



# A systematic comparison of continuous and discrete mixture models

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## Abstract

Modellers are increasingly relying on the use of continuous random coefficients models, such as Mixed Logit, for the representation of variations in tastes across individuals. In this paper, we provide an in-depth comparison of the performance of the Mixed Logit model with that of its far less commonly used discrete mixture counterpart, making use of a combination of real and simulated datasets. The results not only show significant computational advantages for the discrete mixture approach, but also highlight greater flexibility, and show that, across a host of scenarios, the discrete mixture models are able to offer comparable or indeed superior model performance.

*Keywords:* Discrete choice; Mixture models; Random parameters; Discrete and continuous distributions.

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## 1. Introduction and context

Allowing for variations in behaviour across decision makers is one of the most fundamental principles in discrete choice modelling, given that the assumption of a purely homogeneous population cannot in general be seen to be valid.

The typical way of allowing for such variation is through a deterministic approach, linking the taste heterogeneity to variations in socio-demographic factors such as income or trip purpose. While appealing from the point of view of interpretation (and

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especially for forecasting), it is often not possible to represent all variations in tastes in a deterministic fashion, for reasons of data quality, but also due to inherent randomness in choice behaviour. For this reason, random coefficient structures, such as the Mixed Multinomial Logit (MMNL) model, which allow for random variations in behaviour across respondents, have an important advantage in terms of flexibility. In general, such models have the disadvantage that their choice probabilities take on the form of integrals that do not possess a closed form solution, such that numerical processes, typically simulation, are required during estimation and application of the models. This greatly limited the use of these structures for many years after their initial developments. Over recent years, gains in computer speed and the efficiency of simulation based estimation processes (see for example Hess et al. 2006) have however led to increased interest in the MMNL model in particular, by researchers and, to a lesser degree, also practitioners.

Despite the improvements in estimation capability, the cost of using the MMNL model remains high. While this might be acceptable in many cases, another important issue remains, namely the choice of distribution to be used for representing the random variations in tastes across respondents. Here, there is a major risk of producing misleading results when making an inappropriate choice of distribution, as discussed by Hess et al. (2005).

In this paper, we explore an alternative approach, based on the idea of replacing the continuous distribution functions by discrete distributions, spreading the mass among several discrete values. Mathematically, the model structure of a discrete mixture (DM) model is a special case of a latent class model (cf. Kamakura and Russell, 1989; Chintagunta et al., 1991), assigning different coefficient values to different parts of the population of respondents, a concept discussed in the field of transport studies for example by Greene and Hensher (2003) and Lee et al. (2003). Latent class approaches make use of two sub-models, one for class allocation, and one for within class choice. The former models the probability of an individual being assigned to a specific class as a function of attributes of the respondent and possibly of the alternatives in the choice set. The within class model is then used to compute the class-specific choice

probabilities for the different alternatives, conditional on the tastes within that class. The actual choice probability for individual  $n$  and alternative  $i$  is given by a sum of the class-specific choice probabilities, weighted by the class allocation choice probabilities for that specific individual.

The latent class approach is appealing from the point of view that it allows for differences in sensitivities across population groups, where the group allocation can be related to socio-demographic characteristics. However, in practice, it may not always be possible to explain group allocation with the help of a probabilistic model relating the outcome to observed variables. This situation is similar to the case where taste heterogeneity cannot be explained deterministically, leading to a requirement for using random coefficients models. As such, in this paper, we explore the use of models in which the class allocation probabilities are independent of explanatory variables, and are simply given by constants that are to be estimated during model calibration. As such, the resulting model exploits the class membership concept in the context of random coefficients models, with a limited set of possible values for the coefficients.

Thus far, there have seemingly been only two main applications of this approach in the area of transport research, by Gopinath (1995), in the context of mode choice for freight shippers, and by Dong and Koppelman (2003), who made use of discrete mixtures of MNL models in the analysis of mode choice for work trips in New York, referring to the resulting model as the “Mass Point Mixed Logit model”. Although the properties of DM models have been discussed by several other authors (e.g. Wedel et al., 1999), the model structure does not seem to have received widespread exposure or application, despite its many appealing characteristics. Given this observation, part of the aim of this paper is to re-explore the potential advantages of DM models, with the hope of encouraging their more widespread use. Additionally, the paper aims to offer a systematic comparison of the performance of discrete and continuous mixture models across a host of situations, making use of simulated data.

The remainder of this paper is organised as follows. The next section sets out the theory behind DM models. Section 3 presents a case study using real data, while Section

4 uses four different simulated datasets in a systematic comparison of discrete and continuous mixture models. Finally, Section 5 presents the conclusions of the paper.

## 2. Methodology

Let  $x_{in}$  be a vector defining the attributes of alternative  $i$  as faced by respondent  $n$ , and let  $\beta$  be a vector defining the tastes of the decision maker, where, in purely deterministic models,  $\beta$  is constant across respondents. Furthermore, let  $x_n$  be a vector grouping together the individual vectors  $x_{jn}$  across the alternatives contained in the choice set of respondent  $n$ . We can then define  $P_n(i|x_n, C_n, \beta)$  to give the choice probability of alternative  $i$  for individual  $n$ , with a choice set  $C_n$ , conditional on the observed vector  $x_n$ , and for given values for the vector of parameters  $\beta$  (to be estimated).

