INVARIANT MANIFOLDS FOR SINGULARLY PERTURBED PARABOLIC EQUATIONS (*)

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SOMMARIO. - *Consideriamo un sistema in cui un'equazione differenziale è accoppiata a un'equazione di evoluzione singolarmente perturbata:*
$$\begin{cases} \dot{x} = f(t,x,y,\epsilon) \\ \epsilon\dot{y} = A(t,x)y + g(t,x,y,\epsilon) \end{cases}.$$
Dimostreremo che, per ϵ piccolo, il sistema ammette una varietà invariante regolare $C_\epsilon = \{(t,x,y) | y = k(t,x,\epsilon)\}$ e che l'equazione ridotta $\dot{x} = f(t,x,k(t,x,\epsilon),\epsilon)$ è C^r vicina all'"equazione limite" $\dot{x} = f(t,x,0,0)$. Daremo anche una descrizione qualitativa della dinamica vicino alla varietà invariante C_ϵ.

SUMMARY. - *We consider a system in which a differential equation is coupled with a singularly perturbed semilinear evolution equation, namely*
$$\begin{cases} \dot{x} = f(t,x,y,\epsilon) \\ \epsilon\dot{y} = A(t,x)y + g(t,x,y,\epsilon) \end{cases}.$$
We will prove that, for small ϵ, the system admits a smooth invariant manifold $C_\epsilon = \{(t,x,y) | y = k(t,x,\epsilon)\}$ and that the reduced equation $\dot{x} = f(t,x,k(t,x,\epsilon),\epsilon)$ is C^r near to the "limit equation" $\dot{x} = f(t,x,0,0)$. We will also give a qualitative description of the dynamics near the invariant manifold C_ϵ.

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Introduction.

Let us consider the following semilinear parabolic system, depending on the "small" parameter ϵ,

$$(S)_\epsilon \begin{cases} \dot{x} = f(t,x,y,\epsilon) \\ \epsilon\dot{y} = A(t,x)y + g(t,x,y,\epsilon) \end{cases} \tag{1}$$

where $x \in X$, $y \in Y$, and X, Y are Banach spaces.

Let us suppose that $A(t,x) = A_0 + A_1(t,x)$, with $-A_0$ sectorial and $A_1(t,x) \in \mathcal{L}(Y^\alpha, Y)$ (here Y^α is the α -fractional power space generated by $-A_0$). Let us suppose that $g(t,x,0,0) = 0$, $D_y g(t,x,0,0) = 0$, and that the spectrum of $A(t,x)$ is disjoint from a strip containing the imaginary axis, uniformly with respect to (t,x). Finally, let us suppose that $(t,x) \mapsto A_1(t,x)$, f and g verify certain smoothness and boundedness conditions and some further "technical" conditions that we will make precise later. We say that $(S)_\epsilon$ is a "singular perturbation problem"; in fact, for $\epsilon = 0$, $(S)_\epsilon$ is not a parabolic system, but reduces to a differential equation coupled with a nondifferential equation, namely

$$\begin{cases} \dot{x} = f(t,x,y,0) \\ A(t,x)y + g(t,x,y,0) = 0 \end{cases} . \tag{2}$$

It is clear that, if x is a solution of

$$\dot{x} = f(t,x,0,0), \tag{3}$$

then $(x,0)$ satisfies (2). In certain cases (e.g. when Y has finite dimension), the hypotheses on g and on the spectrum of $A(t,x)$ imply that equation

$$A(t,x)y + g(t,x,y,0) = 0 \tag{4}$$

defines implicitly the submanifold $\mathbb{R} \times X$ of $\mathbb{R} \times X \times Y^\alpha$. In such cases, (3) can be considered as an ODE on the submanifold implicitly defined by (4). In general, we ask if any information about the qualitative behaviour of the solutions of $(S)_\epsilon$ (for small ϵ) can be obtained by studying (3). The answer is in part affirmative: in Sections 2 and 3 we will prove that, under suitable hypotheses on A, f and g, and for sufficiently small ϵ, there is an invariant manifold

C_ϵ for the system $(S)_\epsilon$, contained in $\mathbb{R} \times X \times Y^\alpha$ which behaves like a "center manifold" (in the sense that contains all globally defined bounded solutions of $(S)_\epsilon$). This manifold will be the graph of a map k_ϵ (regular and bounded) defined on $\mathbb{R} \times X$ with values in Y^α; also, k_ϵ will depend smoothly on the parameter ϵ and $k_\epsilon(t,x) \to 0$ when $\epsilon \to 0$, uniformly with respect to (t,x), along with its derivatives (up to a certain order). Then, for sufficiently small ϵ , the "reduced" equation

$$\dot{x} = f(t,x,k(t,x,\epsilon),\epsilon) \tag{5}$$

will be "near" to (3). As a consequence, we will be able to apply bifurcation theory to equilibrium points and periodic solutions of (3) and obtain informations about qualitative behaviour of $(S)_\epsilon$.

In Sections 4 and 5 we will analyze the behaviour of solutions of $(S)_\epsilon$ "near" to the manifold C_ϵ, for small ϵ . Namely, we will obtain that, if $Re\sigma\,(A\,(t,x)) > 0$, uniformly with respect to (t,x), then C_ϵ is "attractive", i.e. each solution starting near C_ϵ tends esponentially, as $t \to \infty$, to some solution lying on C_ϵ. More generally (without supposing $Re\sigma\,(A\,(t,x)) > 0$), we will obtain that, for each $(t_0,x_0,y_0) \in C_\epsilon$, there are local "stable" and an "unstable" manifolds contained in $X \times Y^\alpha$; such manifolds are the sets of the points near (x_0,y_0) which are the value at t_0 of a solution of (1) attracted (respectively as $t \to \infty$ and $t \to -\infty$) by the solution whose value at t_0 is (x_0,y_0).

In Section 1 we recall some results about abstract evolution equations; for a complete treatment of the subject, the reader is referred to [He], Chaps. 1,3,7. In proving smoothness of invariant manifolds, we shall use an abstract regularity Theorem for solutions of fixed points equations on a scale of Banach spaces, due to K. Rybakowski ([Ry1]).

Problems of type (1) have been richly studied, specially in finite dimension. The first work in which singular perturbations have been treated from a geometric point of view seems to be Fenichel's paper [Fe]; the techniques developed by Fenichel however do not generalize to semigroups. Some recent papers (specially [V] and [VG]) have developed a different approach to the general problem of invariant manifolds, based on the use of spaces of functions of exponential growth. This approach works for semilinear parabolic equations as well as for ODE, once the fundamental notions about semigroups are well understood. Following this approach, K. Sakamoto developes in [Sa] a complete treatment of singularly perturbed autonomous ODE; along this line, K. Rybakowski gives in [Ry2] a slightly different (and simpler) proof of existence and smoothness of an invariant manifold for $(S)_\epsilon$ based on the abstract result in [Ry1].

In infinite dimension and with $-A_0$ sectorial, problem (2) has been studied in this form by D. Henry ([He], Ch. 9, page 295), who obtained existence and C^1-smoothness of C_ϵ. In this paper we generalize the results of [Sa] to singular perturbed nonautonomous semilinear parabolic equations, improving Henry's results.

In dealing with stability and asymptotic phase we essentially follow the "direct" tecnique developed by D. Henry; a different possible approach to this problem is to prove existence of invariant foliation (as in [CLL]) and then to state the asymptotic phase as a corollary.

Finally, we point out that here we are mainly concerned with existence and smoothness of invariant manifolds for *abstract* evolution equations. Applications to singularly perturbed parabolic PDE will be given elsewhere.

Before starting,we observe that, when $\epsilon > 0$, we can rescale the time variable (setting $\tau := t/\epsilon$), and obtain the "equivalent" system

$$(F)_\epsilon \begin{cases} \frac{dx}{d\tau} = \epsilon f(\epsilon\tau, x, y, \epsilon) \\ \frac{dy}{d\tau} = A(\epsilon\tau, x)y + g(\epsilon\tau, x, y, \epsilon) \end{cases} . \tag{6}$$

This is a "regular" perturbation problem; however, for $\epsilon = 0$ the system reduces to

$$(F)_0 \begin{cases} \frac{dx}{d\tau} = 0 \\ \frac{dy}{d\tau} = A(0, x)y + g(0, x, y, 0) \end{cases} . \tag{7}$$

System (7) has X as an invariant manifold, but such invariant manifold is only a set of equilibrium points: so reducing $(F)_0$ to the invariant manifold would not be of any help in studying qualitative behaviour of $(F)_\epsilon$ for $\epsilon > 0$.

1. Preliminaries.

In this Section we recall some results about semilinear parabolic equations, cited from [He].

1.1. Generalized Gronwall inequality.

THEOREM 1.1.1. ([He,]Lemma 7.1.1) *Let* $b \geq 0$, $\beta > 0$ *be real numbers; let* $t \longmapsto a(t)$ *be a non-negative function, locally integrable on* $[0, T)$ *for some* $0 < T \leq \infty$; *let* $t \longmapsto u(t)$ *be a non-negative function, locally integrable on* $[0, T)$, *such that, for* $t \in [0, T)$,

$$u(t) \leq a(t) + b \int_0^t (t-s)^{\beta-1} u(s) ds.$$

Then, for $t \in [0, T)$,

$$u(t) \leq a(t) + \vartheta \int_0^t E'_\beta(\vartheta(t-s)) a(s) ds,$$

where $\vartheta := (b\Gamma(\beta))^{1/\beta}$; $E_\beta(z) := \sum_{n=o}^{\infty} \frac{z^{n\beta}}{\Gamma(n\beta+1)}$, $E'_\beta(w) := \frac{dE_\beta}{dz}(w)$ $(z, w \in \mathbb{C})$. If $a(t) \equiv a$, then $u(t) \leq aE_\beta(\vartheta t)$. Finally, there exists $\widetilde{M} > 0$ such that $|E_\beta(z)| \leq \frac{\widetilde{M}}{\beta} |e^z|$. ♢

1.2. Solution operators.

Let X be a Banach space, let A be sectorial in X, and let $0 \leq \alpha < 1$; let $t \mapsto B(t) : (t_0, t_1) \to \mathcal{L}(X^\alpha, X)$ be locally Hoelder continuous. By [He] , Th. 7.1.3, we know that there exists a family of "solution operators" $\{T(t, \tau), t_0 < \tau \leq t < t_1\}$ such that:

1. $\{T(t, \tau), t \geq \tau\}$ is strongly continuous with respect to (t, τ), with values in $\mathcal{L}(X^\beta, X^\beta)$ for each $0 \leq \beta < 1$;
2. for each $\tau \in (t_0, t_1)$ and $x_0 \in X$, $T(t, \tau)x_0$ is the solution of

$$\dot{x}\ (t) + Ax(t) = B(t)x(t)$$

on (τ, t_1);

3. if $t \geq s \geq \tau$, $T(\tau,\tau) = I$ and $T(t,s)T(s,\tau) = T(t,\tau)$;

4. for each $\left[\widetilde{t_0}, \widetilde{t_1}\right] \subseteq (t_0, t_1)$there is a constant C, depending only on A, α, $\widetilde{t_1} - \widetilde{t_0}$, and $\sup_{t \in \left[\widetilde{t_0}, \widetilde{t_1}\right]} \|B(t)\|_{\mathcal{L}(X^\alpha, X)}$, such that, for $x \in D(A^2)$, $\widetilde{t_0} \leq \tau \leq t \leq \widetilde{t_1}$ and for $0 \leq \beta, \gamma, \vartheta \leq 1$,

   (a) $|T(t,\tau)x|_\beta \leq \frac{C}{1-\beta}(t-\tau)^{(\gamma-\beta)_-} |x|_\gamma$, when $\beta < 1$, and where $(\gamma - \beta)_- := \min\{\gamma - \beta, 0\}$;

   (b) $|T(t,\tau)x - x|_{\ \beta} \leq \frac{C}{(1-\beta)\vartheta}(t-\tau)^\vartheta |x|_{\beta+\vartheta}$, when $\theta > 0$, $\beta + \vartheta \leq 1$;

   (c) $|T(t+h,\tau)x - T(t,\tau)x|_\beta \leq \frac{C}{1-\beta-\vartheta}(\frac{1}{\vartheta}+\frac{1}{1-\beta})h^\vartheta(t-\tau)^{(\gamma-\beta-\vartheta)_-} |x|_\gamma$, when $t+h > t > \tau, 0 < \vartheta$, and $\beta + \vartheta < 1$;

   (d) $|T(t,\tau)x - T(t,\tau-h)x|_\beta \leq C(\frac{1}{\vartheta(1-\beta)} + \frac{1}{1+\gamma-\alpha-\vartheta})$ $h^\vartheta(t-\tau)^{(\gamma-\beta-\vartheta)_-} |x|_\gamma$, when $t > \tau > \tau - h$, $0 < \vartheta$, $\beta < 1$ and $1 + \gamma > \alpha + \vartheta$.

Note that, since $D(A^2)$ is dense in X^β for each $0 \leq \beta \leq 2$, these estimates hold not only for $x \in D(A^2)$, but for $x \in X^\gamma$, $X^{\beta+\vartheta}$, X^γ, X^γ respectively.

Theorem 1.2.1. ([He], Th. 7.1.4) *Let A, $B(t)$ be as above; let $\tau \in (t_0, t_1)$, $x_0 \in X$, $f : (\tau, t_1) \to X$ locally Hoelder continuous, with $\int_\tau^{\tau+\rho} |f(\sigma)| d\sigma < \infty$ for any $\rho > 0$. Then there is a unique solution of*

$$\begin{cases} \dot{x}\ (t) + A(t)x(t) = f(t), & \tau < t < t_1 \\ x(\tau) = x_0 \end{cases},$$

where $A(t) := A - B(t)$. The solution is given by

$$x(t) = T(t,\tau)x_0 + \int_\tau^t T(t,s)f(s)ds. \tag{8}$$

Moreover, $x : (\tau, t_1) \to X^\alpha$ is locally Hoelder continuous, and $x(t) \to x_0$ in X as $t \to \tau$. Finally, if $0 \leq \beta < 1$, $x_0 \in X^\beta$ and

$\int_\tau^t (t-s)^{-\beta} |f(s)|\, ds \to 0$ as $t \to \tau^+$, then $x(t) \to x_0$ in X^β as $t \to \tau^+$. If we assume only that $f : (\tau, t_1) \to X$ is continuous, (8) defines a continuous map from (τ, t_1) to X^α as well. $\diamondsuit$

1.3. Exponential dichotomies.

Let A and $B(t)$ be as above, with $B(t)$ defined on $(-\infty, +\infty)$. let $T(t,s) \in \mathcal{L}(X,X)$ be the solution operator (defined for $t \geq s$; $t, s \in \mathbb{R}$).

DEFINITION 1.3.1. *Let $Z \subseteq X$ with continuous dense inclusion. Set $A(t) := A + B(t)$, we say that the equation $\dot{x}\,(t) + A(t)x(t) = 0$ has an exponential dichotomy on $(-\infty, +\infty)$ with respect to Z, with exponent $\mu > 0$ and limit M_0, if $T(t,s) \in \mathcal{L}(Z,Z)$ for each $t \geq s$, and if it has projections $P(t) \in \mathcal{L}(Z,Z)$, $t \in \mathbb{R}$, such that:*

1. *$T(t,s)P(s) = P(t)T(t,s)$ for each $t \geq s$ in $\mathbb{R}$;*
2. *the restriction $T(t,s)\,|R(P(s))$ $(t \geq s)$ is an isomorphism of $R(P(s))$ on $R(P(t))$; we define $T(s,t)$ (for $s \leq t$) as the inverse map of $T(t,s)$ from $R(P(t))$ to $R(P(s))$;*
3. *$\|T(t,s)(I - P(s))\|_{\mathcal{L}(Z,Z)} \leq M_0 e^{-\mu(t-s)}$ for $t \geq s$ in $\mathbb{R}$;*
4. *$\|T(t,s)P(s)\|_{\mathcal{L}(Z,Z)} \leq M_0 e^{-\mu(s-t)}$ for $s \geq t$ in $\mathbb{R}$, where $T(t,s)$ is the operator defined in 2.*

In what follows, we always have $Z = X^\gamma$ for any $0 \leq \gamma < 1$. See Th. 1.3.2.
If $P(t) \equiv 0$, we say that the dichotomy is trivial.
From the definition, it follows immediately that, for $t, s, \tau \in \mathbb{R}$, it holds:

1. $T(t,s)T(s,\tau)P(\tau) = T(t,\tau)P(\tau)$;
2. $T(t,s)P(s) = P(t)T(t,s)$ on $R(P(s))$.

The main properties of exponential dichotomies are collected in the following Proposition:

PROPOSITION 1.3.2. ([He], Lemma 7.6.2) *Let A_0 be a sectorial operator in a Banach space X; let $t \mapsto A_1(t) : \mathbb{R} \to \mathcal{L}(X^\alpha, X)$ be bounded and locally Hoelder continuous. Set $A(t) := A_0 + A_1(t)$, we have:*

1. *If $0 \leq \gamma < 1$, then $\dot{x}(t) + A(t)x(t) = 0$ has an exponential dichotomy on $\mathbb{R}$ with respect to X iff has an exponential dichotomy with respect to X^γ.*

2. *If $\dot{x}(t) + A(t)x(t) = 0$ has a dichotomy on $\mathbb{R}$ (with respect to X or X^α) with limit M_0 and exponent μ, and $P(t)$ is the relative projection, then exist constants M_1, M_2, M_3, M_4, M_5 (depending only on α, δ, γ, A_0, α, the constants M_0, μ of the dichotomy, and $\sup_{t \in \mathbb{R}} \|A_1(t)\|_{\mathcal{L}(X^\alpha, X)}$) such that:*

   (a) $|T(t,s)P(s)x|_\gamma \leq M_1 e^{-\mu(s-t)} |x|_\delta$ for $s \geq t$, when $0 \leq \gamma < 1$ and $\delta \geq 0$;

   (b) $|T(t,s)(I - P(s))x|_\gamma \leq M_2 e^{-\mu(t-s)} \max\left\{1, (t-s)^{\delta-\gamma}\right\} |x|_\delta$ for $t > s$ and $0 \leq \delta \leq \gamma < 1$;

   (c) $|P(t_1)x - P(t_2)x|_\gamma \leq M_3 |t_1 - t_2|^\delta |x|_\gamma$ if $0 < \delta < 1 - \gamma \leq 1$;

   (d) if $U(t,s) := \begin{cases} T(t,s)(I - P(s)) \text{ for } t > s \\ -T(t,s)P(s) \text{ for } t \leq s \end{cases}$,

   set $\omega_\gamma(\sigma) := \begin{cases} \sigma^{-\gamma} \text{ for } 0 < \sigma \leq 1 \\ 1 \text{ elsewhere} \end{cases}$,

   then $|U(t,s)x|_\gamma \leq M_1 \omega_\gamma(t-s) e^{-\mu|t-s|} |x|$;

   (e) $|U(t+h,s)x - U(t,s)x|_\gamma \leq M_4 |h|^\delta \omega_{\gamma+\delta}(t-s) e^{-\mu|t-s|} |x|$ if $0 < \delta < 1 - \alpha \leq 1$, $|h| \leq 1$, $|t-s| \leq |t+h-s|$ and s is not between t and $t+h$;

   (f) $|U(t,s-h)x - U(t,s)x|_\gamma \leq M_5 |h|^\delta \omega_{\gamma+\delta}(t-s) e^{-\mu|t-s|} |x|$ if $0 < \delta < 1 - \alpha \leq 1$, $0 \leq \gamma < 1$, $|h| \leq 1$, $|t-s| \leq |t+h-s|$ and t is not between s and $s-h$.

REMARK.

2.(f) implies that the map $s \mapsto U(t,s) : \mathbb{R} \to \mathcal{L}(X, X^\alpha)$ is locally Hoelder continuous on $(-\infty, t) \cup (t, +\infty)$ so that it is locally integrable on this set. By 2.(d) it follows that this map is integrable in a neighborhood of t . So it is locally integrable on $\mathbb{R}$.

Definition 1.3.3. *Let X be a Banach space. Let $\rho \in \mathbb{R}$. Let us define, for $I := \mathbb{R}$, $[0, +\infty)$, $(-\infty, 0]$ and, for $x : I \to X$,*

$$|x|_{I,\rho} := \sup_{t \in I} e^{-\rho|t|} |x(t)|,$$

$$BC^\rho(I, X) := \left\{ x : I \to X \middle| x \text{ such that } |x|_{I,\rho} < \infty \right\}.$$

Each $BC^\rho(I, X)$ is a Banach space with the norm $|\cdot|_{I,\rho}$.

Now we give some representation results for exponentially growing solutions of linear nonhomogeneus parabolic equations, whose simple proof is implicitly contained in [He], Ch.7, and therefore is left to the reader.

Theorem 1.3.4. *Let A_0, $A_1(t)$, $A(t)$ be as in Prop. 1.3.2. Let us assume that $\dot{x} + A(t)x = 0$ has an exponential dichotomy on $\mathbb{R}$ with exponent μ and limit M. Let $0 < \rho_1 < \rho_2 \le \mu$ and let $f \in BC^{\rho_1}(\mathbb{R}, X)$. Set*

$$\psi(t) := \int_{-\infty}^{+\infty} U(t,s) f(s) ds. \tag{9}$$

Then the integral in (9) converges absolutely in X^α for each $t \in \mathbb{R}$, $\psi \in BC^{\rho_2}(\mathbb{R}, X^\alpha)$ and

$$|\psi|_{BC^{\rho_2}(\mathbb{R}, X^\alpha)} \le M \frac{2+\mu}{1-\alpha} \frac{1}{\mu - \rho_1} |f|_{BC^{\rho_1}(\mathbb{R}, X)}. \tag{10}$$

If f is locally Hoelder continuous, then ψ is the unique solution of $\dot{x}(t) + A(t)x(t) = f(t)$ in $BC^{\rho_2}(\mathbb{R}, X^\alpha)$.

