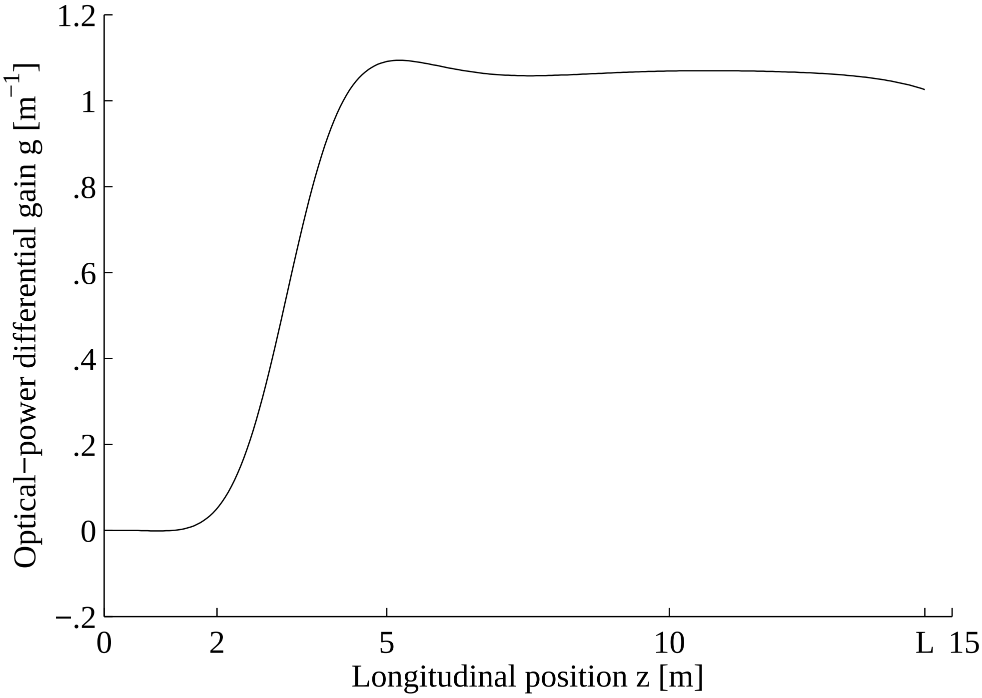

Figure 8.4: Optical-power differential gain (GENESIS 1.3 simulation).

is negligible and the optical beam only diffracts. Then, guided modes undergo exponential gain; lower-order modes exhibit higher growth rates and dominate gradually the optical beam: this results in a progressive reduction of r.m.s. radius and increase of g. Beyond $z = 5$ m, the fundamental guided mode dominates fully the optical beam; radius and gain reach a stable level. A modest saturation is evident near the undulator end: there, g exhibits a slight reduction and, as a consequence, a less effective optical guiding lets diffraction increase slightly the radius again. An estimate for differential gain and gain length of the fundamental guided mode is obtained by reading the plot at $z = 10$ m; we find

$$L_g \text{ (sim)} = 94 \text{ cm}$$

A comparison to previous results yields

$$L_g \text{ (1D)} << L_g \text{ (sim)} \approx L_g \text{ (SGM)}$$

which confirms the importance of optical guiding and the validity of our approach.

The accuracy in predicting the gain length demonstrated by SGM approximation encourages further analysis. Besides the gain length, formula (7.17)

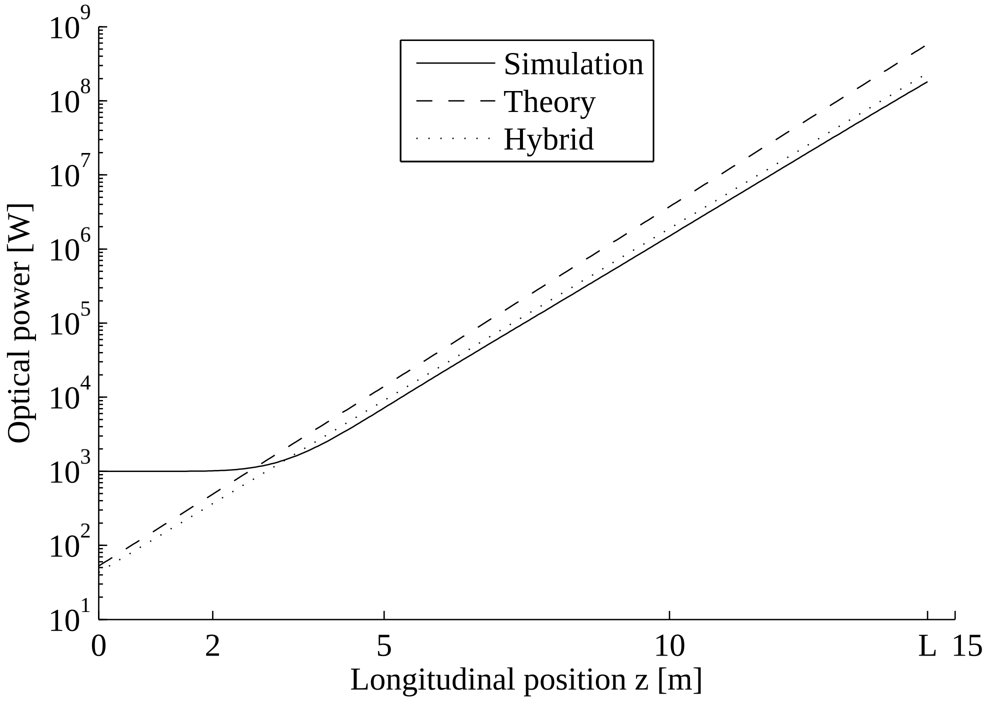

Figure 8.5: Optical-power evolution (vertical axis in log scale).