Proof. Prop. 1.3.2 and simple integration. $\diamondsuit$

Theorem 1.3.5. *Let $A_0, A_1(t), A(t), M, \mu$ be as in Th. 1.3.4, so the following estimates hold:*

$$|T(t,s)(I-P(s))x|_\alpha \le Me^{-\mu(t-s)}\omega_\alpha(t-s)\,|x| \text{ for } t>s$$

$$|T(t,s)(I-P(s))x|_\alpha \le Me^{-\mu(t-s)}\,|x|_\alpha$$

$$|T(t,s)P(s)x|_\alpha \le Me^{-\mu(t-s)}\,|x| \ .$$

Let $\rho \in \mathbb{R}, |\rho| < \mu$; let $f \in BC^\rho([0,+\infty), X)$ be locally Hoelder continuous.
The following facts are equivalent:

a) $\psi \in BC^\rho([0,+\infty), X^\alpha)$ and $\psi(\cdot - t_0)$ is a solution of $\dot{x}(t) + A(t)x(t) = f(t-t_0)$ on $(t_0,+\infty)$;

b) there is $x_0 \in X^\alpha$ such that, for $t \ge t_0$,

$$\psi(t-t_0) = T(t,t_0)(I-P(t_0))x_0 + \int_{t_0}^{+\infty} U(t,s)f(s-t_0)ds. \tag{11}$$

Moreover, even if f is not locally Hoelder continuous, (11) defines a map belonging to $BC^\rho([0,+\infty))$, for which the following estimate holds:

$$|\psi|_{BC^\rho([0,+\infty),X^\alpha)} \le M\,|x_0|_\alpha + \\ +M\left(\frac{1}{1-\alpha}+\frac{1}{\mu-\rho}+\frac{1}{\mu+\rho}\right)|f|_{BC^\rho([0,+\infty),X)} \ . \tag{12}$$

Proof. Prop. 1.3.2 and simple integration. Note that, if $\rho < 0$, we have $|\psi|_\rho \le M\,|x_0|_\alpha + M\frac{2+\mu}{1-\alpha}\frac{1}{\mu+\rho}\,|f|$. ◊

THEOREM 1.3.6. *Let $A_0, A_1(t), A(t), M, \mu$ be as in Th. 1.3.5. Let $|\rho| < \mu$ and let $f \in BC((-\infty,0], X)$, be locally Hoelder continuous. Then the following facts are equivalent:*

a) $\psi \in BC^\rho((-\infty,0], X^\alpha)$ and $\psi(\cdot - t_0)$ is a solution of $\dot{x} + A(t)x = f(t-t_0)$ on $(-\infty,t_0)$.

b) there is $x_0 \in X^\alpha$ such that, for $t \leq t_0$,

$$\psi(t-t_0) = T(t,t_0)P(t_0)x_0 + \int_{-\infty}^{t_0} U(t,s)f(s-t_0)ds. \quad (13)$$

Moreover, even if f is not locally Hoelder continuous, (13) defines a map belonging to $BC^\rho((-\infty,0], X^\alpha)$, for which the following estimate holds:

$$|\psi|_\rho \leq M|x_0|_\alpha + M\left(\frac{1}{1-\alpha} + \frac{1}{\mu+\rho} + \frac{1}{\mu-\rho}\right)|f|_\rho . \quad (14)$$

Proof. First, we observe that, for $t < t_0$ and $\xi \in X$, $t \mapsto T(t,t_0)P(t_0)\xi$ solves

$$\begin{cases} \dot{x}(t) - A(t)x(t) = 0, \ t < t_0 \\ x(t_0) = P(t_0)\xi \end{cases} .$$

In fact, for $t_1 \leq t < t_0$, we have

$$T(t,t_0)P(t_0)\xi = T(t,t_1)T(t_1,t_0)P(t_0)\xi.$$

Now, one can argue as in Th. 1.3.5. ◊

The following Theorem will be used only in the proof of Th. 3.1.2 below.

THEOREM 1.3.7. *Let A_0 be a sectorial operator. Let $t \mapsto A_{1,a}(t)$, $t \mapsto A_{1,b}(t)$ be locally Hoelder continuous and bounded, from $\mathbb{R}$ to $\mathcal{L}(X^\alpha, X)$. Let $A_a(t) := A_0 + A_{1,a}(t)$, $A_b(t) := A_0 + A_{1,b}(t)$, and let $T_a(t,s)$, $T_b(t,s)$ be the solution operators, respectively for*

$$\dot{x} + A_a(t)x = 0 \quad (15)$$

and

$$\dot{x} + A_b(t)x = 0. \quad (16)$$

Let us suppose that both (15) and (16) have an exponential dichotomy, with projections $P_a(s)$, $P_b(s)$; limits M_a, M_b; exponents μ_a,

μ_b; Green functions $U_a(t,s), U_b(t,s)$ respectively. Then for $t,s\in\mathbb{R}$, $\xi\in X$, we have the folowing formula:

$$(U_a(t,s)-U_b(t,s))\xi=\int_{-\infty}^{+\infty}U_b(t,p)(A_{1,b}(p)-A_{1,a}(p))U_a(p,s)\xi dp. \tag{17}$$

Moreover, the following estimate hold:

$$\begin{aligned} &|(U_a(t,s)-U_b(t,s))\xi|_\alpha\le\\ \le &\int_{-\infty}^{+\infty}|U_b(t,p)(A_{1,b}(p)-A_{1,a}(p))U_a(p,s)\xi|_\alpha\, dp\le\\ &\le \text{const. } \omega_\alpha(t-s)e^{\rho|t-s|}\,|\xi|\,. \end{aligned} \tag{18}$$

$\diamondsuit$

Proof. The proof is based on the fact that, setting

$$x(t):=U_a(t,s)\xi=\begin{cases} T_a(t,s)(I-P_a(s))\xi,\ t>s\\ -T_a(t,s)P_a(s)\xi,\ t\le s\end{cases},$$

we have

$$\begin{aligned} \tfrac{dx}{dt}(t)&=-A_a(t)x(t)=-A_b(t)x(t)+\left(A_{1,b}(t)-A_{1,a}(t)\right)x(t)=\\ &=-A_b(t)x(t)+\left(A_{1,b}(t)-A_{1,a}(t)\right)T_a(t,s)(I-P_a(s))\xi,\ t>s;\\ x(s)&=(I-P_a(s))\xi, \end{aligned}$$

and

$$\begin{aligned} \tfrac{dx}{dt}(t)&=-A_b(t)x(t)+\left(A_{1,b}(t)-A_{1,a}(t)\right)T_a(t,s)P_a(s)\xi,\ t\le s;\\ x(s)&=-P_a(s)\xi. \end{aligned}$$

So, we can apply Th. 1.3.6 and 1.3.5. $\diamondsuit$

1.4. Uniform exponential dichotomy Theorem.

The following Theorem gives sufficient conditions for an exponential dichotomy. Actually, it is a consequence of a more general Theorem ([He], teor. 7.6.12), and the reader is referred to [He] for a complete treatment of the subject.

THEOREM 1.4.1. ([He], Th. 7.6.13) *Let A_0 be a sectorial operator in X, and let $0 \leq \alpha < 1$; let (Λ, d) be a metric space; let $\lambda \mapsto A_1(\lambda) : \Lambda \to \mathcal{L}(X^\alpha, X)$ be locally Hoelder continuous, bounded, uniformly continuous, with range contained in a compact subset of $\mathcal{L}(X^1, X)$. Let us suppose that A_0 has compact resolvent. Finally, set $A(\lambda) := A_0 - A_1(\lambda)$, let us suppose that, for some $\mu > 0$, $\sigma(A(\lambda)) \subseteq \{z \in \mathbb{C} \ ||Rez| \geq \mu\}$. Let $0 < \mu_1 < \mu$. Then there exist $\epsilon > 0$, $M_1 \geq 1$ such that the equation*

$$\dot{x} + A(\lambda(t))x = 0$$

has an exponential dichotomy on $\mathbb{R}$ with exponent μ_1 and limit M_1, for each locally Hoelder continuous map $\lambda : \mathbb{R} \to \Lambda$ verifying the following condition:

$$(*) \text{ if } |t - s| \leq 1, \text{ then } d(\lambda(t), \lambda(s)) \leq \epsilon. \qquad \diamondsuit$$

1.5. A technical Lemma.

In the sequel, the following technical Lemma, whose simple proof is left to the reader, will be very useful.

LEMMA 1.5.1. ([Ry2], Lemma9) *Let X, Y, Z be normed spaces. Let $A \subseteq Y$ be convex. Let $\psi : A \to Z$ be continuous and bounded. Let $\varphi : X \to C^0(J, A)$ (with $J = \mathbb{R}$, or $J = [0, +\infty)$, or $J = (-\infty, 0]$) be a map satisfying the following condition:*

$$(*) \left[\begin{array}{c} \text{for every compact interval } \subseteq J, \text{the map} \\ X \times I \ni (\xi, t) \mapsto \varphi(\xi)(t) \in A \\ \text{is continuous in } \xi, \text{ uniformly with respect to } t \in I \end{array} \right].$$

Then for all $\xi \in X$ and every $\alpha > 0$,

a) $\lim\limits_{h \to 0} \sup\limits_{t \in J} e^{-\alpha|t|} |\psi(\varphi(\xi + h)(t)) - \psi(\varphi(\xi)(t))| = 0;$

b) $\lim\limits_{h \to 0} \sup\limits_{t \in J} e^{-\alpha|t|} \sup\limits_{\vartheta \in [0,1]} |\psi(\vartheta\varphi(\xi + h)(t) + (1 - \vartheta)\varphi(\xi)(t)) - \psi(\varphi(\xi)(t)| = 0.$

Assumption $(*)$ is satisfied if φ is continuous from X to $BC^\eta(J, Y)$ for some $\eta \in \mathbb{R}$. $\diamondsuit$

2. Setting of the problem.

2.1. Main hypotheses and first results.

Let us consider the following singularly perturbed system:

$$(S)_\epsilon \begin{cases} \dot{x} = f(t,x,y,\epsilon) \\ \epsilon \dot{y} = A(t,x)y + g(t,x,y,\epsilon) \end{cases} .$$

We make the following assumptions:
(SP)

1. X and Y are Banach spaces, $x \in X$ and $y \in Y$;

2. there exist $r \in \mathbb{N}$, $\epsilon_3, \delta_0 \in \mathbb{R}$, $r \geq 2$ and $\epsilon_3, \delta_0 > 0$, such that, if $B^\alpha_{\delta_0}(0)$ is the open δ_0-ball centerd at 0 in Y^α, we have:

   (a) $A(t,x) = A_0 + A_1(t,x)$, where $-A_0$ is a sectorial operator in Y; if Y^α is the "fractional-power space" generated by $-A_0$, the map
   $$(t,x) \mapsto A_1(t,x) : \mathbb{R} \times X \to \mathcal{L}(Y^\alpha, Y)$$
   is of class C^r_b;

   (b) $f \in C^r_b(\mathbb{R} \times X \times B^\alpha_{\delta_0}(0) \times (0,\epsilon_3), X)$ and $g \in C^r_b(\mathbb{R} \times X \times B^\alpha_{\delta_0}(0) \times (0,\epsilon_3), Y)$ (Note that this assumption implies that, for $0 \leq n \leq r-1$, $D^n g : \mathbb{R} \times X \times B^\alpha_{\delta_0}(0) \times (0,\epsilon_3) \to \mathcal{L}^n(\mathbb{R} \times X \times Y^\alpha \times \mathbb{R}, Y)$ and $D^n f : \mathbb{R} \times X \times B^\alpha_{\delta_0}(0) \times (0,\epsilon_3) \to \mathcal{L}^n(\mathbb{R} \times X \times Y^\alpha \times \mathbb{R}, X)$ are globally Lipschitz continuous, and therefore can be extended continuously to $\mathbb{R} \times X \times B^\alpha_{\delta_0}[0] \times [0,\epsilon_3]$: we denote these extensions still as $D^n f$ and $D^n g$);

3. $g(t,x,0,0) = 0$ for all $(t,x) \in \mathbb{R} \times X$;

4. $D_y g(t,x,0,0) = 0$ for all $(t,x) \in \mathbb{R} \times X$;

5. A_0 has compact resolvent;

6. For some $\mu > 0$, and for all $(t,x) \in \mathbb{R} \times X$, $|\mathrm{Re}\sigma(A(t,x))| \geq 2\mu$.

The first result regards the linear part of system $(S)_\epsilon$.

LEMMA 2.1.1. *There are real* ϵ_2 *and* $\tilde{N}$, $\epsilon_3 \geq \epsilon_2 > 0$, $\tilde{N} > 0$, *such that, for each* $0 < \epsilon \leq \epsilon_2$, *and for all* $x \in C^1(\mathbb{R}, X)$ *with* $\sup_{t\in\mathbb{R}} |\dot{x}(t)| \leq \tilde{N}$, *and all* $t_0 \in \mathbb{R}$, *the equation*

$$\dot{y} = A(t_0 + \epsilon t, x(t))y$$

has an exponential dichotomy with limit M *and exponent* μ *independent of* ϵ, x *and* t_0.

Proof. We apply Th. 1.4.1. First of all, we observe that

$$\{A_1(t,x)\,|(t,x) \in \mathbb{R} \times X\}$$

is bounded in $\mathcal{L}(Y^\alpha, Y)$. Since A_0 has compact resolvent, the inclusion $Y^1 \hookrightarrow Y^\alpha$ is compact (see [He], Ch.1). By the Ascoli-Arzelà Theorem, it is easily seen that also the inclusion $\mathcal{L}(Y^\alpha, Y) \hookrightarrow \mathcal{L}(Y^1, Y)$ is compact. So

$$\{A_1(t,x)\,|(t,x) \in \mathbb{R} \times X\}$$

is in a compact subset of $\mathcal{L}(Y^1, Y)$.
Now we set $\Lambda := \mathbb{R} \times X$, and supply it with a metric space structure, by mean of the product norm. Since the map $(t,x) \mapsto A_1(t,x)$ is of class C_b^r, with $r \geq 2$, it is globally Lipschitz. By Th. 1.4.1, there are $q > 0$, $M > 0$, such that, if $\lambda : \mathbb{R} \to \Lambda$ verifies:

$$(*)\ d(\lambda(t), \lambda(s)) \leq q \text{ if } |t-s| \leq 1,$$

the equation

$$\dot{y} = A(\lambda(t))y$$

has an exponential dichotomy on $\mathbb{R}$ with exponent μ and limit M.
Now, if $\epsilon > 0$, $x \in C_b^r$ and $\sup_{t\in\mathbb{R}} |\dot{x}(t)| < \infty$, if $t_0 \in \mathbb{R}$, and if $|t-s| \leq 1$, we have

$$d((t_0 + \epsilon t, x(t)), (t_0 + \epsilon s, x(s))) \leq \epsilon\,|t-s| + \sup_{\tau\in\mathbb{R}} |\dot{x}(\tau)|\,|t-s| \leq$$

$$\leq \epsilon + \sup_{\tau \in \mathbb{R}} |\dot{x}(\tau)| \, .$$

Setting $\epsilon_2 := q/2$, $\tilde{N} := q/2$, we obtain the conclusion. $\diamondsuit$

Let us consider now, for $\epsilon > 0$,

$$(S)_\epsilon \begin{cases} \dot{x} = f(t, x, y, \epsilon) \\ \epsilon \dot{y} = A(t, x)y + g(t, x, y, \epsilon) \end{cases} .$$

If we rescale the time variable, by mean of the map $t \mapsto \epsilon t$, we obtain that $(S)_\epsilon$ is equivalent to

$$(F)_\epsilon \begin{cases} \dot{x} = \epsilon f(\epsilon t, x, y, \epsilon) \\ \dot{y} = A(\epsilon t, x)y + g(\epsilon t, x, y, \epsilon) \end{cases} ,$$

in the sense that:

1. if $(x, y) : (t_1, t_2) \subseteq \mathbb{R} \to X \times Y$ is a solution of $(S)_\epsilon$, then $(\hat{x}, \hat{y}) : (\frac{t_1}{\epsilon}, \frac{t_2}{\epsilon}) \to X \times Y$ defined by

$$\begin{aligned} \hat{x}(t) &:= x(\epsilon t) \\ \hat{y}(t) &:= y(\epsilon t) \ , \end{aligned}$$

   is a solution of $(F)_\epsilon$;

2. conversely, if $(x, y) : (t_1, t_2) \subseteq \mathbb{R} \to X \times Y$ is a solution of $(F)_\epsilon$, then $(\check{x}, \check{y}) : (\epsilon t_1, \epsilon t_2) \to X \times Y$, defined by

$$\begin{aligned} \check{x}(t) &:= x\left(\frac{1}{\epsilon} t\right) \\ \check{y}(t) &:= y\left(\frac{1}{\epsilon} t\right) , \end{aligned}$$

   is a solution of $(S)_\epsilon$.

We call $(S)_\epsilon$ the "slow" system and $(F)_\epsilon$ the "fast" system.

2.2. Some properties of the map g.

Now we analyze some properties of the map g that can be directly deduced from hypotheses **(SP)**2.(b), **(SP)**3 and **(SP)**4. As we have already seen, since $g \in C_b^r(\mathbb{R} \times X \times B_{\delta_0}^\alpha(0) \times (0, \epsilon_3))$, for every $0 \le j \le r-1$, $D^j g$ can be extended continuously to $\mathbb{R} \times X \times B_{\delta_0}^\alpha(0) \times [0, \epsilon_3]$. Also, we agreed to denote these extensions again as $D^j g$. Moreover, for $0 \le j \le r-1$, the map

$$D^j g : \mathbb{R} \times X \times B_{\delta_0}^\alpha[0] \times [0, \epsilon_3] \to \mathcal{L}^j(\mathbb{R} \times X \times Y^\alpha \times \mathbb{R}, Y)$$

is Lipschitz continuous; hence, if a sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$, $\epsilon_n \to 0$, is given, then

$$\left\{D^j g(t, x, y, \epsilon_n)\right\}_{n\in\mathbb{N}}$$

is a Cauchy sequence in $\mathcal{L}^j(\mathbb{R}\times X\times Y^\alpha\times\mathbb{R}, Y)$, uniformly with respect to $(t, x, y) \in \mathbb{R} \times X \times B_{\delta_0}^\alpha[0]$. So, for $0 \le j \le r-1$, $D^j g(t, x, y, \epsilon_n) \to D^j g(t, x, y, 0)$ uniformly with respect to $(t, x, y) \in \mathbb{R} \times X \times B_{\delta_0}^\alpha[0]$. Therefore the map $(t, x, y) \to D^j g(t, x, y, 0)$ belongs to $C_b^{r-1}(\mathbb{R}\times X\times B_{\delta_0}^\alpha(0), Y)$, and $(t, x, y) \mapsto D_{(t,x,y)}^{r-1} g(t, x, y, 0)$ is Lipschitz continuous. We assumed that

$$\begin{aligned} &g(t, x, 0, 0) \equiv 0 \\ &D_y g(t, x, 0, 0) \equiv 0. \end{aligned}$$

Obviously, also $D_x g(t, x, 0, 0) \equiv 0$. By the Mean Value Theorem, we conclude that there is a constant $C \in \mathbb{R}$, $C \ge 0$, such that, for all $(t, x, y, \epsilon) \in \mathbb{R} \times X \times B_{\delta_0}^\alpha(0) \times [0, \epsilon_3]$, the following estimates hold:

$$\begin{aligned} &|g(t, x, y, \epsilon)| \le C(\epsilon + |y|_\alpha^2); \\ &|D_x g(t, x, y, \epsilon)| \le C(\epsilon + |y|_\alpha); \\ &|D_y g(t, x, y, \epsilon)| \le C(\epsilon + |y|_\alpha). \end{aligned} \tag{19}$$

2.3. Strategy of work.

Our goal is to build an invariant manifold for $(S)_\epsilon$. Namely, we shall construct a map $k : \mathbb{R} \times X \times (0, \epsilon_0) \to Y^\alpha$, of class C_b^r, with $k(t, x, \epsilon) \underset{\epsilon\to 0}{\longrightarrow} 0$ uniformly with respect to $(t, x) \in \mathbb{R} \times X$, such that the set

$$C_\epsilon := \{(t, x, y) \,|y = k(t, x, \epsilon)\} \subseteq \mathbb{R} \times X \times Y^\alpha$$

is an invariant manifold of $(S)_\epsilon$. As a consequence, we will have that, since $k(t, x, \epsilon) \underset{\epsilon\to 0}{\longrightarrow} 0$ uniformly with respect to (t, x) and since, for

$0 \le j \le r-1$, $D^j k$ is Lipschitz continuous, then $D^j k(t,x,\epsilon) \to 0$ uniformly with respect to (t,x) for $0 \le j \le r-1$; therefore $k(\cdot,\cdot,\epsilon) \to 0$ in $C_b^{r-1}(\mathbb{R} \times X, Y^\alpha)$. So we'll have that

$$f(t,x,k(t,x,\epsilon),\epsilon) \underset{\epsilon\to 0}{\longrightarrow} f(t,x,0,0)$$
$$D_{(t,x)}\left[(t,x) \mapsto f(t,x,k(t,x,\epsilon),\epsilon)\right] \underset{\epsilon\to 0}{\longrightarrow} D_{(t,x)} f(t,x,0,0)$$

uniformly with respect to $(t,x) \in \mathbb{R} \times X$; i.e. the reduced equation

$$\dot{x} = f(t,x,k(t,x,\epsilon),\epsilon)$$

"tends" to the equation

$$\dot{x} = f(t,x,0,0).$$

Let us suppose now that we have built an invariant manifold for $(F)_\epsilon$, $\epsilon > 0$. Namely, let us suppose that we have constructed a map $h : \mathbb{R} \times X \times (0,\epsilon_0) \to Y^\alpha$ such that, for all $(t_0,x_0) \in \mathbb{R} \times X$, there is a unique solution (x,y) of $(F)_\epsilon$ on $(-\infty,+\infty)$, with $x(t_0) = x_0$ and $y(t) = h(t,x(t),\epsilon)$ for all $t \in (-\infty,+\infty)$. Let us set

$$k(t,x,\epsilon) := h\left(\frac{t}{\epsilon}, x, \epsilon\right).$$

It is very easy to check that h defines an invariant manifold for $(S)_\epsilon$. Conversely, if $k : \mathbb{R} \times X \times (0,\epsilon_0) \to Y^\alpha$ defines an invariant manifold for $(S)_\epsilon$, then $h(t,x,\epsilon) := k(\epsilon t, x, \epsilon)$ defines an invariant manifold for $(F)_\epsilon$.

In the next section we will construct an invariant manifold for $(F)_\epsilon$, and hence, as we have just seen, one for $(S)_\epsilon$.

3. Existence and smoothness of the invariant manifold.

In this Section we prove existence and smoothness of an invariant manifold for systems of type $(S)_\epsilon$ assuming hypotheses **(SP)**.

3.1. Existence.

We start with some "technical" results.

LEMMA 3.1.1. *Let* ϵ_2 *and* $\tilde{N}$ *be as in Lemma 2.1.1.*

1. Let $\rho \in \mathbb{R}$, $\rho > 0$; let $x_1 \in BC^{\rho}(\mathbb{R}, X)$ and, for $t \in \mathbb{R}$, let $x(t) := \int_0^t x_1(s)ds$. Then $x \in BC^{\rho}(\mathbb{R}, X)$ and $|x|_\rho \leq \frac{1}{\rho}|x_1|_\rho$.

2. Let $x \in C^1(\mathbb{R},X)$ with $\sup_{t\in\mathbb{R}} |\dot{x}(t)| \leq \tilde{N}$. Let $\epsilon \leq \epsilon_2$. Let $0 \leq \rho_1 \leq \rho_2 < \mu$. Let $\psi_1 \in BC^{\rho_1}(\mathbb{R}, Y)$. For $t \in \mathbb{R}$, let

$$\psi(t) := \int_{-\infty}^{+\infty} U(t, s; t_0, x(\cdot), \epsilon)\psi_1(s)ds.$$

Then ψ is well defined and belongs to $BC^{\rho_2}(\mathbb{R}, Y^\alpha)$, and the following estimate holds:

$$|\psi|_{\rho_2} \leq K\frac{1}{\mu - \rho_1}|\psi_1|_{\rho_1}$$

where $K := M\frac{2+\mu}{1-\alpha}$.

Moreover, if ψ_1 is locally Hoelder continuous, ψ is the unique solution of

$$\dot{y} = A(t_0 + \epsilon t, x(t))y + \psi_1(t)$$

in $BC^{\rho_2}(\mathbb{R}, Y^\alpha)$.

Proof.

1. For every $t \in \mathbb{R}$ we have:

$$e^{-\rho|t|}|x(t)| \leq e^{-\rho|t|}\left|\int_0^t |x_1(s)|\,ds\right| \leq e^{-\rho|t|}\left|\int_0^t e^{\rho|s|}|x_1|_\rho\,ds\right| \leq$$
$$\leq e^{-\rho|t|}\frac{1}{\rho}e^{\rho|t|}|x_1|_\rho = \frac{1}{\rho}|x_1|_\rho.$$

2. Point 2. is simply a reformulation of Th. 1.3.4. ◊

Lemma 3.1.2. *Let* $0 < \rho < \frac{\mu}{2}$ *and let* $w \in BC^\rho(\mathbb{R}, Y)$. *Let* $x, x^* \in C^1(\mathbb{R}, X)$, *with* $\sup_{t\in\mathbb{R}} |\dot{x}(t)| \le \tilde{N}$, $\sup_{t\in\mathbb{R}} |\dot{x}^*(t)| \le \tilde{N}$. *Let* $t_0, t_0^* \in \mathbb{R}$, *and let* $0 < \epsilon, \epsilon^* \le \epsilon_2$. *Let*

$$\tilde{w}(t) := \int_{-\infty}^{+\infty} U(t, s; t_0, x(\cdot), \epsilon) w(s) ds.$$

Then

$$\begin{aligned}\tilde{w}(t) = \int_{-\infty}^{+\infty} U(t, s; t_0^*, x^*(\cdot), \epsilon^*) &\left[(A_1(t_0 + \epsilon s, x(s)) + \right.\\ &\left. - A_1(t_0^* + \epsilon^* s, x^*(s)))\tilde{w}(s) + w(s)\right] ds.\end{aligned}$$

Proof. By Th. 1.3.7,

$$\begin{aligned}\tilde{w}(t) &= \int_{-\infty}^{+\infty} U(t, s; t_0, x(\cdot), \epsilon) w(s) ds = \\ &= \int_{-\infty}^{+\infty} \{U(t, s; t_0^*, x^*(\cdot), \epsilon^*) w(s) + \\ &+ \int_{-\infty}^{+\infty} U(t, p; t_0^*, x^*(\cdot), \epsilon^*)\left[-A_1(t_0^* + \epsilon^* p, x^*(p)) + \right.\\ &\left. + A_1(t_0 + \epsilon p, x(p))\right] \\ &U(p, s; t_0, x(\cdot), \epsilon) w(s) dp\} ds = \\ &= \int_{-\infty}^{+\infty} \{U(t, s; t_0^*, x^*(\cdot), \epsilon^*) w(s) + \\ &+ \int_{-\infty}^{+\infty} U(t, s; t_0^*, x^*(\cdot), \epsilon^*)\left[A_1(t_0 + \epsilon s, x(s)) + \right.\\ &\left. - A_1(t_0^* + \epsilon^* s, x^*(s))\right] \\ &U(s, p; t_0, x(\cdot), \epsilon) w(p) dp\} ds = \\ &= \int_{-\infty}^{+\infty} U(t, s; t_0^*, x^*(\cdot), \epsilon^*)\left[w(s) + (A_1(t_0 + \epsilon s, x(s)) + \right.\\ &\left. - A_1(t_0^* + \epsilon^* s, x^*(s)))\tilde{w}(s)\right] ds.\end{aligned}$$

Here we have used Fubini and Tonelli Theorems. This was possible thanks to estimate (18): infact, we have

$$\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}|U(t,s;t_0^*,x^*(\cdot),\epsilon^*)(A_1(t_0+\epsilon s,x(s))+ \right.$$
$$-A_1(t_0^*+\epsilon^* s,x^*(s)))$$
$$\left. U(s,p;t_0,x(\cdot),\epsilon)w(p)|\right]dsdp\le$$
$$\le\int_{-\infty}^{+\infty}\text{const. }\omega_\alpha(t-p)e^{-\frac{\mu}{2}|t-p|}\left|w(p)\right|dp\le$$
$$\le\int_{-\infty}^{+\infty}\text{const. }\omega_\alpha(t-p)e^{-\frac{\mu}{2}|t-p|}\left|w\right|_\rho e^{\rho|p|}dp<\infty.$$

◊

Lemma 3.1.3. *Let $x:\mathbb{R}\to X$, $y:\mathbb{R}\to Y^\alpha$ be continuous, with $\sup_{t\in\mathbb{R}}|y(t)|_\alpha<\delta_0$. Let $N:=\sup_{(t,x,y,\epsilon)}|f(t,x,y,\epsilon)|$. Let $0<\epsilon_1\le\epsilon_2$ be such that $\epsilon_1 N\le\tilde{N}$. Let $0<\epsilon\le\epsilon_1$. Then the following facts are equivalent:*

1. $(x(\cdot-t_0),y(\cdot-t_0))$ is a solution on $\mathbb{R}$ of

$$(F)_\epsilon\begin{cases}\dot{x}=\epsilon f(\epsilon t,x,y,\epsilon)\\ \dot{y}=A(\epsilon t,x(t))y+g(\epsilon t,x,y,\epsilon)\end{cases}.$$

2. $x\in C^1(\mathbb{R},X)$, $\sup_{t\in\mathbb{R}}|\dot{x}(t)|\le\epsilon_1 N$, and, for some $\xi\in X$, some $t_0\in\mathbb{R}$, and all $t\in\mathbb{R}$,

$$x(t)=\xi+\int_0^t\epsilon f(\epsilon(s+t_0),x(s),y(s),\epsilon)ds$$
$$y(t)=\int_{-\infty}^{+\infty}U(t,s;\epsilon t_0,x,\epsilon)g(\epsilon(t_0+s),x(s),y(s),\epsilon)ds.$$

Proof.
1.⇒ 2.
If 1. holds, for every $t\in\mathbb{R}$

$$x(t-t_0)=x(0)+\int_{t_0}^t\epsilon f(\epsilon s,x(s-t_0),y(s-t_0),\epsilon)ds=$$
$$=x(0)+\int_0^{t-t_0}\epsilon f(\epsilon(s+t_0),x(s),y(s),\epsilon)ds$$

and so

$$x(t) = x(0) + \int_0^t \epsilon f(\epsilon(s+t_0), x(s), y(s), \epsilon) ds.$$

As for y, by hypothesis the map $s \mapsto g(\epsilon s, x(s-t_0), y(s-t_0), \epsilon) : \mathbb{R} \to Y$ is locally Hoelder continuous (see [He], Ch. 3); moreover, by the estimates 19, the map $s \mapsto g(\epsilon s, x(s-t_0), y(s-t_0), \epsilon)$ belongs to $BC^0(\mathbb{R}, Y)$. By Lemma 3.1.1, for each $t \in \mathbb{R}$,

$$y(t-t_0) = \int_{-\infty}^{+\infty} U(t, s; 0, x(\cdot - t_0), \epsilon) g(\epsilon s, x(s-t_0), y(s-t_0), \epsilon) ds =$$

$$= \int_{-\infty}^{+\infty} U(t, s+t_0; 0, x(\cdot - t_0), \epsilon) g(\epsilon(s+t_0), x(s), y(s), \epsilon) ds.$$

So

$$y(t) = \int_{-\infty}^{+\infty} U(t_0+t, t_0+s; 0, x(\cdot - t_0), \epsilon) g(\epsilon(s+t_0), x(s), y(s), \epsilon) ds.$$

Let us set

$$\begin{aligned} \psi(\lambda) &:= T(\lambda, t_0+s; 0, x(\cdot - t_0), \epsilon) y \\ \tilde{\psi}(\lambda) &:= \psi(\lambda + t_0). \end{aligned}$$

By definition, ψ solves

$$\begin{cases} \frac{du}{d\lambda} = A(\epsilon\lambda, x(\lambda - t_0)) u \\ u(t_0+s) = y \end{cases} ;$$

moreover, $\tilde{\psi}$ solves

$$\begin{cases} \frac{du}{d\lambda} = A(\epsilon(\lambda + t_0), x(\lambda)) u \\ u(s) = y \end{cases} .$$

Therefore, for all $y \in Y$, we have

$$T(t, s; \epsilon t_0, x(\cdot), \epsilon) y = \tilde{\psi}(t) = \psi(t+t_0) = T(t+t_0, s+t_0; 0, x(\cdot - t_0), \epsilon) y.$$

So

$$y(t) = \int_{-\infty}^{+\infty} U(t, s; \epsilon t_0, x(\cdot), \epsilon) g(\epsilon(s+t_0), x(s), y(s), \epsilon) ds.$$

2.$\Rightarrow$1.

If 2. holds,

$$\frac{d}{ds}\left[s \mapsto x(s-t_0)\right](t) = \epsilon f(\epsilon t, x(t-t_0), y(t-t_0), \epsilon).$$

As for y, we know that the map $s \mapsto g(\epsilon s, x(s-t_0), y(s-t_0), \epsilon)$ is continuous and bounded, i.e. belongs to $BC^0(\mathbb{R}, Y)$. Arguing as in the first part of the proof, it is easily seen that

$$y(t-t_0) = \int_{-\infty}^{+\infty} U(t,s;0,x(\cdot - t_0),\epsilon) g(\epsilon s, x(s-t_0), y(s-t_0), \epsilon) ds.$$

It can be easily shown (cf. the proof of Th. 1.3.4), that, for $t \geq s$,

$$y(t-t_0) = T(t,s;0,x(\cdot - t_0),\epsilon) y(s-t_0) +$$
$$+ \int_s^t T(t,p;0,x(\cdot - t_0),\epsilon) g(\epsilon p, x(p-t_0), y(p-t_0), \epsilon) dp.$$

By estimates at page 155, it can be shown that y is locally Hoelder continuous, and so, by Th. 1.2.1, $(x(\cdot - t_0), y(\cdot - t_0))$ is a solution of $(F)_\epsilon$.

NOTE: we could not have applied directly Lemma 3.1.1, as it should have seemed more natural, because we didn't know a-priori if $s \mapsto g(\epsilon s, x(s-t_0), y(s-t_0), \epsilon)$ were locally Hoelder continuous.

◇

LEMMA 3.1.4. *Let* $N := \sup|f|$, $N_1 := \max\{\sup|D_x f|, \sup|D_y f|$, $M_2 := \sup|DA_1|$. *There exist* $b, \rho, \epsilon_0, \delta \in \mathbb{R}$ *such that:*

$$b > 2,\ 0 < \epsilon_0 \leq \epsilon_1,\ \rho > 2N_1\epsilon_0,\ 0 < \delta \leq \delta_0,\ \rho r b < \mu/2,$$

$$C(\epsilon_0 + \delta^2) < \tfrac{\mu}{K}\delta,$$

$$0 < \kappa_1 := \tfrac{K}{\mu - \rho r b} \max\left\{M_2\delta, C(\epsilon_0+\delta^2), C(\epsilon_0+\delta)\right\} < \tfrac{1}{4}.$$

Proof.

1. Fix $b > 2$;
2. choose $\rho < \frac{1}{2}\frac{\mu}{rb}$ $\left(\Rightarrow \frac{1}{\mu - \rho r b} < \frac{1}{\mu/2}\right)$;

3. choose δ such that: $0 < \delta \leq \delta_0$; $\delta < \frac{1}{2}\frac{1}{4}\frac{\mu/2}{KM_2}$; $\delta^2 < \frac{1}{2}\frac{1}{4}\frac{\mu/2}{KC}$; $\delta < \frac{1}{2}\frac{1}{4}\frac{\mu/2}{KC}$;

4. finally, choose ϵ_0 such that: $0 < \epsilon_0 \leq \epsilon_1$; $\epsilon_0 < \frac{\rho}{2N_1}$; $\epsilon_0 < \frac{1}{2}\frac{1}{4}\frac{\mu/2}{KC}$; $\epsilon_0 < \frac{\mu}{KC}\delta - \delta^2$.

With this choice, it is easily seen that all our requests are satisfied.
$\diamondsuit$

THEOREM 3.1.5. *Let N, N_1, M_2, ρ, ϵ_0, δ, b be as in Lemma 3.1.4; in particular, with this choice, we have*

$$0 < \frac{K}{\mu-\rho}\max\left\{M_2\delta, C(\epsilon_0+\delta^2), C(\epsilon_0+\delta)\right\} < \frac{1}{4}.$$

Then for each $0 < \epsilon \leq \epsilon_0$, for all $\xi \in X$, $t_0 \in \mathbb{R}$, there is a unique couple

$$(x, y) = (\varphi_1(t_0, \xi, \epsilon), \varphi_2(t_0, \xi, \epsilon))$$

of maps, with $x \in C^1(\mathbb{R}, X)$, $y \in C^0(\mathbb{R}, Y^\alpha)$, and $|\dot{x}|_0 := \sup_{t\in\mathbb{R}} |\dot{x}(t)| \leq \epsilon_0 N$, $|y|_0 := \sup_{t\in\mathbb{R}} |y(t)| \leq \delta$, stisfying, for $t \in \mathbb{R}$, the following integral system:

$$\left\{\begin{array}{l} x(t) = \xi + \displaystyle\int_0^t \epsilon f(\epsilon(s+t_0), x(s), y(s), \epsilon)ds \\ y(t) = \displaystyle\int_{-\infty}^{+\infty} U(t, s; \epsilon t_0, x, \epsilon) g(\epsilon(s+t_0), x(s), y(s), \epsilon)ds \end{array}\right. . \quad (20)$$

Proof. We introduce the Banach space $BC^\rho(\mathbb{R}, X) \times BC^\rho(\mathbb{R}, Y^\alpha)$ supplied with the product norm $|(x,y)|_\rho := |x|_\rho + |y|_\rho$.
Let

$$\mathcal{A} := \left\{(x,y)\middle| x \in C^1(\mathbb{R}, X), y \in C^0(\mathbb{R}, Y^\alpha), |\dot{x}|_0 \leq \epsilon_0 N, |y|_0 \leq \delta\right\}.$$

Since $\rho > 0$, we have that $\mathcal{A} \subseteq BC^\rho(X) \times BC^\rho(Y^\alpha)$. We define

$$\mathcal{F}_1 : \mathbb{R} \times X \times (0, \epsilon_0) \times \mathcal{A} \to BC^\rho(X)$$
$$\mathcal{F}_2 : \mathbb{R} \times X \times (0, \epsilon_0) \times \mathcal{A} \to BC^\rho(Y^\alpha).$$

in this way:

$$\mathcal{F}_1(t_0,\xi,\epsilon,x,y)(t) := \xi + \int_0^t \epsilon f(\epsilon(t_0+s),x(s),y(s),\epsilon)ds$$
$$\mathcal{F}_2(t_0,\xi,\epsilon,x,y)(t) := \int_{-\infty}^{+\infty} U(t,s;\epsilon t_0,x,\epsilon)g(\epsilon(t_0+s),x(s),y(s),\epsilon)ds.$$

Clearly, $\mathcal{F}_1(t_0,\xi,\epsilon,x,y) \in C^1(\mathbb{R},X)$ and $|\mathcal{F}_1(t_0,\xi,\epsilon,x,y)'|_0 \leq \epsilon N$. Moreover, by Lemma 3.1.1 and by our choice of the constants (Lemma 3.1.4),

$$\begin{aligned} |\mathcal{F}_2(t_0,\xi,\epsilon,x,y)|_0 &\leq \tfrac{K}{\mu} \sup_{s\in\mathbb{R}} |g(\epsilon(t_0+s),x(s),y(s),\epsilon)| \leq \\ &\leq \tfrac{K}{\mu} C(\epsilon_0+\delta^2) < \delta. \end{aligned}$$

Therefore $\mathcal{F} := (\mathcal{F}_1,\mathcal{F}_2) : \mathcal{A} \to \mathcal{A}$.
Now, for $(x,y),(x^*,y^*) \in \mathcal{A}$,

$$\begin{aligned} &\mathcal{F}_1(t_0,\xi,\epsilon,x,y)(t) - \mathcal{F}_1(t_0,\xi,\epsilon,x^*,y^*)(t) = \\ &= \int_0^t \epsilon \left[f(\epsilon(t_0+s),x(s),y(s),\epsilon) - f(\epsilon(t_0+s),x^*(s),y^*(s),\epsilon) \right] ds \end{aligned}$$

and so, by Lemma 3.1.1,

$$\begin{aligned} &|\mathcal{F}_1(t_0,\xi,\epsilon,x,y) - \mathcal{F}_1(t_0,\xi,\epsilon,x^*,y^*)|_\rho \leq \\ &\leq \tfrac{1}{\rho}\epsilon \sup_{s\in\mathbb{R}} e^{-\rho|s|} |f(\epsilon(t_0+s),x(s),y(s),\epsilon) - f(\epsilon(t_0+s),x^*(s),y^*(s),\epsilon)| \leq \\ &\leq \tfrac{\epsilon_0}{\rho} N_1(|x-x^*|_\rho + |y-y^*|_\rho). \end{aligned}$$

moreover, since $\rho < \mu/2$, by Lemma 3.1.2,

$$\begin{aligned} &\mathcal{F}_2(t_0,\xi,\epsilon,x,y)(t) - \mathcal{F}_2(t_0,\xi,\epsilon,x^*,y^*)(t) = \\ &= \int_{-\infty}^{+\infty} U(t,s;\epsilon t_0,x^*,\epsilon) \left[(A_1(\epsilon(t_0+s),x(s)) - A_1(\epsilon(t_0+s),x^*(s))) \right. \\ &\qquad \mathcal{F}_2(t_0,\xi,\epsilon,x,y)(s) + \\ &\qquad \left. + g(\epsilon(t_0+s),x(s),y(s),\epsilon) - g(\epsilon(t_0+s),x^*(s),y^*(s),\epsilon) \right] ds \end{aligned}$$

and so, by Lemma 3.1.1,

$$\begin{aligned}
&|\mathcal{F}_2(t_0,\xi,\epsilon,x,y)-\mathcal{F}_2(t_0,\xi,\epsilon,x^*,y^*)|_\rho \leq \\
&\leq \tfrac{K}{\mu-\rho}\sup_{s\in\mathbb{R}} e^{-\rho|s|}\,|(A_1(\epsilon(t_0+s),x(s))- \\
&-A_1(\epsilon(t_0+s),x^*(s)))\mathcal{F}_2(t_0,\xi,\epsilon,x,y)(s)+ \\
&+g(\epsilon(t_0+s),x(s),y(s),\epsilon)-g(\epsilon(t_0+s),x^*(s),y^*(s),\epsilon)| \leq \\
&\leq \tfrac{K}{\mu-\rho}\left[M_2\delta\,|x-x^*|_\rho + C(\epsilon_0+\delta)(|x-x^*|_\rho+|y-y^*|_\rho)\right] \leq \\
&\leq \tfrac{K}{\mu-\rho}2\max\{M_2\delta, C(\epsilon_0+\delta)\}\,(|x-x^*|_\rho+|y-y^*|_\rho).
\end{aligned}$$

Therefore

$$|\mathcal{F}(t_0,\xi,\epsilon,x,y)-\mathcal{F}(t_0,\xi,\epsilon,x^*,y^*)|_\rho \leq \kappa\,|(x,y)-(x^*,y^*)|_\rho \tag{21}$$

where $\kappa := \left(\frac{\epsilon_0}{\rho}N_1+2\kappa_1\right)<1$.