(in combination with (8.3)) yields optical-power evolution along the whole undulator length. Figure 8.5 shows $P(z)$ as predicted from SGM approximation, along with the simulative result. Consistent with the differential gain g reported in figure 8.4, initially the simulated power sticks to seed power. Growth starts around $z = 3$ m, and exponential gain is evident beyond $z = 5$ m; saturation is unappreciable. Formula (7.17) predicts an exponential gain, so SGM approximation yields a straight line: the latter describes the evolution of the power carried by the fundamental guided mode. Initially, the optical beam is dominated by diffractive contributions, so the contribution of this mode to optical power is not evident. Consistent with simulative result, the fundamental guided mode overcomes the seed at about $z = 3$ m; from there on, SGM prediction and simulative result are quite close. Besides a slightly different slope, which is due to the approximation in the SGM gain length, the two plots exhibit a relative vertical shift. The latter comes from approximations on initial value in formula (7.17). In order to highlight all error contributions, we consider a function proportional to $\exp(gz)$, with g set to its simulative value at $z = 10$ m. This cancels slope errors. Now, the initial value in formula (7.17) is $c_1 E_0$. Thus, the power P_0 carried by the fundamental guided mode at $z = 0$ is equal to the seed power P_s multiplied by two factors. The first is $|E_0|^2/2\eta_0 P_s$; it accounts for the fact that only a fraction of P_s is carried by the fundamental guided mode. The second is $|c_1|^2$; it accounts for what in one-dimensional FEL theory is usally called "launch loss". Within our SGM approximation, each guided mode behaves as the whole optical beam in one-dimensional theory: it is composed by three "longitudinal" modes (the three exponentials in formula (7.16)). Each of them carries a fraction of guided-mode power; one of them undergoes exponential growth and eventually dominates the guided mode. As a consequence, only a fraction of guided-mode power is amplified. In order to split the two factors in P_0/P_s, we evaluate numerically the fraction of P_s carried by the fundamental guided mode, i.e. the initial value E_0. As stated by formula (6.38), E_0 involves a scalar product between seed and guided mode; the transverse profile of the fundamental guided mode exhibits phase variations (see figure 8.3), on which the result depends. Within SGM approximation, phase is neglected, so E_0 is affected by errors. Now we evaluate the scalar product from simulated transverse profiles; launch loss is still given by theory. The resulting optical-power evolution is reported in figure 8.5. An improvement to our SGM approximation may be obtained by introducing a phase on waveguide-mode transverse profile $e(\vec{\rho})$, but such an extension is beyond the scope of the present work.

8.4 Conclusions

The main concepts underlying the FEL physical phenomenon may be fully understood by means of a one-dimensional, steady-state classical theory; such a theory allows defining the most relevant parameters of the system and provides a prediction for many quantities of practical interest, such as the optical-power gain length. However, real FELs exhibit time-dependent and three-dimensional phenomena as well. As pointed out in section 1.3.2, time-dependent phenomena result in a gain reduction (i.e., a gain-length increase); besides this, the optical-pulse evolution is significantly modified with respect to what is found from steady-state theory. The phenomenon may induce an useful FEL tunability. Three-dimensional phenomena share with time-dependent ones a considerable gain degradation, which is due to diffraction of light and the consequent optical-power loss and reduced coupling to the electron beam. Yet, as pointed out in section 1.3.3, optical guiding may occur, which reduces diffraction and power loss, thereby counteracting degradation of electron-light coupling and gain.

In the present work, we have shown that both tunability and optical guiding in FEL amplifiers can be taken into account in a relatively simple way. If the seed pulse is short enough, its frequency is shifted by a mechanism which shares many similarities with frequency pulling in conventional lasers. The phenomenon is known by the same name and is ruled by a similar law, as we have proved in chapter 5. On the other hand, the main consequences of optical guiding are deduced from a simple (virtual) waveguide model. Only the dominant guided mode is taken into account, and its transverse profile is approximated by a shape which is set a priori; yet, the model yields and accurate prediction for optical-power evolution on high-gain regime.

Our work opens the way to further developements. In chapter 5, frequency pulling is studied within a high-gain amplifier with an ideal electron beam. Useful extensions would be obtained by considering a lower gain and a warm beam, as well as other FEL configurations, such as the HGHG and CHG FELs considered in references [33, 34]. As far as the waveguide model of chapter 6 is concerned, relevant improvements are related to the introduction of a phase on waveguide-mode transverse profile and to finding a way to estimate the r.m.s. radius of the guided mode. Also, it may be seen whether the model works in warm-beam and non-linear cases. Lastly, a time-dependent extension would allow studying frequency pulling within a three-dimensional framework. We remark that warm-beam, non-linear and time-dependent extensions are not feasible within the approach of references [41, 42], since these rely on a cold linear plasma model within a monochromatic e.m. field.

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