Let $\xi\in X$, $\epsilon\in(0,\epsilon_0)$, $t_0\in\mathbb{R}$ be fixed; let $(x_0,y_0)\in\mathcal{A}$. For each $k\geq 0$, let us set

$$(x_{k+1},y_{k+1}) := \mathcal{F}(t_0,\xi,\epsilon,x_k,y_k).$$

For $m>n$ we have:

$$|(x_m,y_m)-(x_n,y_n)|_\rho \leq (\kappa^{m-1}+\ldots+\kappa^n)\,|(x_1,y_1)-(x_0,y_0)|_\rho\,,$$

hence $\{(x_k,y_k)\}_{k\in\mathbf{N}}$ is a Cauchy sequence in $BC^\rho(X)\times BC^\rho(Y^\alpha)$. So it has limit $(x,y)=(x(t_0,\xi,\epsilon),y(t_0,\xi,\epsilon))$ in $BC^\rho(X)\times BC^\rho(Y^\alpha)$. In particular, $(x_k(s),y_k(s))\to(x(s),y(s))$ uniformly on the compact intervals of $\mathbb{R}$. By the continuity of f,

$$\begin{aligned}
\mathcal{F}_1(t_0,\xi,\epsilon,x,y)(t) &= \xi+\int_0^t \epsilon f(\epsilon(t_0+s),\lim_{k\to\infty}x_k(s),\lim_{k\to\infty}y_k(s),\epsilon)ds = \\
&= \xi+\int_0^t \epsilon\lim_{k\to\infty} f(\epsilon(t_0+s),x_k(s),y_k(s),\epsilon)ds.
\end{aligned}$$

Since in the last expression the limit is uniform on the compact intervals of $\mathbb{R}$, we have

$$\begin{aligned}
\mathcal{F}_1(t_0,\xi,\epsilon,x,y)(t) &= \xi+\lim_{k\to\infty}\int_0^t \epsilon f(\epsilon(t_0+s),x_k(s),y_k(s),\epsilon)ds = \\
&= \lim_{k\to\infty}\mathcal{F}_1(t_0,\xi,\epsilon,x_k,y_k)(t) = \lim_{k\to\infty}x_{k+1}(t) = x(t).
\end{aligned}$$

As a first consequence, we obtain that $x \in C^1(\mathbb{R}, X)$ and $|\dot{x}|_0 \le \epsilon_0 N$. On the other side, since $y_k(t) \to y(t)$ uniformly on the compacts, y is continuous, and $|y|_0 \le \delta$. Therefore $(x, y) \in \mathcal{A}$.
Now, we observe that:

1. $(x_{k+1}, y_{k+1}) \to (x, y)$ in $BC^\rho(X) \times BC^\rho(Y^\alpha)$;

2. $(x_{k+1}, y_{k+1}) = \mathcal{F}(t_0, \xi, \epsilon, x_k, y_k)$;

3. $\mathcal{F}(t_0, \xi, \epsilon, x_k, y_k) \to \mathcal{F}(t_0, \xi, \epsilon, x, y)$ in $BC^\rho(X) \times BC^\rho(Y^\alpha)$, since $\mathcal{F} : \mathcal{A} \to BC^\rho(X) \times BC^\rho(Y^\alpha)$ is continuous, and $(x_k, y_k) \to (x, y) \in \mathcal{A}$.

Therefore $(x, y) = \mathcal{F}(t_0, \xi, \epsilon, x, y)$, i.e. (x, y) is a solution of (20) belonging to $\mathcal{A}$. If (x^*, y^*) is another solution of (20), with $|\dot{x}^*|_0 \le \epsilon_0 N$, $|y^*|_0 \le \delta$, we have:

$$\begin{aligned} |(x^*, y^*) - (x, y)|_\rho = |\mathcal{F}(t_0, \xi, \epsilon, x^*, y^*) - \mathcal{F}(t_0, \xi, \epsilon, x, y)|_\rho \le \\ \le \kappa \, |(x^*, y^*) - (x, y)|_\rho \, . \end{aligned}$$

Since $\kappa < 1$, it must be $(x, y) = (x^*, y^*)$.

NOTE: we could not have applied the contraction principle, because $\mathcal{A}$ is not a closed subset of $BC^\rho(X) \times BC^\rho(Y^\alpha)$. ◊

Now we are able to construct an invariant center-like manifold for $(F)_\epsilon$, and consequently one for $(S)_\epsilon$. Let φ_1, φ_2 be the maps obtained in Th. 3.1.5. We set

$$h(t_0, \xi, \epsilon) := \varphi_2(t_0, \xi, \epsilon)(0).$$

We want to show that h defines an invariant manifold for $(F)_\epsilon$. Let us fix $(t_0, \xi) \in \mathbb{R} \times X$ and set

$$\begin{aligned} \tilde{x}(t) &:= \varphi_1(t_0, \xi, \epsilon)(t - t_0) \\ \tilde{y}(t) &:= \varphi_2(t_0, \xi, \epsilon)(t - t_0). \end{aligned}$$

By Lemma 3.1.3, $(\tilde{x}(t), \tilde{y}(t))$ is a solution of $(F)_\epsilon$ on $(-\infty, +\infty)$. Moreover

$$(\tilde{x}(t_0), \tilde{y}(t_0)) = (x_0, h(t_0, \xi, \epsilon))$$

and, for all $t \in \mathbb{R}$,

$$\tilde{y}(t) = \varphi_2(t_0, \xi, \epsilon)(t - t_0) \ .$$

Now, we set

$$\begin{aligned} \tilde{\tilde{x}}(\lambda) &:= \varphi_1(t_0, \xi, \epsilon)(\lambda + t - t_0) \\ \tilde{\tilde{y}}(\lambda) &:= \varphi_2(t_0, \xi, \epsilon)(\lambda + t - t_0). \end{aligned}$$

Clearly $(\tilde{\tilde{x}}, \tilde{\tilde{y}}) \in \mathcal{A}$; moreover

$$(\tilde{\tilde{x}}(\cdot - t), \tilde{\tilde{y}}(\cdot - t)) = (\varphi_1(t_0, \xi, \epsilon)(\cdot - t_0), \varphi_2(t_0, \xi, \epsilon)(\cdot - t_0))$$

is a solution of $(F)_\epsilon$ on $\mathbb{R}$, and $\tilde{\tilde{x}}(0) = \varphi_1(t_0, \xi, \epsilon)(t - t_0)$. Then, by Th. 3.1.3, $(\tilde{\tilde{x}}, \tilde{\tilde{y}})$ solves the integral system

$$\begin{cases} x(p) = \varphi_1(t_0, \xi, \epsilon)(t - t_0) + \displaystyle\int_0^p \epsilon f(\epsilon(s + t), x(s), y(s), \epsilon) ds \\ y(p) = \displaystyle\int_{-\infty}^{+\infty} U(p, s; \epsilon t, x, \epsilon) g(\epsilon(s + t), x(s), y(s), \epsilon) ds \end{cases} .$$

By the uniqueness result in Th. 3.1.5, we have that

$$(\tilde{\tilde{x}}, \tilde{\tilde{y}}) = (\varphi_1(t, \varphi_1(t_0, \xi, \epsilon)(t - t_0), \epsilon), \varphi_2(t, \varphi_1(t_0, \xi, \epsilon)(t - t_0), \epsilon)).$$

Since

$$\begin{aligned} \tilde{\tilde{x}}(0) &= \varphi_1(t_0, \xi, \epsilon)(t - t_0) \\ \tilde{\tilde{y}}(0) &= \varphi_2(t_0, \xi, \epsilon)(t - t_0), \end{aligned}$$

we obtain

$$\begin{aligned} \varphi_1(t_0, \xi, \epsilon)(t - t_0) &= \varphi_1(t, \varphi_1(t_0, \xi, \epsilon)(t - t_0), \epsilon)(0) \\ \varphi_2(t_0, \xi, \epsilon)(t - t_0) &= \varphi_2(t, \varphi_1(t_0, \xi, \epsilon)(t - t_0), \epsilon)(0) \end{aligned} \tag{22}$$

and therefore

$$\tilde{y}(t) = h(t, \tilde{x}(t), \epsilon).$$

So we have just proved that h defines an invariant manifold for $(F)_\epsilon$. These arguments show also that each solution of $(F)_\epsilon$ belonging to $\mathcal{A}$ must lie on this invariant manifold.

As we have seen on page 168, the map

$$k(t_0, \xi, \epsilon) := h\left(\frac{t_0}{\epsilon}, \xi, \epsilon\right) = \varphi_2\left(\frac{t_0}{\epsilon}, \xi, \epsilon\right)(0)$$

defines an invariant manifold for $(S)_\epsilon$. We are mainly interested in proving smoothness and boundedness and studying the behaviour as $\epsilon \to 0$ of k rather than of h. So it is natural to introduce the maps

$$\psi_1(t_0, \xi, \epsilon) := \varphi_1\left(\tfrac{t_0}{\epsilon}, \xi, \epsilon\right)$$

$$\psi_2(t_0, \xi, \epsilon) := \varphi_2\left(\tfrac{t_0}{\epsilon}, \xi, \epsilon\right)$$

and to study the proerties of $\psi := (\psi_1, \psi_2)$ rather than those of $\varphi := (\varphi_1, \varphi_2)$. For sake of clarity, we point out that:

1. The map $t \mapsto (\psi_1(t_0, \xi, \epsilon)(t - t_0/\epsilon), \psi_2(t_0, \xi, \epsilon)(t - t_0/\epsilon))$ is a solution of $(F)_\epsilon$, and its value at t_0/ϵ is $\left(\xi, h\left(\frac{t_0}{\epsilon}, \xi, \epsilon\right)\right)$.

2. The map $t \mapsto (\psi_1(t_0, \xi, \epsilon)(t/\epsilon - t_0/\epsilon), \psi_2(t_0, \xi, \epsilon)(t/\epsilon - t_0/\epsilon))$ is a solution of $(S)_\epsilon$, and its value at t_0 is $(\xi, h(t_0, \xi, \epsilon))$.

We end this Section with two very important results.

PROPOSITION 3.1.6.

$$\lim_{\epsilon \to 0} |\psi_2(t_0, \xi, \epsilon)|_\rho = 0$$

uniformly with respect to $(t_0, \xi) \in \mathbb{R} \times X$.

Proof. By definition,

$$\begin{gathered} \psi_2(t_0, \xi, \epsilon)(t) = \\ = \int_{-\infty}^{+\infty} U(t, s; t_0, \psi_1(t_0, \xi, \epsilon), \epsilon) \\ g(t_0 + \epsilon s, \psi_1(t_0, \xi, \epsilon)(s), \psi_2(t_0, \xi, \epsilon)(s), \epsilon) ds \end{gathered}$$

and so, by Lemma 3.1.1,

$$\begin{gathered} |\psi_2(t_0, \xi, \epsilon)|_\rho \leq \tfrac{K}{\mu - \rho} \sup_{s \in \mathbb{R}} e^{-\rho|s|} \,| \\ g(t_0 + \epsilon s, \psi_1(t_0, \xi, \epsilon)(s), \psi_2(t_0, \xi, \epsilon)(s), \epsilon)| \leq \\ \leq \tfrac{K}{\mu - \rho} \sup_{s \in \mathbb{R}} e^{-\rho|s|} C(\epsilon + |\psi_2(t_0, \xi, \epsilon)(s)|^2) \leq \tfrac{KC}{\mu - \rho}\epsilon + \tfrac{KC\delta}{\mu - \rho} |\psi_2(t_0, \xi, \epsilon)|_\rho . \end{gathered}$$

Since $\frac{KC\delta}{\mu-\rho} < 1$, we obtain that

$$|\psi_2(t_0, \xi, \epsilon)|_\rho \leq \left(1 - \frac{KC\delta}{\mu - \rho}\right)^{-1} \frac{KC}{\mu - \rho}\epsilon,$$

hence the conclusion. ◊

REMARK.

As a consequence of this Proposition we have that $|k(t_0, \xi, \epsilon)|_\alpha \underset{\epsilon\to 0}{\longrightarrow} 0$, uniformly with respect to $(t_0, \xi) \in \mathbb{R} \times X$.

PROPOSITION 3.1.7. *The map* $\psi : \mathbb{R} \times X \times (0, \epsilon_0) \to BC^\rho(X) \times BC^\rho(Y^\alpha)$ *is globally Lipschitz continuous.*

Proof. We know that ψ_1 and ψ_2 satisfy:

$$\psi_1(t_0, \xi, \epsilon)(t) = \xi + \int_0^t \epsilon f(t_0 + \epsilon s, \psi_1(t_0, \xi, \epsilon)(s), \psi_2(t_0, \xi, \epsilon)(s), \epsilon) ds$$
$$\psi_2(t_0, \xi, \epsilon)(t) = \int_{-\infty}^{+\infty} U(t, s; t_0, \psi_1(t_0, \xi, \epsilon), \epsilon) \, g(t_0 + \epsilon s, \psi_1(t_0, \xi, \epsilon)(s), \psi_2(t_0, \xi, \epsilon)(s), \epsilon) ds$$

i.e. $\psi = (\psi_1, \psi_2)$ satisfies

$$\psi(t_0, \xi, \epsilon) = \mathcal{F}\left(\tfrac{t_0}{\epsilon}, \xi, \epsilon, \psi(t_0, \xi, \epsilon)\right).$$

Then we set

$$\mathcal{G}_i(t_0, \xi, \epsilon, x, y) := \mathcal{F}_i\left(\tfrac{t_0}{\epsilon}, \xi, \epsilon, x, y\right); i := 1, 2$$
$$\mathcal{G} := (\mathcal{G}_1, \mathcal{G}_2).$$

Clearly,

$$\mathcal{G} : \mathbb{R} \times X \times (0, \epsilon_0) \times \mathcal{A} \to BC^\rho(X) \times BC^\rho(Y^\alpha)$$

is a contraction on $\mathcal{A}$, uniform with respect to (t_0, ξ, ϵ) (cf. (21)). If we show that $\mathcal{G}$ is Lipschitz continuous in (t_0, ξ, ϵ), with Lipschitz constant L independent of $(x, y) \in \mathcal{A}$, then clearly ψ is Lipschitz continuous. In fact

$$|\psi(t_0, \xi, \epsilon) - \psi(t_0^*, \xi^*, \epsilon^*)|_\rho =$$
$$= |\mathcal{G}(t_0, \xi, \epsilon, \psi(t_0, \xi, \epsilon)) - \mathcal{G}(t_0^*, \xi^*, \epsilon^*, \psi(t_0^*, \xi^*, \epsilon^*))|_\rho \leq$$
$$\leq |\mathcal{G}(t_0, \xi, \epsilon, \psi(t_0, \xi, \epsilon)) - \mathcal{G}(t_0^*, \xi^*, \epsilon^*, \psi(t_0, \xi, \epsilon))|_\rho +$$
$$+ |\mathcal{G}(t_0^*, \xi^*, \epsilon^*, \psi(t_0, \xi, \epsilon)) - \mathcal{G}(t_0^*, \xi^*, \epsilon^*, \psi(t_0^*, \xi^*, \epsilon^*))|_\rho \leq$$
$$\leq L\,|(t_0, \xi, \epsilon) - (t_0^*, \xi^*, \epsilon^*)| + \kappa\,|\psi(t_0, \xi, \epsilon) - \psi(t_0^*, \xi^*, \epsilon^*)|.$$

Since $\kappa < 1$, it follows that ψ is Lipschitz continuous. So we have to prove that $\mathcal{G}_1$ and $\mathcal{G}_2$ are Lipschitz continuous.

$$\begin{gathered}\mathcal{G}_1(t_0,\xi,\epsilon,x,y)(t) - \mathcal{G}_1(t_0^*,\xi^*,\epsilon^*,x,y)(t) = \\ = \xi - \xi^* + (\epsilon-\epsilon^*)\int_0^t f(t_0+\epsilon s, x(s), y(s), \epsilon)ds + \\ +\epsilon^*\int_0^t \left[f(t_0+\epsilon s, x(s), y(s), \epsilon) - f(t_0^*+\epsilon^* s, x(s), y(s), \epsilon^*)\right] ds\end{gathered}$$

and therefore

$$\begin{gathered}\left|\mathcal{G}_1(t_0,\xi,\epsilon,x,y) - \mathcal{G}_1(t_0^*,\xi^*,\epsilon^*,x,y)\right|_\rho \le \\ \le |\xi-\xi^*| + |\epsilon-\epsilon^*|\, N \sup_{t\in\mathbb{R}} |t|\, e^{-\rho|t|} + \\ +\epsilon_0 \sup|Df| \sup_{t\in\mathbb{R}} e^{-\rho|t|}\left|\int_0^t \left(|t_0-t_0^*| + |s|\,|\epsilon-\epsilon^*| + |\epsilon-\epsilon^*|\right) ds\right| \le \\ \le \text{const.}\left(|\xi-\xi^*| + |t_0-t_0^*| + |\epsilon-\epsilon^*|\right).\end{gathered}$$

As for $\mathcal{G}_2$, since $\rho < \mu/2$, by Lemmas 3.1.2 and 3.1.1, we have:

$$\begin{gathered}\mathcal{G}_2(t_0,\xi,\epsilon,x,y)(t) - \mathcal{G}_2(t_0^*,\xi^*,\epsilon^*,x,y)(t) = \\ \int_{-\infty}^{+\infty} U(t,s;t_0^*,x,\epsilon^*)\left[(A_1(t_0+\epsilon s, x(s)) + \right. \\ -A_1(t_0^*+\epsilon^* s, x(s)))\mathcal{G}_2(t_0,\xi,\epsilon,x,y)(s) + \\ \left. +g(t_0+\epsilon s, x(s), y(s), \epsilon) - g(t_0^*+\epsilon^* s, x(s), y(s), \epsilon^*)\right] ds\end{gathered}$$

and therefore

$$\begin{gathered}\left|\mathcal{G}_2(t_0,\xi,\epsilon,x,y) - \mathcal{G}_2(t_0^*,\xi^*,\epsilon^*,x,y)\right|_\rho \le \\ \le \sup_{t\in\mathbb{R}} e^{-\rho|t|}\left[M_2(|t_0-t_0^*| + |\epsilon-\epsilon^*|\,|t|)\delta + \right. \\ \left. +\sup|Dg|\left(|t_0-t_0^*| + |\epsilon-\epsilon^*|\,|t| + |\epsilon-\epsilon^*|\right]\right. \le \text{const.}(|t_0-t_0^*| + |\epsilon-\epsilon^*|).\end{gathered}$$

$\diamondsuit$

Remark.

An immediate consequence of this Proposition is that the map k is Lipschitz continuous. Infact $k = ev_0 \circ \psi_2$, where $ev_0 : BC^\rho(Y^\alpha) \to Y^\alpha$ is the "evaluation map at 0", which is linear and bounded.

3.2. Smoothness.

In this section we prove that the map k belongs to $C^r_b(\mathbb{R}\times X\times(0,\epsilon_0),Y^\alpha)$. Namely, we shall prove that

$$\psi\in C^r_b(\mathbb{R}\times X\times(0,\epsilon_0),BC^{\rho rb}(\mathbb{R},X)\times BC^{\rho rb}(\mathbb{R},Y^\alpha)).$$

We shall apply Theorem 2.1 in [Ry1]. First, we proceed heurystically: ψ satisfies

$$\psi(t_0,\xi,\epsilon)=\mathcal{G}(t_0,\xi,\epsilon,\psi(t_0,\xi,\epsilon)).$$

Differentiating formally this expression, we obtain that $D^k\psi$ should be expressed as as a finite sum involving the derivatives $D^j_{form.}\mathcal{G}$ evaluated at $(t_0,\xi,\epsilon,\psi(t_0,\xi,\epsilon))$. Since

$$\mathcal{G}_1(t_0,\xi,\epsilon,x,y)(t):=\xi+\int_0^t\epsilon f(t_0+\epsilon s,x(s),y(s),\epsilon)ds,$$

it is natural to set

$$\begin{gathered}D_{form.}\mathcal{G}_1(t_0,\xi,\epsilon,x,y)\,(t_0^1,\xi^1,\epsilon^1,x^1,y^1)\,(t):=\\=\xi^1+\int_0^t(\epsilon^1f(t_0+\epsilon s,x(s),y(s),\epsilon)+\\+\epsilon Df(t_0+\epsilon s,x(s),y(s),\epsilon)\,(t_0^1+\epsilon^1s,x^1(s),y^1(s),\epsilon^1))ds\end{gathered}$$

and, for $j>1$,

$$\begin{gathered}D^j_{form.}\mathcal{G}_1(t_0,\xi,\epsilon,x,y)\left[(t_0^i,\xi^i,\epsilon^i,x^i,y^i)\right]_{i=1,\dots,j}(t):=\\ \int_0^t\Big(\epsilon D^jf(t_0+\epsilon s,x(s),y(s),\epsilon)\left[(t_0^i+\epsilon^is,x^i(s),y^i(s),\epsilon^i)\right]_{i=1,\dots,j}+\\ \sum_{l=1}^j\epsilon^lD^{j-1}f(t_0+\epsilon s,x(s),y(s),\epsilon)\left[(t_0^i+\epsilon^is,x^i(s),y^i(s),\epsilon^i)\right]_{i=1,\dots,j}^{i\neq l}\Big)ds.\end{gathered}$$

As for $D^j_{form.}\mathcal{G}_2$, we observe that, if $s\mapsto g(t_0+\epsilon s,x(s),y(s),\epsilon)$ were locally Hoelder continuous, by Lemma 3.1.1 we should have:

$$\frac{d}{dt}\mathcal{G}_2(t_0,\xi,\epsilon,x,y)(t)=A(t_0+\epsilon t,x(t))\mathcal{G}_2(t_0,\xi,\epsilon,x,y)(t)+$$

$$+g(t_0+\epsilon t,x(t),y(t),\epsilon).$$

By formal differentiation we obtain:

$$\frac{d}{dt}\left(D^{j}_{form}\mathcal{G}_2(t_0,\xi,\epsilon,x,y)\left[(t_0^i,\xi^i,\epsilon^i,x^i,y^i)\right]_{i=1,\dots,j}\right)(t)=$$
$$=A(t_0+\epsilon t,x(t))D^{j}_{form.}\mathcal{G}_2(t_0,\xi,\epsilon,x,y)\left[(t_0^i,\xi^i,\epsilon^i,x^i,y^i)\right]_{i=1,\dots,j}(t)+$$
$$+\sum_{(N,M)\in S'}D^{\#N}_{1.}A(t_0+\epsilon t,x(t))\left[(t_0^i+\epsilon^i t,x^i(t))\right]_{i\in N}$$
$$D^{\#M}_{form.}\mathcal{G}_2(t_0,\xi,\epsilon,x,y)\left[(t_0^i,\xi^i,\epsilon^i,x^i,y^i)\right]_{i\in M}(t)+$$
$$+D^j g(t_0+\epsilon t,x(t),y(t),\epsilon)\left[(t_0^i+\epsilon^i t,x^i(t),y^i(t),\epsilon^i)\right]_{i=1,\dots,j}$$

where

$$S\ :=\{(N,M)\,|N,M\subseteq\{1,...,j\}\,,N\cap M=\emptyset,N\cup M=\{1,...,j\}\}$$
$$S':=\{(N,M)\in S\,|N\neq\emptyset\}$$

Lemma 3.1.3 suggests to set:

$$D^{0}_{form.}\mathcal{G}_2(t_0,\xi,\epsilon,x,y)\left[\left(t_0^i,\xi^i,\epsilon^i,x^i,y^i\right)\right]_{i\in\emptyset}:=\mathcal{G}_2(t_0,\xi,\epsilon,x,y)$$

and, for $1\leq j\leq r$,

$$D^{j}_{form.}\mathcal{G}_2(t_0,\xi,\epsilon,x,y)\left[(t_0^i,\xi^i,\epsilon^i,x^i,y^i)\right]_{i=1,\dots,j}(t):=$$
$$=\int_{-\infty}^{+\infty}U(t,s;t_0,x,\epsilon)\Big[\sum_{(N,M)\in S'}D^{\#N}A_1(t_0+\epsilon s,x(s))$$
$$\left[(t_0^i+\epsilon^i s,x^i(s))\right]_{i\in N}D^{\#M}_{form.}\mathcal{G}_2(t_0,\xi,\epsilon,x,y)\left[(t_0^i,\xi^i,\epsilon^i,x^i,y^i)\right]_{i\in M}(s)+$$
$$+D^j g(t_0+\epsilon s,x(s),y(s),\epsilon)\left[(t_0^i+\epsilon^i s,x^i(s),y^i(s),\epsilon^i)\right]_{i=1,\dots,j}\Big]ds\ .$$

From now, to simplify notations, we write z and z^i instead of (x,y) and (x^i,y^i) respectively. For $k:=1,2$, let us set

$$f_{0,k}(t_0,\xi,\epsilon):=\mathcal{G}_k(t_0,\xi,\epsilon,\psi(t_0,\xi,\epsilon))$$

and, for $j\geq 1$,

$$f_{j,k}(t,\xi,\epsilon)\left[\left(t_0^i,\xi^i,\epsilon^i,z^i\right)\right]_{i=1,\dots,j}:=D^{j}_{form.}\mathcal{G}_k(t_0,\xi,\epsilon,\psi(t_0,\xi,\epsilon))$$
$$\left[\left(t_0^i,\xi^i,\epsilon^i,z^i\right)\right]_{i=1,\dots,j}.$$

Note that we have not specified the domain of the $f_{j,k}$-s. This will be made clear in the following Lemmas, which describe the main properties of the $f_{j,k}$-s.

Let, $N, N_1, M_2, b, \rho, \epsilon_0, \delta, \kappa_1$ be the constants obtained in Lemma 3.1.4, $\kappa := \frac{N_1}{\rho}\epsilon_0 + 2\kappa_1 < 1$, $c := \max\left\{1, \sup_{s\in\mathbb{R}} e^{-\rho|s|}\,|s|\right\}$ and let $a \in \mathbb{R}$, $0 < a < \mu/2$. Let us set $\bar{M} := \sup_{j,t,x}\left|D^j A_1(t,x)\right|$.

Lemma 3.2.1. *There is a constant $C' = C'(a)$ such that, for each $j \in \{1, ..., r\}$, for every j-tuple $(\eta_1, ..., \eta_j)$ of real positive numbers, with $\rho \le \eta_i$ $(i := 1, ..., j)$ and $\eta := \eta_1 + ... + \eta_j \le a$, for all $\xi, \xi^1, ..., \xi^j \in X$, all $t_0, t_0^1, ..., t_0^1 \in \mathbb{R}$, all $\epsilon \in (0, \epsilon_0)$, $\epsilon^1, ..., \epsilon^j \in \mathbb{R}$, and all $z^1, ..., z^j$, with $z^i \in BC^{\eta_i}(X) \times BC^{\eta_i}(Y^\alpha)$ ($i := 1, ..., j$), and for $k = 1, 2$, the map*

$$f_{j,k}(t_0, \xi, \epsilon)\left[\left(t_0^i, \xi^i, \epsilon^i, z^i\right)\right]_{i=1,...,j}$$

is well defined, belongs to $BC^\eta(X)$ for $k = 1$, to $BC^\eta(Y^\alpha)$ for $k = 2$, and

$$\left|f_{j,k}(t_0, \xi, \epsilon)\left[\left(t_0^i, \xi^i, \epsilon^i, z^i\right)\right]_{i=1,...,j}\right|_\eta \le C' \prod_{i=1}^{j}\left(\left|\left(t_0^i, \xi^i, \epsilon^i\right)\right| + \left|z^i\right|_{\eta_i}\right).$$

Proof.
Case $k = 1$: we apply Lemma 3.1.1, a).
Case $k = 2$: induction on $j := 1, ..., r$ and Lemma 3.1.1, b). ♢

Lemma 3.2.2. *Let $j \in \{0, ..., r\}$, let $(\eta_1, ..., \eta_j)$ be a j-tuple of real positive numbers, with $\eta_i \ge \rho$ $(j := 1, ..., r)$; set $\eta := \rho$ if $j = 0$, $\eta := \rho + \eta_1 + ... + \eta_j$ if $j \ge 1$, let $\eta \le a$. Then, for each $\zeta > \eta$ and for $k = 1, 2$,*

$$\Big|(f_{j,k}(t_0 + \vartheta, \xi + h, \epsilon + \chi) - f_{j,k}(t_0, \xi, \epsilon))$$

$$\left[\left(t_0^i, \xi^i, \epsilon^i, z^i\right)\right]_{i=1,...,j}\Big|_\zeta \xrightarrow[(\vartheta,h,\chi)\to 0]{} 0$$

uniformly with respect to $(t_0^i, \xi^i, \epsilon^i, z^i)$ with $|t_0^i| + |\xi^i| + |\epsilon^i| + |z^i|_{\eta_i} \le 1$.

Proof.
Case $k = 1$.
We set $D^{-1}f := 0$, and, for $j := 0, ..., r$,

$$B(\vartheta, h, \chi) := \Big| \left(f_{j,1}(t_0 + \vartheta, \xi + h, \epsilon + \chi) - f_{j,1}(t_0, \xi, \epsilon) \right)$$

$$\left[\left(t_0^i, \xi^i, \epsilon^i, z^i \right) \right]_{i=1,...,j} \Big|_{\zeta} .$$

By applying Lemma 3.1.1, by adding and subtracting terms, and by Lemma 1.5.1, we obtain

$$\begin{gathered} B(\vartheta, h, \chi) \leq |h| + \tfrac{(2c)^j \epsilon_0 \bar{M}}{\zeta} \sup_{s \in \mathbb{R}} e^{\left(-\zeta + \sum_{i=1}^{j} \eta_i\right)|s|} \Big(|\chi| + \\ \left| + D^j f\left((t_0 + \vartheta) + (\epsilon + \chi)s, \psi(t_0 + \vartheta, \xi + h, \epsilon + \chi)(s), \epsilon + \chi \right) + \right. \\ \left. - D^j f\left(t_0 + \epsilon s, \psi(t_0, \xi, \epsilon)(s), \epsilon \right) \right| + \\ + j \left| D^{j-1} f\left((t_0 + \vartheta) + (\epsilon + \chi)s, \psi(t_0 + \vartheta, \xi + h, \epsilon + \chi)(s), \epsilon + \chi \right) + \right. \\ \left. - D^{j-1} f\left(t_0 + \epsilon s, \psi(t_0, \xi, \epsilon) \right) \right| \Big) \underset{(\vartheta, h, \chi) \to 0}{\longrightarrow} 0. \end{gathered}$$

Case $k = 2$: induction on $j := 0, ..., r$.
If $j = 0$, by Lipschitz continuity of ψ, since $\zeta > \eta := \rho$, we have

$$\begin{gathered} \left| f_{0,2}(t + \vartheta, \xi + h, \epsilon + \chi) - f_{0,2}(t, \xi, \epsilon) \right|_{\zeta} = \\ = \left| \psi_2(t_0 + \vartheta, \xi + h, \epsilon + \chi) - \psi_2(t_0, \xi, \epsilon) \right|_{\zeta} \leq \\ \leq \left| \psi_2(t_0 + \vartheta, \xi + h, \epsilon + \chi) - \psi_2(t_0, \xi, \epsilon) \right|_{\rho} \underset{(\vartheta, h, \chi) \to 0}{\longrightarrow} 0. \end{gathered}$$

Now, let $1 \leq j \leq r$, and let us suppose that, for each n, $0 \leq n \leq j-1$, the assertion is true; for j we have

$$\begin{gathered} f_{j,2}(t_0 + \vartheta, \xi + h, \epsilon + \chi) \left[\left(t_0^i, \xi^i, \epsilon^i, z^i \right) \right]_{i=1,...,j} (t) = \\ = \int_{-\infty}^{+\infty} U\left(t, s; t_0 + \vartheta, \psi_1(t_0 + \vartheta, \xi + h, \epsilon + \chi), \epsilon + \chi \right) w_1(s) ds \end{gathered}$$

where

$$w_1(s) :=$$
$$= \sum_{(N,M)\in S'} D^{\# N} A_1 \left((t_0+\vartheta)+(\epsilon+\chi)s, \psi_1(t_0+\vartheta,\xi+h,\epsilon+\chi)(s)\right)$$
$$\left[(t_0^i+\epsilon^i s, x^i(s))\right]_{i\in N}$$
$$f_{\# M,2}(t_0+\vartheta,\xi+h,\epsilon+\chi)\left[(t_0^i,\xi^i,\epsilon^i,z^i)\right]_{i\in M}(s)+$$
$$+D^j g\left((t_0+\vartheta)+(\epsilon+\chi)s, \psi_1(t_0+\vartheta,\xi+h,\epsilon+\chi)(s),\epsilon+\chi\right)$$
$$\left[(t_0^i+\epsilon^i s, z^i(s),\epsilon^i)\right]_{i=1,\dots,j}.$$

By Lemma 3.2.1, $w_1 \in BC^{\sum_{i=1}^{j}\eta_i}(Y) \subseteq BC^a(Y)$. Since $0<a<\mu/2$, we can apply Lemma 3.1.2, to obtain

$$f_{j,2}(t_0+\vartheta,\xi+h,\epsilon+\chi)\left[(t_0^i,\xi^i,\epsilon^i,z^i)\right]_{i=1,\dots,j}(t) =$$
$$= \int_{-\infty}^{+\infty} U\left(t,s;t_0,\psi_1(t_0,\xi,\epsilon),\epsilon\right) w_2(s)ds$$

where

$$w_2(s) :=$$
$$= \left(A_1((t_0+\vartheta)+(\epsilon+\chi)s, \psi_1(t_0+\vartheta,\xi+h,\epsilon+\chi)(s))+\right.$$
$$\left.-A_1(t_0+\epsilon s, \psi_1(t_0,\xi,\epsilon)(s))\right)$$
$$f_{j,2}(t_0+\vartheta,\xi+h,\epsilon+\chi)\left[(t_0^i,\xi^i,\epsilon^i,z^i)\right]_{i=1,\dots,j}(s)+w_1(s).$$

So

$$\left(f_{j,2}(t_0+\vartheta,\xi+h,\epsilon+\chi)-f_{j,2}(t_0,\xi,\epsilon)\right)\left[(t_0^i,\xi^i,\epsilon^i,z^i)\right]_{i=1,\dots,j}(t) =$$
$$= \int_{-\infty}^{+\infty} U\left(t,s;t_0,\psi_1(t_0,\xi,\epsilon),\epsilon\right) w(s)ds$$

where

$$\begin{aligned} & w(s) := \\ & = (A_1((t_0+\vartheta)+(\epsilon+\chi)s, \psi_1(t_0+\vartheta,\xi+h,\epsilon+\chi)(s)) \\ & -A_1(t_0+\epsilon s, \psi_1(t_0,\xi,\epsilon)(s))) \\ & f_{j,2}(t_0+\vartheta,\xi+h,\epsilon+\chi)\left[(t_0^i,\xi^i,\epsilon^i,z^i)\right]_{i=1,\dots,j}(s) \\ & + \sum_{(N,M)\in S'} \left\{B^1_{(N,M)}(s)+B^2_{(N,M)}(s)\right\} + \\ & + \left(D^j g\left((t_0+\vartheta)+(\epsilon+\chi)s, \psi(t_0+\vartheta,\xi+h,\epsilon+\chi)(s), \epsilon+\chi\right) + \right. \\ & -D^j g\left(t_0+\epsilon s, \psi(t_0,\xi,\epsilon)(s),\epsilon\right)\left[(t_0^i+\epsilon^i s, z^i(s),\epsilon^i)\right]_{i=1,\dots,j} \end{aligned}$$

with

$$\begin{aligned} & B^1_{(N,M)}(s) := \\ & = D^{\# N} A_1\left((t_0+\vartheta)+(\epsilon+\chi)s, \psi_1(t_0+\vartheta,\xi+h,\epsilon+\chi)(s)\right) \\ & \left[(t_0^i+\epsilon^i s, x^i(s))\right]_{i\in N} \\ & f_{\# M,2}(t_0+\vartheta,\xi+h,\epsilon+\chi)\left[(t_0^i,\xi^i,\epsilon^i,z^i)\right]_{i\in M}(s) + \\ & -D^{\# N} A_1\left(t_0+\epsilon s, \psi_1(t_0,\xi,\epsilon)(s)\right)\left[(t_0^i+\epsilon^i s, x^i(s))\right]_{i\in N} \\ & f_{\# M,2}(t_0+\vartheta,\xi+h,\epsilon+\chi)\left[(t_0^i,\xi^i,\epsilon^i,z^i)\right]_{i\in M}(s) \end{aligned}$$

and with

$$\begin{aligned} B^2_{(N,M)}(s) &:= D^{\# N} A_1\left(t_0+\epsilon s, \psi_1(t_0,\xi,\epsilon)(s)\right)\left[(t_0^i+\epsilon^i s, x^i(s))\right]_{i\in N} \\ & f_{\# M,2}(t_0+\vartheta,\xi+h,\epsilon+\chi)\left[(t_0^i,\xi^i,\epsilon^i,z^i)\right]_{i\in M}(s) - \\ & -D^{\# N} A_1\left(t_0+\epsilon s, \psi_1(t_0,\xi,\epsilon)(s)\right)\left[(t_0^i+\epsilon^i s, x^i(s))\right]_{i\in N} \\ & f_{\# M,2}(t_0,\xi,\epsilon)\left[(t_0^i,\xi^i,\epsilon^i,z^i)\right]_{i\in M}(s). \end{aligned}$$

By Lemma 3.1.1,

$$\begin{aligned} & \left|\left(f_{j,2}(t_0+\vartheta,\xi+h,\epsilon+\chi) - f_{j,2}(t_0,\xi,\epsilon)\right)\left[\left(t_0^i,\xi^i,\epsilon^i,z^i\right)\right]_{i=1,\dots,j}\right|_\zeta \\ & \le \frac{K}{\mu-\zeta}\,|w|_\zeta\,. \end{aligned}$$

So, we have to estimate the norm $|\cdot|_\zeta$ of the terms in w. This is easily done using Lemma 1.5.1, Lemma 3.2.1 and the inductive hypothesis. $\diamondsuit$

LEMMA 3.2.3. Let $j \in \{0, ..., r-1\}$, let $(\eta_1, ..., \eta_j)$ be a j-tuple of positive real numbers, with $\eta_i \geq \rho$ $(i := 1, ..., j)$; set $\eta := 2\rho$ if $j = 0$, $\eta := 2\rho + \eta_1 + ... + \eta_j$ if $j \neq 0$, let $\eta \leq a$. Then, for each $\zeta > \eta$ and for $k = 1, 2$ we have that

$$\Big|(f_{j,k}(t_0+\vartheta, \xi+h, \epsilon+\chi) - f_{j,k}(t_0, \xi, \epsilon))\left[(t_0^i, \xi^i, \epsilon^i, z^i)\right]_{i=1,...,j} + \\ -f_{j+1,k}(t_0, \xi, \epsilon)\,(\vartheta, k, \chi, \psi(t_0+\vartheta, \xi+h, \epsilon+\chi) \\ -\psi(t_0, \xi, \epsilon))\left[(t_0^i, \xi^i, \epsilon^i, z^i)\right]_{i=1,...,j}\Big|_\zeta = \\ = o\left(|(\vartheta, h, \chi)|\right)$$

as $(\vartheta, h, \chi) \to 0$, uniformly with respect to $(t_0^i, \xi^i, \epsilon^i, z^i)$ with $|t_0^i| + |\xi^i| + |\epsilon^i| + |z^i|_{\eta_i} \leq 1$.

Proof.

Case $k = 1$.

Let us set

$$B(t) := (f_{j,2}(t_0+\vartheta, \xi+h, \epsilon+\chi) - f_{j,2}(t_0, \xi, \epsilon) + \\ -f_{j+1,2}(t_0, \xi, \epsilon)\,(\vartheta, k, \chi, \psi(t_0+\vartheta, \xi+h, \epsilon+\chi) + \\ -\psi(t_0, \xi, \epsilon)))\left[(t_0^i, \xi^i, \epsilon^i, z^i)\right]_{i=1,...,j}(t);$$

by definition of $f_{j,2}$ and $f_{j+1,2}$, recasting the terms in this expression, we have

$$\begin{gathered}B(t)=\int_0^t\Big\{\epsilon\left(D^jf((t_0+\vartheta)+(\epsilon+\chi)s,\right.\\ \psi(t_0+\vartheta,\xi+h,\epsilon+\chi)(s),\epsilon+\chi)+\\ -D^jf(t_0+\epsilon s,\psi(t_0,\xi,\epsilon)(s),\epsilon)+\\ -D^{j+1}f(t_0+\epsilon s,\psi(t_0,\xi,\epsilon)(s),\epsilon)\\ (\vartheta+\chi s,\psi(t_0+\vartheta,\xi+h,\epsilon+\chi)(s)-\psi(t_0,\xi,\epsilon)(s),\chi))\\ \left[(t_0^i+\epsilon^i s,z^i(s),\epsilon^i)\right]_{i=1,\dots,j}+\sum_{l=1}^j B^l(s)+\\ +\chi\left(D^jf((t_0+\vartheta)+(\epsilon+\chi)s,\psi(t_0+\vartheta,\xi+h,\epsilon+\chi)(s),\epsilon+\chi)+\right.\\ \left.-D^jf(t_0+\epsilon s,\psi(t_0,\xi,\epsilon)(s),\epsilon)\right)\left[(t_0^i+\epsilon^i s,z^i(s),\epsilon^i)\right]_{i=1,\dots,j}\Big\}ds\end{gathered}$$

where

$$\begin{gathered}B^l(s):=\sum_{l=1}^j\epsilon_l\Big(D^{j-1}f((t_0+\vartheta)+(\epsilon+\chi)s,\\ \psi(t_0+\vartheta,\xi+h,\epsilon+\chi)(s),\epsilon+\chi)+\\ -D^{j-1}f(t_0+\epsilon s,\psi(t_0,\xi,\epsilon)(s),\epsilon)\\ -D^jf(t_0+\epsilon s,\psi(t_0,\xi,\epsilon)(s),\epsilon)\\ (\vartheta+\chi s,\psi(t_0+\vartheta,\xi+h,\epsilon+\chi)(s)\\ -\psi(t_0,\xi,\epsilon)(s),\chi\left[(t_0^i+\epsilon^i s,z^i(s),\epsilon^i)\right]_{i=1,\dots,j}^{i\neq l}\Big).\end{gathered}$$

By Lemma 3.1.1, we reduce to estimate the norm $|\cdot|_\zeta$ of the addends in the above expressions: this is easily done by using the Mean Value Theorem, Lemma 1.5.1 and Lemma 3.2.2 and the fact that ψ is Lipschitz continuous.

Case $k=2$: induction on $j:=0,...,r-1$.

Let $j\in\{0,...,r-1\}$, and let us suppose that the assert is true for each j', $0\leq j'<j$. Since $0<a<\mu/2$, applying Lemma 3.2.1 and

Lemma 3.1.2, and adding and subtracting terms, we obtain

$$\begin{aligned}
&(f_{j,2}(t_0+\vartheta,\xi+h,\epsilon+\chi)-f_{j,2}(t_0,\xi,\epsilon)+\\
&-f_{j+1,2}(t_0,\xi,\epsilon)\,(\vartheta,h,\chi,\psi(t_0+\vartheta,\xi+h,\epsilon+\chi)+\\
&-\psi(t_0,\xi,\epsilon)))\left[(t_0^i,\xi^i,\epsilon^i,z^i)\right]_{i=1,\dots,j}(t)=\\
&=\int_{-\infty}^{+\infty}U(t,s;t_0,\psi_1(t_0,\xi,\epsilon),\epsilon)w_1(s)ds
\end{aligned}$$

where

$$w_1(s)=D(s)+\sum_{(N,M)\in S'}E_{(N,M)}(s)+Q(s)$$

with

$$\begin{aligned}
D(s):=\ &(A_1((t_0+\vartheta)+(\epsilon+\chi)s,\psi_1(t_0+\vartheta,\xi+h,\epsilon+\chi)(s))+\\
&-A_1(t_0+\epsilon s,\psi_1(t_0,\xi,\epsilon)(s)))\,f_{j,2}(t_0+\vartheta,\xi+h,\epsilon+\chi)\\
&\left[(t_0^i,\xi^i,\epsilon^i,z^i)\right]_{i=1,\dots,j}(s)+\\
&-DA_1(t_0+\epsilon s,\psi_1(t_0,\xi,\epsilon)(s))\\
&(\vartheta+\chi s,\psi_1(t_0+\vartheta,\xi+h,\epsilon+\chi)(s)-\psi_1(t_0,\xi,\epsilon)(s))\\
&f_{j,2}(t_0,\xi,\epsilon)\left[(t_0^i,\xi^i,\epsilon^i,z^i)\right]_{i=1,\dots,j}(s);
\end{aligned}$$

$$\begin{aligned}
E_{(N,M)}(s)=\ &\big(D^{\#N}A_1((t_0+\vartheta)+(\epsilon+\chi)s,\psi_1(t_0+\vartheta,\xi+h,\epsilon+\chi)(s))+\\
&-D^{\#N}A_1(t_0+\epsilon s,\psi_1(t_0,\xi,\epsilon)(s))+\\
&-D^{\#N+1}A_1(t_0+\epsilon s,\psi_1(t_0,\xi,\epsilon)(s))\,(\vartheta+\chi s,\psi_1(t_0+\vartheta,\xi+h,\epsilon+\chi)(s)+\\
&-\psi_1(t_0,\xi,\epsilon)(s)))\left[(t_0^i+\epsilon^i s,x^i(s))\right]_{i\in N}f_{\#M,2}(t_0,\xi,\epsilon)\left[(t_0^i,\xi^i,\epsilon^i,z^i)\right]_{i\in M}(s)+\\
&+D^{\#N}A_1(t_0+\epsilon s,\psi_1(t_0,\xi,\epsilon)(s))\left[(t_0^i+\epsilon^i s,x^i(s))\right]_{i\in N}\\
&(f_{\#M,2}(t_0+\vartheta,\xi+h,\epsilon+\chi)-f_{\#M,2}(t_0,\xi,\epsilon)-f_{\#M+1,2}(t_0,\xi,\epsilon)\\
&(\vartheta,h,\chi,\psi(t_0+\vartheta,\xi+h,\epsilon+\chi)-\psi(t_0,\xi,\epsilon)))\left[(t_0^i,\xi^i,\epsilon^i,z^i)\right]_{i\in M}(s)+\\
&+\big(D^{\#N}A_1((t_0+\vartheta)+(\epsilon+\chi)s,\psi_1(t_0+\vartheta,\xi+h,\epsilon+\chi)(s))+\\
&-D^{\#N}A_1(t_0+\epsilon s,\psi_1(t_0,\xi,\epsilon)(s))\big)\left[(t_0^i+\epsilon^i s,x^i(s))\right]_{i\in N}\\
&(f_{\#M,2}(t_0+\vartheta,\xi+h,\epsilon+\chi)-f_{\#M,2}(t_0,\xi,\epsilon))\left[(t_0^i,\xi^i,\epsilon^i,z^i)\right]_{i\in M}(s).
\end{aligned}$$

and

$$Q(s) := (D^j g((t_0+\vartheta)+(\epsilon+\chi)s, \psi(t_0+\vartheta,\xi+h,\epsilon+\chi)(s), \epsilon+\chi)+$$
$$-D^j g(t_0+\epsilon s, \psi(t_0,\xi,\epsilon)(s),\epsilon)+$$
$$-D^{j+1}g(t_0+\epsilon s, \psi(t_0,\xi,\epsilon)(s),\epsilon)\,(\vartheta+\chi s, \quad \psi(t_0+\vartheta,\xi+h,\epsilon+\chi)(s)+$$
$$-\psi(t_0,\xi,\epsilon)(s), \chi\,[(t_0^i+\epsilon^i s, x^i(s),\epsilon^i)]_{i=1,\dots,j}\,.$$

By Lemma 3.1.1, we reduce to estimate the norm $|\cdot|_\zeta$ of the single addends of w_1. This is easily done by applying the Mean Value Theorem, Lemma 3.2.2, and Lemma 1.5.1, and by using the inductive hypothesis and the fact that ψ is Lipschitz continuous. ♢

LEMMA 3.2.4. *For each ζ with $\rho \le \zeta \le \rho r b$, for all $\epsilon \in (0,\epsilon_0)$, for all $\xi \in X$, for all $t_0 \in \mathbb{R}$, and for all $z^1 \in BC^\rho(X) \times BC^\rho(Y^\alpha)$,*

$$\left|f_{1,1}\left(t_0,\xi,\epsilon\right)\left(0,0,0,z^1\right)\right|_\zeta + \left|f_{1,2}\left(t_0,\xi,\epsilon\right)\left(0,0,0,z^1\right)\right|_\zeta \le \kappa\left|z^1\right|_\zeta .$$

Proof. the assert is a direct consequence of the definition of $f_{1,1}$, $f_{1,2}$ and κ and of Lemma 3.1.1. ♢

Now we are able to apply Th. 2.1 of [Ry1] and to prove that ψ is smooth.

THEOREM 3.2.5.

$$\psi \in C^r\left(\mathbb{R}\times X\times(0,\epsilon_0), BC^{\rho r b}(X)\times BC^{\rho r b}(Y^\alpha)\right)$$

and $D^j\psi$ is bounded for $1 \le j \le r$.

Proof. We apply Th. 2.1 of [Ry1], with

1. $U := \mathbb{R}\times X\times(0,\epsilon_0)$, $E_s := BC^{\rho s}(X)\times BC^{\rho s}(Y^\alpha)$, $s \in \{1,2,\dots,r\}\cup\{b,2b,\dots,rb\}$;
2. $f_j\left(t_0,\xi,\epsilon\right)\left[(t_0^i,\xi^i,\epsilon^i,z^i)\right]_{i=1,\dots,j} :=$
   $= \Big(f_{j,1}\left(t_0,\xi,\epsilon\right)\left[(t_0^i,\xi^i,\epsilon^i,z^i)\right]_{i=1,\dots,j}, f_{j,2}\left(t_0,\xi,\epsilon\right)$
   $\left[(t_0^i,\xi^i,\epsilon^i,z^i)\right]_{i=1,\dots,j}$, $\quad j := 1,\dots,r$;

3. $M := C'(a)$, for $a := \rho r b$.

(H1) is trivially satisfied;
(H2)1. and **(H2)**2. are satisfied by Lemma 3.2.1;
(H2)3. is satisfied by Lemma 3.2.2, provided $b > 2$;
(H2)4. and **(H2)**5. are satisfied by Lemma 3.2.3, provided $b > 2$;
(H2)6. is satisfied by Lemma 3.2.4. ◊

We have defined

$$k\left(t_0, \xi, \epsilon\right) := \psi_2\left(t_0, \xi, \epsilon\right)(0) = \left(\pi_{Y^\alpha} \circ ev_0 \circ \psi\right)\left(t_0, \xi, \epsilon\right)$$

where $\pi_{Y^\alpha} : X \times Y^\alpha \to Y^\alpha$ is the projection on the first factor, and $ev_0 : BC^{\rho r b}(X) \times BC^{\rho r b}(Y^\alpha) \to X \times Y^\alpha$ is the evaluation map at 0 (i.e. $ev_0(\beta) := \beta(0)$). So $k \in C^r\left(\mathbb{R} \times X \times (0, \epsilon_0), Y^\alpha\right)$, and $D^j k : \mathbb{R} \times X \times (0, \epsilon_0) \to \mathcal{L}^j\left(\mathbb{R} \times X \times \mathbb{R}, Y^\alpha\right)$ is bounded for $0 \le j \le r$. Finally, since $k\left(t_0, \xi, \epsilon\right) \underset{\epsilon \to 0}{\longrightarrow} 0$ uniformly with respect to $\left(t_0, \xi\right)$, we can conclude that $k\left(\cdot, \cdot, \epsilon\right) \underset{\epsilon \to 0}{\longrightarrow} 0$ in $C_b^{r-1}\left(\mathbb{R} \times X, Y^\alpha\right)$(cf. page 167, when we discussed the properties of g).

Remark.

More general results about regularity can be obtained, just looking more carefully at the proof of Th. A in [Ry1]. For instance, if f and g are of class C^r with respect to (x, y), but only locally Hoelder continuous in t and Hoelder continuous in ϵ (together with their (x, y)-derivatives), we still obtain existence of an invariant manifold, which now is the graph of a map, again of class C^r with respect to x, locally Hoelder continuous in t and Hoelder continuous in ϵ (together with its x-derivatives).

4. Stability of the invariant manifold.

As usual, we consider the system

$$(S)_\epsilon \begin{cases} \dot{x} = f(t, x, y, \epsilon) \\ \epsilon \dot{y} = A(t, x) y + g(t, x, y, \epsilon) \end{cases} .$$

We assume hypotheses **(SP)**. Moreover, in this section we assume that the spectrum of $A(t,x)$ satisfies

$$Re\sigma(A(t,x)) \leq -2\mu.$$

This implies that equation

$$\dot{y} = A(t_0 + \epsilon t, x(t))y$$

has a trivial dichotomy (see Lemma 2.1.1, and [He], Ths. 7.4.1, 7.4.2). In particular, if $\epsilon_2, \tilde{N}, M$ and μ are as in Lemma 2.1.1, then for each $x \in C^1(\mathbb{R}, X)$ with $|\dot{x}|_0 \leq \tilde{N}$ and for each $0 < \epsilon \leq \epsilon_2$, and for $t \geq s$, it holds

$$|T(t,s;t_0,x,\epsilon)y|_\alpha \leq Me^{-\mu(t-s)}(t-s)^{-\alpha}\,|y| \,\mathrm{forall} y \in Y;$$

$$|T(t,s;t_0,x,\epsilon)y|_\alpha \leq Me^{-\mu(t-s)}\,|y|_\alpha \text{ forall } y \in Y^\alpha.$$

Finally, let $b, \rho, \epsilon_0, \delta$ be as in Lemma 3.1.4, with the further condition

$$\epsilon_0 + \delta < \frac{1}{MC}\frac{\mu^{1/(1-\alpha)}}{2^{1/(1-\alpha)}\Gamma(1-\alpha)}.$$

First of all, we recall that for every $(t_0, x_0, y_0) \in \mathbb{R} \times X \times B^\alpha_\delta$ there exists a unique maximal solution of $(S)_\epsilon$,

$$(x,y) : [t_0, t_1) \to X \times B^\alpha_\delta$$

with $(x(t_0), y(t_0)) = (x_0, y_0)$. In this Section we show that, if ϵ and $|y_0 - k(t_0, x_0, \epsilon)|_\alpha$ are sufficiently small, then such solution is defined on $[0, +\infty)$, and satisfies

$$|y(t) - k(t, x(t), \epsilon)|_\alpha \leq \mathrm{const.}e^{-\frac{\mu}{2\epsilon}(t-t_0)}\,|y_0 - k(t_0, x_0, \epsilon)|_\alpha \,.$$

We say that C_ϵ is uniformly exponentially attractive.

4.1. Attractivity of C_ϵ.

We procede by steps.

LEMMA 4.1.1. *Let* $(t_0, x_0, y_0) \in \mathbb{R} \times X \times B^\alpha_{\delta_0}$ *and let* $(x, y) : [t_0, t_1) \to X \times B^\alpha_\delta$ *be the maximal solution of* $(S)_\epsilon$ *with* $(x(t_0), y(t_0)) = (x_0, y_0)$. *Let us set*

$$\hat{x}(t) := x(\epsilon t)$$
$$\hat{y}(t) := y(\epsilon t);$$

(therefore $(\hat{x}, \hat{y}) : [t_0/\epsilon, t_1/\epsilon) \to X \times B^\alpha_{\delta_0}$ *is the maximal solution of* $(F)_\epsilon$*, with* $(\hat{x}(t_0/\epsilon),\ \hat{y}(t_0/\epsilon)) = (x_0, y_0)$*). Let* $\varphi := (\varphi_1, \varphi_2)$ *be the map constructed in Th. 3.1.5, and let L be a Lipschitz constant for k. Then, for* $t_0/\epsilon \le s \le t < t_1/\epsilon$*, it holds*

$$|\hat{x}(s) - \varphi_1(t, \hat{x}(t), \epsilon)(s - t)| \le$$

$$\le \epsilon N_1(1 + L) \int_s^t e^{\epsilon N_1(2+L)(p-s)} \left|\hat{y}(p) - k(\epsilon p, \hat{x}(p), \epsilon)\right|_\alpha dp.$$

Proof. Definition of φ_1 and Th. 1.1.1 (Gronwall inequality). $\diamondsuit$

LEMMA 4.1.2. *Let us assume the same hypotheses as in Lemma 4.1.1; then, for* $s \le t_0/\epsilon \le t < t_1/\epsilon$*, it holds*

$$\left|\varphi_1(t, \hat{x}(t), \epsilon)(s - t) - \varphi_1\left(\tfrac{t_0}{\epsilon}, \hat{x}\left(\tfrac{t_0}{\epsilon}\right), \epsilon\right)\left(s - \tfrac{t_0}{\epsilon}\right)\right| \le$$

$$\le \epsilon N_1(1 + L) \int_{t_0/\epsilon}^t e^{\epsilon N_1(2+L)(p-s)} \left|\hat{y}(p) - k\left(\epsilon p, \hat{x}(p), \epsilon\right)\right|_\alpha dp.$$

Proof. For $s \le t_0/\epsilon \le t < t_1/\epsilon$, we have

$$\varphi_1(t, \hat{x}(t), \epsilon)(s - t) = \varphi_1(t, \hat{x}(t), \epsilon)\left(\tfrac{t_0}{\epsilon} - t\right) +$$

$$+ \int_{t_0/\epsilon}^s \epsilon f\left(\epsilon p, \varphi_1(t, \hat{x}(t), \epsilon)(p - t), \varphi_2(t, \hat{x}(t), \epsilon)(p - t), \epsilon\right) dp$$

and

$$\varphi_1\left(\tfrac{t_0}{\epsilon}, \hat{x}\left(\tfrac{t_0}{\epsilon}\right), \epsilon\right)\left(s - \tfrac{t_0}{\epsilon}\right) = \hat{x}\left(\tfrac{t_0}{\epsilon}\right) +$$

$$+ \int_{t_0/\epsilon}^s \epsilon f\left(\epsilon p, \varphi_1\left(\tfrac{t_0}{\epsilon}, \hat{x}\left(\tfrac{t_0}{\epsilon}\right), \epsilon\right)\left(p - \tfrac{t_0}{\epsilon}\right), \varphi_2\left(\tfrac{t_0}{\epsilon}, \hat{x}\left(\tfrac{t_0}{\epsilon}\right), \epsilon\right)\left(p - \tfrac{t_0}{\epsilon}\right), \epsilon\right) dp.$$

Therefore

$$\begin{gathered}\left|\varphi_1(t,\hat{x}(t),\epsilon)(s-t)-\varphi_1\left(\tfrac{t_0}{\epsilon},\hat{x}\left(\tfrac{t_0}{\epsilon}\right),\epsilon\right)\left(s-\tfrac{t_0}{\epsilon}\right)\right|\leq\\ \leq\left|\hat{x}\left(\tfrac{t_0}{\epsilon}\right)-\varphi_1(t,\hat{x}(t),\epsilon)\left(\tfrac{t_0}{\epsilon}-t\right)\right|+\\ +\epsilon N_1\left|\int_{t_0/\epsilon}^{s}\left(\left|\varphi_1\left(\tfrac{t_0}{\epsilon},\hat{x}\left(\tfrac{t_0}{\epsilon}\right),\epsilon\right)\left(p-\tfrac{t_0}{\epsilon}\right)+\right.\right.\\ -\varphi_1(t,\hat{x}(t),\epsilon)(p-t)\left|+\right|\varphi_2\left(\tfrac{t_0}{\epsilon},\hat{x}\left(\tfrac{t_0}{\epsilon}\right),\epsilon\right)\left(p-\tfrac{t_0}{\epsilon}\right)+\\ \left.\left.\left|-\varphi_2(t,\hat{x}(t),\epsilon)(p-t)\right|_\alpha\right)\right|dp\Bigg|\ .\end{gathered}$$

By Th. 1.1.1, we get

$$\begin{gathered}\left|\varphi_1(t,\hat{x}(t),\epsilon)(s-t)-\varphi_1\left(\tfrac{t_0}{\epsilon},\hat{x}\left(\tfrac{t_0}{\epsilon}\right),\epsilon\right)\left(s-\tfrac{t_0}{\epsilon}\right)\right|\leq\\ \leq e^{\epsilon N_1(t_0/\epsilon-s)}\left|\hat{x}\left(\tfrac{t_0}{\epsilon}\right)-\varphi_1(t,\hat{x}(t),\epsilon)\left(\tfrac{t_0}{\epsilon}-t\right)\right|+\\ +\epsilon N_1 L\left|\int_{t_0/\epsilon}^{s}e^{\epsilon N_1(p-s)}\right.\\ \left.\varphi_1(t,\hat{x}(t),\epsilon)(p-t)-\varphi_1\left(\tfrac{t_0}{\epsilon},\hat{x}\left(\tfrac{t_0}{\epsilon}\right),\epsilon\right)\left(p-\tfrac{t_0}{\epsilon}\right)dp\right|.\end{gathered}$$

By multiplying both sides by $e^{\epsilon N_1 s}$, applying again Th. 1.1.1, and multiplying both sides by $e^{-\epsilon N_1 s}$, we get

$$\begin{gathered}\left|\varphi_1(t,\hat{x}(t),\epsilon)(s-t)-\varphi_1\left(\tfrac{t_0}{\epsilon},\hat{x}\left(\tfrac{t_0}{\epsilon}\right),\epsilon\right)\left(s-\tfrac{t_0}{\epsilon}\right)\right|\leq\\ \leq e^{\epsilon N_1(t_0/\epsilon-s)}\left|\hat{x}\left(\tfrac{t_0}{\epsilon}\right)-\varphi_1(t,\hat{x}(t),\epsilon)\left(\tfrac{t_0}{\epsilon}-t\right)\right|e^{\epsilon N_1 L(t_0/\epsilon-s)}=\\ =e^{\epsilon N_1(1+L)(t_0/\epsilon-s)}\left|\hat{x}\left(\tfrac{t_0}{\epsilon}\right)-\varphi_1(t,\hat{x}(t),\epsilon)\left(\tfrac{t_0}{\epsilon}-t\right)\right|.\end{gathered}$$

Finally, by Lemma 4.1.1, we get

$$\begin{gathered}\left|\varphi_1(t,\hat{x}(t),\epsilon)(s-t)-\varphi_1\left(\tfrac{t_0}{\epsilon},\hat{x}\left(\tfrac{t_0}{\epsilon}\right),\epsilon\right)\left(s-\tfrac{t_0}{\epsilon}\right)\right|\leq\\ \leq\epsilon N_1(1+L)e^{\epsilon N_1(1+L)(t_0/\epsilon-s)}\int_{t_0/\epsilon}^{t}e^{\epsilon N_1(2+L)(p-t_0/\epsilon)}\\ \left|\hat{y}(p)-k\left(\epsilon p,\hat{x}(p),\epsilon\right)\right|_\alpha dp\leq\\ \leq\epsilon N_1(1+L)\int_{t_0/\epsilon}^{t}e^{\epsilon N_1(2+L)(p-s)}\left|\hat{y}(p)-k\left(\epsilon p,\hat{x}(p),\epsilon\right)\right|_\alpha dp.\end{gathered}$$

$\diamondsuit$

Lemma 4.1.3. *Assume the hypotheses of the previous Lemmas. There exists $\tilde{M} = \tilde{M}(\alpha)$ and ϵ^* such that, if $\epsilon < \epsilon^*$, and for $t_0 \leq t < t_1$,*

$$|y(t) - k(t, x(t), \epsilon)|_\alpha \leq \frac{M\tilde{M}}{1-\alpha} |y_0 - k(t_0, x_0, \epsilon)|_\alpha \, e^{-\frac{\mu}{2\epsilon}(t-t_0)}.$$

Proof. We procede by steps.
Preliminary step.
Let $t_0 < \tilde{t}_1 < t_1$; let us set

$$\tilde{x}(t) := \begin{cases} x_0 + \dot{x}(t_0)(t - t_0) \text{ for } t < t_0 \\ x(t) \text{ for } t_0 \leq t \leq \tilde{t}_1 \\ x(\tilde{t}_1) + \dot{x}(\tilde{t}_1)(t - t_1) \text{ for } t > \tilde{t}_1 \end{cases}$$

It is easily seen that $\tilde{x}$ is of class C^1 on $\mathbb{R}$ and that $\left|\frac{d}{dt}\widehat{\tilde{x}}\right|_0 \leq \epsilon_0 N$.

1^{st} step.
By definition of φ_2 and by Lemma 3.1.2, it follows that, for $t_0/\epsilon \leq t < \tilde{t}_1/\epsilon$,

$$\begin{gathered} \varphi_2(t, \hat{x}(t), \epsilon)(\tau) = \\ = \int_{-\infty}^{\tau} T(\tau, \sigma; \epsilon t, \varphi_1(t, \hat{x}(t), \epsilon)(\cdot), \epsilon) g\,(\epsilon(t+\sigma), \varphi_1(t, \hat{x}(t), \epsilon)(\sigma), \\ \varphi_2(t, \hat{x}(t), \epsilon)(\sigma), \epsilon)\, d\sigma = \\ = \int_{-\infty}^{\tau} T(\tau, \sigma; \epsilon t, \widehat{\tilde{x}}(\cdot + t), \epsilon)\, [(A_1(\epsilon(t+\sigma), \varphi_1(t, \hat{x}(t), \epsilon)(\sigma)) + \\ - A_1(\epsilon(t+\sigma), \widehat{\tilde{x}}(t+\sigma)) \\ \varphi_2(t, \hat{x}(t), \epsilon)(\sigma) + g\,(\epsilon(t+\sigma), \varphi_1(t, \hat{x}(t), \epsilon)(\sigma), \\ \varphi_2(t, \hat{x}(t), \epsilon)(\sigma), \epsilon)]\, d\sigma = \\ = \int_{-\infty}^{\tau+t} T(\tau, \sigma - t; \epsilon t, \widehat{\tilde{x}}(\cdot + t), \epsilon)\, [(A_1(\epsilon\sigma, \varphi_1(t, \hat{x}(t), \epsilon)(\sigma - t)) + \\ - A_1(\epsilon\sigma, \widehat{\tilde{x}}(\sigma))\Big) \\ \varphi_2(t, \hat{x}(t), \epsilon)(\sigma - t) + g\,(\epsilon\sigma, \varphi_1(t, \hat{x}(t), \epsilon)(\sigma - t), \\ \varphi_2(t, \hat{x}(t), \epsilon)(\sigma - t), \epsilon)]\, d\sigma. \end{gathered}$$

Then, by adding and subtracting terms, and recalling the definition of k (see also the proof of Lemma 3.1.3), for $t_0/\epsilon \leq t < \tilde{t}_1/\epsilon$, we have:

$$k(\epsilon t, x(\epsilon t), \epsilon) = T(t, t_0/\epsilon; 0, \widehat{\tilde{x}}(\cdot), \epsilon) k(t_0, x_0, \epsilon) +$$
$$+ \int_{t_0/\epsilon}^{t} T(t, \sigma; 0, \widehat{\tilde{x}}(\cdot), \epsilon)$$
$$\Big[\Big(A_1(\epsilon\sigma, \varphi_1(t, \hat{x}(t), \epsilon)(\sigma - t)) - A_1(\epsilon\sigma, \widehat{\tilde{x}}(\sigma)) \Big)$$
$$k\,(\epsilon\sigma, \varphi_1(t, \hat{x}(t), \epsilon)(\sigma - t), \epsilon) -$$
$$+ g\,(\epsilon\sigma, \varphi_1(t, \hat{x}(t), \epsilon)(\sigma - t), k\,(\epsilon\sigma, \varphi_1(t, \hat{x}(t), \epsilon)(\sigma - t), \epsilon)\,, \epsilon)]\, d\sigma +$$
$$+ \int_{-\infty}^{t_0/\epsilon} T(t, \sigma; 0, \widehat{\tilde{x}}(\cdot), \epsilon)\, [(A_1(\epsilon\sigma, \varphi_1(t, \hat{x}(t), \epsilon)(\sigma - t)) +$$
$$- A_1(\epsilon\sigma, \varphi_1(t_0/\epsilon, \hat{x}(t_0/\epsilon), \epsilon)(\sigma - t_0/\epsilon))) +$$
$$k\,(\epsilon\sigma, \varphi_1(t, \hat{x}(t), \epsilon)(\sigma - t), \epsilon) +$$
$$+ A_1(\epsilon\sigma, \varphi_1(t_0/\epsilon, \hat{x}(t_0/\epsilon), \epsilon)(\sigma - t_0/\epsilon)) +$$
$$(k\,(\epsilon\sigma, \varphi_1(t, \hat{x}(t), \epsilon)(\sigma - t), \epsilon) +$$
$$- k\,(\epsilon\sigma, \varphi_1(t_0/\epsilon, \hat{x}(t_0/\epsilon), \epsilon)(\sigma - t_0/\epsilon), \epsilon)) + A_1(\epsilon\sigma, \widehat{\tilde{x}}(\sigma))$$
$$(k\,(\epsilon\sigma, \varphi_1(t_0/\epsilon, \hat{x}(t_0/\epsilon), \epsilon)(\sigma - t_0/\epsilon), \epsilon) +$$
$$- k\,\,(\epsilon\sigma, \varphi_1(t, \hat{x}(t), \epsilon)(\sigma - t), \epsilon)) +$$
$$+ g\,(\epsilon\sigma, \varphi_1(t, \hat{x}(t), \epsilon)(\sigma - t), k(\epsilon\sigma, \varphi_1(t, \hat{x}(t), \epsilon)(\sigma - t), \epsilon), \epsilon) +$$
$$- g\,(\epsilon\sigma, \varphi_1(t_0/\epsilon, \hat{x}(t_0/\epsilon), \epsilon)(\sigma - t_0/\epsilon),$$
$$k(\epsilon\sigma, \varphi_1(t_0/\epsilon, \hat{x}(t_0/\epsilon), \epsilon)(\sigma - t_0/\epsilon), \epsilon), \epsilon)]\, d\sigma.$$

Moreover, by 1.2.1, and by definition of $\tilde{x}$, for $t_0/\epsilon \le t < \tilde{t}_1/\epsilon$,

$$\hat{y}(t) =$$
$$T(t, t_0/\epsilon; 0, \widehat{\tilde{x}}(\cdot), \epsilon) y_0 + \int_{t_0/\epsilon}^{t} T(t, \sigma; 0, \widehat{\tilde{x}}(\cdot), \epsilon) g(\epsilon\sigma, \hat{x}(\sigma), \hat{y}(\sigma), \epsilon) d\sigma.$$

2^{nd} step.
By definition of $\tilde{x}$, for $t_0/\epsilon \le t < \tilde{t}_1/\epsilon$, we have

$$\begin{gathered}
e^{\mu t} \left|\hat{y}(t) - k(\epsilon t, \hat{x}(t), \epsilon)\right|_\alpha \le M e^{\mu t_0/\epsilon} \left|y_0 - k(t_0, x_0, \epsilon)\right|_\alpha + \\
+ M \left(M_2 \delta + C\left(\epsilon_0 + \delta\right)(1+L)\right) \\
\int_{t_0/\epsilon}^{t} e^{\mu\sigma} (t-\sigma)^{-\alpha} \left|\varphi_1\left(t, \hat{x}\left(t\right), \epsilon\right)(\sigma - t) - \hat{x}(\sigma)\right| d\sigma + \\
+ MC\left(\epsilon_0 + \delta\right) \int_{t_0/\epsilon}^{t} e^{\mu\sigma} (t-\sigma)^{-\alpha} \left|\hat{y}(\sigma) - k(\epsilon\sigma, \hat{x}(\sigma), \epsilon)\right|_\alpha d\sigma + \\
+ M\left(M_2\delta + 2L \sup\left|A_1\right| + C\left(\epsilon_0+\delta\right)(1+L)\right) \\
\int_{-\infty}^{t_0/\epsilon} e^{\mu\sigma} \left(t-\sigma\right)^{-\alpha} \left|\varphi_1\left(t, \hat{x}\left(t\right), \epsilon\right)(\sigma - t) + \right. \\
\left. - \varphi_1\left(\tfrac{t_0}{\epsilon}, \hat{x}\left(\tfrac{t_0}{\epsilon}\right), \epsilon\right)\left(\sigma - \tfrac{t_0}{\epsilon}\right)\right| d\sigma.
\end{gathered}$$

Note that in this last inequality there is no dependence on the choice of $\tilde{t}_1$. So it holds for each $t_0/\epsilon \le t < t_1/\epsilon$.

3^{rd} step.
By Lemmas 4.1.1 and 4.1.2 and by steps 1 and 2 of the present proof, we have that, provided $\mu > \epsilon N_1(2+L)$,

$$\begin{gathered}
e^{\mu t} \left|\hat{y}(t) - k(\epsilon t, \hat{x}(t), \epsilon)\right|_\alpha \le M e^{\mu t_0/\epsilon} \left|y_0 - k(t_0, x_0, \epsilon)\right|_\alpha + \\
+ \Big\{ MC\left(\epsilon_0 + \delta\right) + 2M\left[M_2\delta + C\left(\epsilon_0 + \delta\right)(1+L) + \right. \\
\left. + L\sup\left|A_1\right|\right] \tfrac{\epsilon N_1(1+L)}{\mu - \epsilon N_1(2+L)} \Big\} \\
\int_{t_0/\epsilon}^{t} e^{\mu s} \left(t-s\right)^{-\alpha} \left|\hat{y}(s) - k(\epsilon s, \hat{x}(s), \epsilon)\right|_\alpha ds.
\end{gathered}$$

By Th. 1.1.1, we may conclude that, for sufficiently small ϵ, and for some $\tilde{M} = \tilde{M}(\alpha)$,

$$\left|y(t) - k(t, x(t), \epsilon)\right|_\alpha \le \frac{M\tilde{M}}{1-\alpha} e^{-\frac{\mu}{2\epsilon}(t-t_0)} \left|y_0 - k(t_0, x_0, \epsilon)\right|_\alpha . \qquad \diamondsuit$$

We can resume the above reults in the following theorem.

Theorem 4.1.4. *Let us assume hypotheses of Lemma 4.1.1; moreover, let us assume that $\epsilon < \epsilon^*$ as in Th. 4.1.3. Then, if ϵ*

satisfies also

$$\epsilon < \frac{1}{3}\left(1 - \frac{KC\delta}{\mu-\rho}\right)\left(1 + \frac{M\tilde{M}}{1-\alpha}\right)^{-1}\frac{\mu-\rho}{KC}\delta,$$

if $(t_0, x_0, y_0) \in \mathbb{R} \times X \times Y^\alpha$, and if

$$|y_0|_\alpha < \min\left\{\frac{1}{2}\frac{1-\alpha}{M\tilde{M}}\delta, \delta\right\},$$

then the solution (x, y) of $(S)_\epsilon$ with $(x(t_0), y(t_0)) = (x_0, y_0)$ is defined on $[t_0, \infty)$, with $|y(t)|_\alpha < \delta$ and

$$|y(t) - k(t, x(t), \epsilon)|_\alpha \le \frac{M\tilde{M}}{1-\alpha} e^{-\frac{\mu}{2\epsilon}(t-t_0)} |y_0 - k(t_0, x_0, \epsilon)|_\alpha .$$

Moreover, for all $(t_0, x_0) \in \mathbb{R} \times X$,

$$|k(t_0, x_0, \epsilon)|_\alpha \le \frac{1}{2}\frac{1-\alpha}{M\tilde{M}}\delta$$

and, if $t - t_0 > \max\left\{\frac{2\epsilon}{\mu}\log\left(\frac{6M\tilde{M}}{1-\alpha}\right), 0\right\}$, then

$$|y(t)|_\alpha \le \frac{1}{2}\frac{1-\alpha}{M\tilde{M}}\delta. \qquad \diamondsuit$$

Remark.

This theorem has the following geometric iterpretation: there is a strip $\mathbb{S} := X \times B^\alpha_q(0) \subseteq X \times Y^\alpha$ (where $q := \min\left\{\frac{1}{2}\frac{1-\alpha}{M\tilde{M}}\delta, \delta\right\}$) such that, for sufficiently small ϵ,

1. the manifold $C_{\epsilon,t} := \{(x, k(t, x, \epsilon)) \,|\, x \in X\}$ is contained in $\mathbb{S}$ for all $t \in \mathbb{R}$;

2. for all $(x_0, y_0) \in \mathbb{S}$, and all $t_0 \in \mathbb{R}$, the solution (x, y) of $(S)_\epsilon$ with $(x(t_0), y(t_0)) = (x_0, y_0)$ is defined on $[t_0, \infty)$, and satisfies:

   (a) $|y(t) - k(t, x(t), \epsilon)|_\alpha \le \frac{M\tilde{M}}{1-\alpha} e^{-\frac{\mu}{2\epsilon}(t-t_0)} |y_0 - k(t_0, x_0, \epsilon)|_\alpha$;

   (b) $y(t) \in \mathbb{S}$ for $t > t_0 + \max\left\{\frac{2\epsilon}{\mu}\log\right.$.

4.2. Asymptotic phase.

Let $\mathbb{S}$ the strip defined in the above remark, and let us assume the hypotheses of Th. 4.1.4. In this Section we show that, if ϵ is sufficiently small, there exists a constant $Q = Q(\epsilon) > 0$ such that, if $t_0 \in \mathbb{R}$, $(x_0, y_0) \in \mathbb{S}$ and (x, y) is the solution of $(S)_\epsilon$ with $(x(t_0), y(t_0)) = (x_0, y_0)$ (defined on $[t_0, +\infty)$ by Th. 4.1.4!), then there is a unique solution $(\bar{x}, \bar{y})$ of $(S)_\epsilon$ with $(\bar{x}(t), \bar{y}(t)) \in C_{\epsilon,t}$ for all $t \in \mathbb{R}$, such that, for each $t \geq t_0$,

$$|x(t) - \bar{x}(t)| + |y(t) - \bar{y}(t)|_\alpha \leq Q\,|y_0 - k(t_0, x_0, \epsilon)|_\alpha\, e^{-\frac{\mu}{2\epsilon}(t-t_0)}.$$

We say also that $(S)_\epsilon$ has asymptotic phase. The idea for the construction is this: we look for a constant $Q' = Q'(\epsilon) > 0$ and a map $z \in BC^{-\mu/2\epsilon}(\mathbb{R}_+, X)$, with $|z|_{-\mu/2\epsilon} \leq Q'\,|y_0 - k(t_0, x_0, \epsilon)|$, such that $t \mapsto x(t) + z(t - t_0)$ solves

$$(R)_\epsilon\ \dot{w} = f(t, w, k(t, w, \epsilon), \epsilon)$$

on $(t_0, +\infty)$.
Let $z \in BC^{-\mu/2\epsilon}(\mathbb{R}_+, X)$; we set, for $t \in \mathbb{R}_+$, $z_{t_0}(t) := z(t - t_0)$. It is obvious that $x + z_{t_0}$ solves $(R)_\epsilon$ on $(t_0, +\infty)$ iff

$$\begin{gathered}\dot{z}_{t_0}(t) = \\ = D_x f(t, x(t), y(t), \epsilon) z_{t_0}(t) + [f(t, x(t) + z_{t_0}(t), k(t, x(t) + z_{t_0}(t), \epsilon), \epsilon) + \\ - f(t, x(t), y(t), \epsilon) - D_x f(t, x(t), y(t), \epsilon) z_{t_0}(t)]\end{gathered}$$

on $(t_0, +\infty)$.
Let $W(t, s)$ be the solution operator for

$$\dot{w} = D_x f(t, x(t), y(t), \epsilon) w$$

on $(t_0, +\infty)$. Since $D_x f(t, x(t), y(t), \epsilon) \in \mathcal{L}(X, X)$, $W(t, s)$ is defined for $t_0 < s, t < +\infty$. Moreover, for all $x \in X$, all $t_0 < s, t < +\infty$,

$$W(t, s)x = x + \int_s^t D_x f(\sigma, x(\sigma), y(\sigma), \epsilon) W(\sigma, s) x\, d\sigma,$$

so that

$$|W(t, s)x| \leq |x| + \left|\int_s^t N_1\, |W(\sigma, s)x|\, d\sigma\right|$$

and, by Gronwall inequality,

$$|W(t,s)x| \leq e^{N_1|t-s|}\,|x|\,.$$

LEMMA 4.2.1. *Let $z \in BC^{-\mu/2\epsilon}(\mathbb{R}_+, X)$. the following facts are equivalent:*

1. z satisfies

$$\begin{gathered}\dot z_{t_0}(t) = D_x f(t,x(t),y(t),\epsilon) z_{t_0}(t) + \\ +\left[f(t,x(t)+z_{t_0}(t),k(t,x(t)+z_{t_0}(t),\epsilon),\epsilon)+\right. \\ \left.-f(t,x(t),y(t),\epsilon) - D_x f(t,x(t),y(t),\epsilon) z_{t_0}(t)\right]\end{gathered}$$

   on $(t_0,+\infty)$;

2. z satisfies

$$\begin{gathered}z(t) = \\ = -\int_t^\infty W(t+t_0,\sigma+t_0)\left[f(\sigma+t_0,x(\sigma+t_0)+\right. \\ +z(\sigma),k(t_0,x(\sigma+t_0)+z(\sigma),\epsilon),\epsilon)+ \\ -f(\sigma+t_0,x(\sigma+t_0),y(\sigma+t_0),\epsilon)+ \\ \left.-D_x f(\sigma+t_0,x(\sigma+t_0),y(\sigma+t_0),\epsilon) z(\sigma)\right] d\sigma\end{gathered}$$

   on $(0,+\infty)$.

Proof. The proof is left to the reader. ◇

THEOREM 4.2.2. *Let $t_0, x_0, y_0, \epsilon, (x,y)$ as above. Assume $\epsilon < \frac{\mu}{2}\frac{1}{(3+L)N_1}$. Then there exists a unique $z \in BC^{-\mu/2\epsilon}(\mathbb{R}_+, X)$, such that*

$$\begin{gathered}z(t) = \\ = -\int_t^\infty W(t+t_0,\sigma+t_0)\left[f(\sigma+t_0,x(\sigma+t_0)+\right. \\ +z(\sigma),k(t_0,x(\sigma+t_0)+z(\sigma),\epsilon),\epsilon)+ \\ -f(\sigma+t_0,x(\sigma+t_0),y(\sigma+t_0),\epsilon)+ \\ \left.-D_x f(\sigma+t_0,x(\sigma+t_0),y(\sigma+t_0),\epsilon) z(\sigma)\right] d\sigma.\end{gathered}$$

Moreover

$$|z|_{-\mu/2\epsilon} \leq \frac{\frac{M\tilde{M}}{1-\alpha}}{\frac{\mu}{2\epsilon N_1} - (3+L)} |y_0 - k(t_0, x_0, \epsilon)|_\alpha . \tag{23}$$

Proof. We define $\mathcal{K} : BC^{-\mu/2\epsilon}(\mathbb{R}_+, X) \to BC^{-\mu/2\epsilon}(\mathbb{R}_+, X)$ by

$$\begin{gathered}\mathcal{K}z(t) := \\ = -\int_t^\infty W(t+t_0, \sigma+t_0)\,[f(\sigma+t_0, x(\sigma+t_0)+ \\ +z(\sigma), k(t_0, x(\sigma+t_0)+z(\sigma), \epsilon), \epsilon)+ \\ -f(\sigma+t_0, x(\sigma+t_0), y(\sigma+t_0), \epsilon)+ \\ -D_x f(\sigma+t_0, x(\sigma+t_0), y(\sigma+t_0), \epsilon) z(\sigma)]\, d\sigma.\end{gathered}$$

We have:

$$\begin{gathered}|\mathcal{K}z(t)| \leq \\ \leq \int_t^\infty e^{N_1(\sigma-t)} e^{-\frac{\mu}{2\epsilon}\sigma} \Big(N_1(2+L)\,|z|_{-\mu/2\epsilon} + \\ +N_1 \tfrac{M\tilde{M}}{1-\alpha} |k(t_0, x_0, \epsilon) - y_0|_\alpha\Big)\, d\sigma = \\ = e^{-N_1 t}\Big((2+L)\,|z|_{-\mu/2\epsilon} + \tfrac{M\tilde{M}}{1-\alpha} |k(t_0, x_0, \epsilon) - y_0|_\alpha\Big) \\ \tfrac{N_1}{\mu/2\epsilon - N_1} e^{\left(-\frac{\mu}{2\epsilon}+N_1\right)t} = \\ = \tfrac{N_1}{\mu/2\epsilon - N_1}\Big((2+L)\,|z|_{-\mu/2\epsilon} + \tfrac{M\tilde{M}}{1-\alpha} |k(t_0, x_0, \epsilon) - y_0|_\alpha\Big) e^{-\frac{\mu}{2\epsilon}t}.\end{gathered} \tag{24}$$

So $\mathcal{K}$ is well defined from $BC^{-\mu/2\epsilon}(\mathbb{R}_+, X)$ into itself.
Now, let $z_1, z_2 \in BC^{-\mu/2\epsilon}(\mathbb{R}_+, X)$. We have:

$$\begin{gathered}|(\mathcal{K}z_1 - \mathcal{K}z_2)(t)| = \Big|\int_t^\infty W(t+t_0, \sigma+t_0) \\ [f(\sigma+t_0, x(\sigma+t_0)+z_1(\sigma), k(t_0, x(\sigma+t_0)+z_1(\sigma), \epsilon), \epsilon)+ \\ -f(\sigma+t_0, x(\sigma+t_0)+z_2(\sigma), k(t_0, x(\sigma+t_0)+z_2(\sigma), \epsilon), \epsilon)+ \\ +D_x f(\sigma+t_0, x(\sigma+t_0), y(\sigma+t_0), \epsilon)\,(z_2(\sigma) - z_1(\sigma))]\, d\sigma\Big| \leq \\ \leq \int_t^\infty e^{N_1(\sigma-t)}\Big(N_1\,|z_1 - z_2|_{-\mu/2\epsilon}\, e^{-\frac{\mu}{2\epsilon}\sigma} + N_1 L\,|z_1 - z_2|_{-\mu/2\epsilon}\, e^{-\frac{\mu}{2\epsilon}\sigma} + \\ +N_1\,|z_1 - z_2|_{-\mu/2\epsilon}\, e^{-\frac{\mu}{2\epsilon}\sigma}\Big)\, d\sigma \leq \\ \leq N_1\,(2+L)\int_t^\infty e^{N_1(\sigma-t)} e^{-\frac{\mu}{2\epsilon}\sigma}\, d\sigma\,|z_1 - z_2|_{-\mu/2\epsilon} = \\ = \tfrac{N_1(2+L)}{\mu/2\epsilon - N_1}\,|z_1 - z_2|_{-\mu/2\epsilon}\, e^{-\frac{\mu}{2\epsilon}t}.\end{gathered}$$

Since $\frac{N_1(2+L)}{\mu/2\epsilon - N_1} < 1$, $\mathcal{K}$ is a contraction on $BC^{-\mu/2\epsilon}(\mathbb{R}_+, X)$. By the contraction principle, we obtain the thesis. Estimate (23) follows directly from (24). $\diamondsuit$

Putting together Lemma 4.2.1 and Th. 4.2.2, we may conclude that $(S)_\epsilon$ has asymptotic phase: infact we have only to set

$$\begin{aligned}\bar{x}(t) &:= x(t) + z(t - t_0)\\ \bar{y}(t) &:= k(t, x(t) + z(t - t_0), \epsilon).\end{aligned}$$

5. Stable and unstable manifolds.

Let us turn back to the equivalent systems

$$(S)_\epsilon \begin{cases} \dot{x} = f(t, x, y, \epsilon) \\ \epsilon\dot{y} = A(t, x)y + g(t, x, y, \epsilon) \end{cases}$$

and

$$(F)_\epsilon \begin{cases} \dot{x} = \epsilon f(\epsilon t, x, y, \epsilon) \\ \dot{y} = A(\epsilon t, x)y + g(\epsilon t, x, y, \epsilon) \end{cases} .$$

Let ϵ_0, δ, ρ be as in Lemma 3.1.4. Let us fix $(t_0, x_0, \epsilon) \in \mathbb{R} \times X \times (0, \epsilon_0)$ with ϵ already so small that, for each $(\tau, x, t) \in \mathbb{R} \times X \times \mathbb{R}$,

$$|\varphi_2(\tau, x, \epsilon)(t)|_\alpha \leq \frac{\delta}{4};$$

we reserve, if necessary, to impose some further smallness condition on ϵ. Our goal is to find a closed set $\mathcal{P} \subseteq X \times Y^\alpha$, with

$$(x_0, y_0) := (x_0, k(t_0, x_0, \epsilon)) = (x_0, \varphi_2(t_0/\epsilon, x_0, \epsilon)(0)) \in \mathring{\mathcal{P}},$$

and a stable manifold $W^s(t_0, x_0, \epsilon)$, of class C^r, contained in $\mathcal{P}$, such that

$$W^s(t_0,x_0,\epsilon)=\Big\{(x_1,y_1)\in\mathcal{P}\,|\ \textit{the solution}\ \ (x,y)\ \ \textit{of}\ \ (S)_\epsilon\ \textit{with}$$
$$(x(t_0),y(t_0))=(x_1,y_1)\ \ \textit{exists on}\ [t_0,\infty),\ \textit{with}\ |y(t)|_\alpha<\delta\ \forall t,\ \textit{and}$$
$$\sup_{t\geq t_0} e^{\mu/2\epsilon(t-t_0)}\left(|x(t)-\varphi_1(t_0/\epsilon,x_0,\epsilon)(t/\epsilon-t_0/\epsilon)|+\right.$$
$$\left.+\,|y(t)-\varphi_2(t_0/\epsilon,x_0,\epsilon)(t/\epsilon-t_0/\epsilon)|_\alpha\right)<\infty\Big\}=$$
$$=\Big\{(x_1,y_1)\in\mathcal{P}\,|\ \textit{the solution}\ \ (\hat{x},\hat{y})\ \ \textit{of}\ \ (F)_\epsilon$$
$$\textit{with}\ \ (\hat{x}(t_0/\epsilon),\hat{y}(t_0/\epsilon))=(x_1,y_1)$$
$$\textit{exists on}\ \ [t_0/\epsilon,\infty),\ \ \textit{with}\ \ |\hat{y}(t)|_\alpha<\delta\ \forall t,\ \ \textit{and}$$
$$\sup_{t\geq t_0} e^{\mu/2(t-t_0/\epsilon)}\left(|\hat{x}(t)-\varphi_1(t_0/\epsilon,x_0,\epsilon)(t-t_0/\epsilon)|+\right.$$
$$\left.+\,|\hat{y}(t)-\varphi_2(t_0/\epsilon,x_0,\epsilon)(t-t_0/\epsilon)|_\alpha\right)<\infty\Big\}.$$

We know that the map

$$t\mapsto(\varphi_1(t_0/\epsilon,x_0,\epsilon)(t-t_0/\epsilon),\varphi_2(t_0/\epsilon,x_0,\epsilon)(t-t_0/\epsilon))$$

is the solution of $(F)_\epsilon$ whose value at t_0/ϵ is $(x_0,k(t_0,x_0,\epsilon))$; therefore the map

$$t\mapsto(\varphi_1(t_0/\epsilon,x_0,\epsilon)(t/\epsilon-t_0/\epsilon),\varphi_2(t_0/\epsilon,x_0,\epsilon)(t/\epsilon-t_0/\epsilon))$$

is the solution of $(S)_\epsilon$ whose value at t_0 is $(x_0,k(t_0,x_0,\epsilon))$. Then $W^s(t_0,x_0,\epsilon)$ will be the set of the points $(x_1,y_1)\in\mathcal{P}$, such that the solution of $(S)_\epsilon$ whose value at t_0 is (x_1,y_1) exists always in future, and differs from the solution of $(S)_\epsilon$ whose value at t_0 is $(x_0,k(t_0,x_0,\epsilon))$ by an $O\left(e^{-\mu/2\epsilon(t-t_0)}\right)$.

Similarly we may construct an "unstable" manifold $W^u(t_0,x_0,\epsilon)$

such that

$$W^u(t_0,x_0,\epsilon)=\Big\{(x_1,y_1)\in\mathcal{P}\,|\exists\ \textit{a solution}\ \ (x,y)\ \ \textit{of}\ \ (S)_\epsilon$$
$$\textit{with}\ \ (x(t_0),y(t_0))=(x_1,y_1)\ \ \textit{defined on}\ \ (-\infty,t_0]\,,$$
$$\textit{with}\ \ |y(t)|_\alpha<\delta\ \forall t,\ \ \text{and}$$
$$\sup_{t\le t_0} e^{\mu/2\epsilon(t_0-t)}\left(|x(t)-\varphi_1(t_0/\epsilon,x_0,\epsilon)(t/\epsilon-t_0/\epsilon)|+\right.$$
$$\left.+\,|y(t)-\varphi_2(t_0/\epsilon,x_0,\epsilon)(t/\epsilon-t_0/\epsilon)|_\alpha\right)<\infty\Big\}.$$

Here we are mainly concerned with $W^s(t_0,x_0,\epsilon)$, because as for $W^u(t_0,x_0,\epsilon)$ the arguments are completely analogous.
To simplify notations, since we work with fixed (t_0,x_0,ϵ), we set

$$\begin{aligned}&\varphi_i(t):=\varphi_i(t_0/\epsilon,x_0,\epsilon)(t),\, i:=1,2\\&T(t,s):=T(t,s;t_0,\varphi_1(\cdot),\epsilon)\\&P(s):=P(s;t_0,\varphi_1(\cdot),\epsilon).\end{aligned}$$

We will obtain (for sufficiently small ϵ) $W^s(t_0,x_0,\epsilon)$ as the graph of a map

$$j:R\,(I-P(0))\cap B^\alpha_\gamma\,[k(t_0,x_0,\epsilon)]\to X\times R(P(0)),$$

i.e.

$$W^s(t_0,x_0,\epsilon):=\Big\{(0,\eta)+j(\eta)\,|\eta\in Y^\alpha,(I-P(0))\eta=\eta,$$
$$|\eta-k(t_0,x_0,\epsilon)|_\alpha\le\gamma\Big\},$$

where

$$\gamma:=\left[1-\left(\frac{\epsilon_0N_1}{\mu/2}+\frac{2K}{\mu/2}\max\{M_2\delta,C\,(\epsilon_0+\delta)\}\right)\right]\frac{\delta}{M}$$

First of all, we need the following Lemma, whose simple proof is left to the reader.

Lemma 5.0.3. *If*

$$\epsilon<\gamma\frac{\mu}{MC}\left[1+\left(1-\frac{KC\delta}{\mu-\rho}\right)^{-1}\frac{KC}{\mu-\rho}\right]^{-1},$$

then

$$R(I - P(0)) \cap B^{\alpha}_{\gamma}\left(k(t_0, x_0, \epsilon)\right) \neq \emptyset.$$

Note: this implies that $R(I - P(0)) \cap B^{\alpha}_{\gamma}\left(k(t_0, x_0, \epsilon)\right)$ is a nonempty open subset of $R(I - P(0))$. ♢

5.1. Construction of $W^s(t_0, x_0, \epsilon)$.

LEMMA 5.1.1. *Let* $(x, y) \in BC^{-\mu/2}(\mathbb{R}, X) \times BC^{-\mu/2}(\mathbb{R}, Y^{\alpha})$, *and let* $\eta \in Y^{\alpha}$, *with* $(I - P(0))\eta = \eta$, $|\eta - \phi_2(0)|_{\alpha} \leq \gamma$, *then the following facts are equivalent:*

1. $|\varphi_2(t - t_0/\epsilon) + y(t - t_0/\epsilon)|_{\alpha} < \delta$ for $t \geq t_0/\epsilon$ and the map

$$t \mapsto \left(\varphi_1(t - t_0/\epsilon) + x(t - t_0/\epsilon), \varphi_2(t - t_0/\epsilon) + y(t - t_0/\epsilon)\right)$$

is a solution of $(F)_{\epsilon}$ on $(t_0/\epsilon, \infty)$, with $(I - P(0))(\varphi_2(0) + y(0)) = \eta$;

2. (x, y) solves the integral system

$$\text{(I)} \quad \begin{cases} x(t) = & -\epsilon \displaystyle\int_t^{\infty} [f(t_0 + \epsilon s, \varphi_1(s) + x(s), \varphi_2(s) + y(s), \epsilon) + \\ & -f(t_0 + \epsilon s, \varphi_1(s), \varphi_2(s), \epsilon)]\, ds \\ y(t) = & T(t, 0)(I - P(0))(\eta - \varphi_2(0)) + \\ & + \displaystyle\int_0^{\infty} U(t, s)\,[(A_1(t_0 + \epsilon s, \varphi_1(s) + x(s)) + \\ & -A_1(t_0 + \epsilon s, \varphi_1(s)))\,(\varphi_2(s) + y(s)) + \\ & +g(t_0 + \epsilon s, \varphi_1(s) + x(s), \varphi_2(s) + y(s), \epsilon) + \\ & -g(t_0 + \epsilon s, \varphi_1(s), \varphi_2(s), \epsilon)]\, ds \end{cases}$$

and $|x|_{-\mu/2} + |y|_{-\mu/2} \leq \delta/2$.

Proof. Th. 1.3.5, Lemma 3.1.4 and easy calculation. ♢

THEOREM 5.1.2. *Let us suppose that* ϵ *satisfies the condition in Lemma 5.0.3, and all the other conditions introduced at the beginning of this Section. Then, for every* $\eta \in Y^{\alpha}$, $|\eta - \varphi_2(0)|_{\alpha} \leq \gamma$, $(I -$

$P(0))\eta = \eta$, *there exists a unique couple* $(x,y) = (\chi_1(\eta), \chi_2(\eta))$ *belonging to* $BC^{-\mu/2}(\mathbb{R}_+, X) \times BC^{-\mu/2}(\mathbb{R}_+, Y^\alpha)$, *with* $|x|_{-\mu/2} + |y|_{-\mu/2} \leq \delta/2$, *such that the map*

$$t \mapsto \left(\varphi_1\left(t - t_0/\epsilon\right) + x\left(t - t_0/\epsilon\right), \varphi_2\left(t - t_0/\epsilon\right) + y\left(t - t_0/\epsilon\right)\right)$$

solves the integral system (I) (hence the differential system $(\mathrm{S})_\epsilon$) and satisfies $(I - P(0))\left(\varphi_2(0) + \chi_2(\eta)(0)\right) = \eta$.

Proof. Let us set

$$\begin{aligned} B_{\delta/2}\left[BC^{-\mu/2}(\mathbb{R}_+, X) \times BC^{-\mu/2}(\mathbb{R}_+, Y^\alpha)\right] := \\ = \Big\{(x,y) \in BC^{-\mu/2}(\mathbb{R}_+, X) \times BC^{-\mu/2}(\mathbb{R}_+, Y^\alpha) \mid \\ |x|_{-\mu/2} + |y|_{-\mu/2} \leq \delta/2\Big\} \ . \end{aligned}$$

For $(x,y) \in B_{\delta/2}\left[BC^{-\mu/2}(\mathbb{R}_+, X) \times BC^{-\mu/2}(\mathbb{R}_+, Y^\alpha)\right]$ and $\eta \in Y^\alpha$, with $|\eta - \varphi_2(0)|_\alpha \leq \gamma$, $(I - P(0))\eta = \eta$, let us set

$$\begin{aligned} \mathcal{F}_1(x,y,\eta)(t) := -\epsilon\int_t^\infty \left[f(t_0 + \epsilon\sigma, \varphi_1(\sigma) + x(\sigma), \varphi_2(\sigma) + y(\sigma), \epsilon) + \right. \\ \left. - f(t_0 + \epsilon\sigma, \varphi_1(\sigma), \varphi_2(\sigma), \epsilon)\right] d\sigma \end{aligned}$$

and

$$\begin{aligned} \mathcal{F}_2(x,y,\eta)(t) := T(t,0)(I - P(0))(\eta - \varphi_2(0)) + \\ + \int_0^\infty U(t,\sigma)\left[(A_1(t_0 + \epsilon\sigma, \varphi_1(\sigma) + x(\sigma)) + \right. \\ - A_1(t_0 + \epsilon\sigma, \varphi_1(\sigma)))\left(\varphi_2(\sigma) + y(\sigma)\right) + \\ + g(t_0 + \epsilon\sigma, \varphi_1(\sigma) + x(\sigma), \varphi_2(\sigma) + y(\sigma), \epsilon) + \\ \left. - g(t_0 + \epsilon\sigma, \varphi_1(\sigma), \varphi_2(\sigma), \epsilon)\right] d\sigma. \end{aligned}$$

By Lemma 1.3.5 and Lemma 3.1.4, it easy to show that

$$\begin{aligned} \mathcal{F} := (\mathcal{F}_1(\cdot,\cdot,\eta), \mathcal{F}_2(\cdot,\cdot,\eta)) : \\ B_{\delta/2}\left[BC^{-\mu/2}(\mathbb{R}_+, X) \times BC^{-\mu/2}(\mathbb{R}_+, Y^\alpha)\right] \to \\ \to B_{\delta/2}\left[BC^{-\mu/2}(\mathbb{R}_+, X) \times BC^{-\mu/2}(\mathbb{R}_+, Y^\alpha)\right] \end{aligned}$$

is a contraction, uniform with respect to η, so the thesis follows by the contraction mapping theorem. $\diamondsuit$

REMARK.

It is easy to see that $\mathcal{F}$ is Lipschitz continuous in η, uniformly with respect to (x,y). So the map

$$\eta \mapsto (\chi_1(\eta),\chi_2(\eta)) : R(I-P(0)) \cap B^{\alpha}_{\gamma}[\varphi_2(0)] \to$$

$$\to BC^{-\mu/2}(\mathbb{R}_+,X) \times BC^{-\mu/2}(\mathbb{R}_+,Y^{\alpha})$$

is Lipschitz continuous.

Now we are able to construct the manifold $W^s(t_0,x_0,\epsilon)$. We set

$$\begin{aligned} j(\eta) &:= (\varphi_1(0)+\chi_1(\eta)(0), P(0)(\varphi_2(0)+\chi_2(\eta)(0))) = \\ &= (x_0+\chi_1(\eta)(0), P(0)(k(t_0,x_0,\epsilon)+\chi_2(\eta)(0))) \end{aligned}$$

and

$$W^s(t_0,x_0,\epsilon) := \left\{(0,\eta)+j(\eta) \middle| \eta \in R(I-P(0)) \cap B^{\alpha}_{\gamma}[\varphi_2(0)]\right\}.$$

Since

$$\begin{aligned} (0,\eta)+j(\eta) &= (0,(I-P(0))(\varphi_2(0)+\chi_2(\eta)(0)))+ \\ &+ (\varphi_1(0)+\chi_1(\eta)(0), P(0)(\varphi_2(0)+\chi_2(\eta)(0))) = \\ &= (\varphi_1(0)+\chi_1(\eta)(0), \varphi_2(0)+\chi_2(\eta)(0)) = \\ &= (x_0+\chi_1(\eta)(0), k(t_0,x_0,\epsilon)+\chi_2(\eta)(0)), \end{aligned}$$

we obtain

$$\begin{aligned} &W^s(t_0,x_0,\epsilon) := \\ &= \{(x_0+\chi_1(\eta)(0), k(t_0,x_0,\epsilon)+\chi_2(\eta)(0)) \,|\eta \in \\ &\in R(I-P(0)) \cap B^{\alpha}_{\gamma}[k(t_0,x_0,\epsilon)]\}. \end{aligned}$$

Moreover,

$$(x_0+\chi_1(\eta)(0), k(t_0,x_0,\epsilon)+\chi_2(\eta)(0)) =$$

$$= (\varphi_1(0)+\chi_1(\eta)(0), \varphi_2(0)+\chi_2(\eta)(0))$$

is by definition the value at t_0/ϵ of a solution of $(F)_\epsilon$ defined on $(t_0/\epsilon,\infty)$, whose difference from

$$(\varphi_1(t_0,x_0,\epsilon)(\cdot - t_0/\epsilon), \varphi_2(t_0,x_0,\epsilon)(\cdot - t_0/\epsilon))$$

is exactly
$$(\chi_1(\eta)(\cdot - t_0/\epsilon), \chi_2(\eta)(\cdot - t_0/\epsilon)).$$
If we set
$$\mathcal{P} := \{(x,y) \in X \times Y^\alpha \,|\, |(I-P(0))y - k(t_0,x_0,\epsilon)|_\alpha \le \gamma\},$$
it is easily seen that
$$\begin{aligned} W^s(t_0,x_0,\epsilon) = \Big\{ &(x_1,y_1) \in \mathcal{P} \textit{ such that the solution } (x,y) \textit{ of } (S)_\epsilon \\ &\textit{with } (x(t_0),y(t_0)) = (x_1,y_1) \textit{ exists on } [t_0,\infty), \\ &\textit{with } |y(t)|_\alpha < \delta \forall t \quad \textit{and} \\ &\sup_{t \ge t_0} e^{\frac{\mu}{2\epsilon}(t-t_0)} \left(|x(t) - \varphi_1(t_0/\epsilon, x_0, \epsilon)\,(t/\epsilon - t_0/\epsilon)| + \right. \\ &\left. + |y(t) - \varphi_2(t_0/\epsilon, x_0,\epsilon)\,(t/\epsilon - t_0/\epsilon)|_\alpha\right) < \infty \Big\}. \end{aligned}$$

Smoothness of $W^s(t_0,x_0,\epsilon)$ can be obtained by application of the standard parameter-dependent contraction mapping theorem. The details are left to the reader.

Remark.

It would be interesting trying to understand if the manifolds $W^s(t_0,x_0,\epsilon)$, as (t_0,x_0) varies, "tie together" to form a smooth invariant manifold. The problem is that the map $j_{(t_0,x_0)}$ generating the manifold $W^s(t_0,x_0,\epsilon)$ is defined on an open subset of
$$R(I - P(0; t_0, \varphi_1(t_0/\epsilon, x_0, \epsilon), \epsilon);$$
so, as (t_0,x_0) varies, the domain of $j_{(t_0,x_0)}$ is not constant. Sakamoto solves this problem (in finite dimension!) by showing that the spaces $R(I - P(0; \varphi_1(x_0,\epsilon), \epsilon)$ form a smooth vector-bundle on C_ϵ and constructing each $W^s(x_0,\epsilon)$ as the graph of a map defined on the fiber of x_0; However this technique seems not to generalize to the case of evolution equations in Banach spaces.

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