In a discrete mixture context, the number of possible values for the taste coefficients  $\beta$  is finite. Here, we divide the set of parameters  $\beta$  into two sets;  $\bar{\beta}$  represents a part of  $\beta$  containing deterministic parameters, while  $\hat{\beta}$  is a set of  $K$  random parameters that have a discrete distribution. Within this set, the parameter  $\hat{\beta}_k$  has  $m_k$  mass points  $\hat{\beta}_k^j$ ,  $j = 1, \dots, m_k$ , each of them associated with a probability  $\pi_k^j$ , where we impose the conditions that<sup>1</sup>

$$0 \leq \pi_k^j \leq 1, \quad k = 1, \dots, K; j = 1, \dots, m_k, \quad (1)$$

and

$$\sum_{j=1}^{m_k} \pi_k^j = 1, \quad k = 1, \dots, K. \quad (2)$$

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<sup>1</sup>These constraints can be avoided by setting  $\pi_i = \frac{e^{\alpha_i}}{\sum_{j=1}^J e^{\alpha_j}}$ , where  $\alpha_j$  with  $j = 1, \dots, J$  are estimated

without constraints. While avoiding the need for constraints, this formulation becomes highly non-linear and difficult to handle in estimation.

For each realisation  $\hat{\beta}_1^{j_1}, \dots, \hat{\beta}_K^{j_K}$  of  $\hat{\beta}$ , the choice probability is given by

$$P_n(i | x_n, C_n, \beta = \langle \bar{\beta}, \hat{\beta}_1^{j_1}, \dots, \hat{\beta}_K^{j_K} \rangle), \quad (3)$$

where the deterministic part of  $\bar{\beta}$  stays constant across realisations of the vector  $\hat{\beta}$ .

The unconditional choice probability for alternative  $i$  and decision maker  $n$  can now be written straightforwardly as a mixture over the discrete distributions of the various elements contained in  $\hat{\beta}$  as:

$$\begin{aligned} & P_n(i | x_n, C_n, \bar{\beta}, \hat{\beta}, \pi) \\ &= \sum_{j_1=1}^{m_1} \dots \sum_{j_K=1}^{m_K} P_n(i | x_n, C_n, \beta = \langle \bar{\beta}, \hat{\beta}_1^{j_1}, \dots, \hat{\beta}_K^{j_K} \rangle) \pi_1^{j_1} \dots \pi_K^{j_K}, \end{aligned} \quad (4)$$

where  $\bar{\beta}$ ,  $\hat{\beta}$  and  $\pi$  ( $\pi = \langle \pi_1^1, \dots, \pi_1^{m_1}, \dots, \pi_K^1, \dots, \pi_K^{m_K} \rangle$ ) are vectors of parameters to be estimated in a regular maximum likelihood estimation procedure. An obvious advantage of this approach is that, if the model (3) used inside the mixture has a closed form, then so does the DM itself.

In this paper, we focus on the simple case where the underlying choice model is of MNL form; however, the form given in equation (4) is appropriate for any underlying model (e.g. Nested Logit). The approach can easily be extended to the case of combined discrete and continuous random taste heterogeneity, by partitioning  $\beta$  into three parts; the above defined parts  $\bar{\beta}$  and  $\hat{\beta}$ , and an additional part  $\tilde{\beta}$ , whose elements follow continuous distributions<sup>2</sup>. This however leads to a requirement to use simulation, as with all continuous mixture models.

Finally, a treatment of repeated choice observations analogous to the standard continuous mixture treatment, with tastes varying across individuals, but not across observations for the same individual, is made possible by replacing the conditional choice probabilities for individual observations in equation (4) by probabilities for sequences of choices, and by using the resulting DM term inside the log-likelihood function.

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<sup>2</sup>This approach can then also be used to include error components for correlation or heteroscedasticity.

Several issues arise in the estimation of DM models. Firstly, the non-concavity of the log likelihood function does not allow the identification of a global maximum, even for discrete mixtures of MNL. Given the potential presence of a high number of local maxima, performing several estimations from various starting points is advisable. Also, it is good practice to use starting values other than 0 or 1 for the  $\pi_k^j$  parameters. Secondly, constrained maximum likelihood must be used to account for constraints (1) and (2). Thirdly, clustering of mass points (for example around the mode of the true distribution) is a frequent phenomenon with DM models, and the use of additional bounds on the mass points can be useful, based on the definition of (potentially mutually exclusive) a priori intervals for the individual mass points. In this context, a heuristic is needed to determine the optimal number of support points in actual applications. Some of these issues have caused problems in past applications of DM models, see for example Dong and Koppelman (2003). Given these problems in past research, the results of our analysis should ideally be reconfirmed in future work.

For the purpose of this analysis, the model was coded into BIOGEME (Bierlaire, 2003), where various constraints on the parameters can be imposed to address the issues described above. This also allows modellers to test the validity of specific assumptions, such as a mass at zero for the VTTS, a concept discussed for example by Cirillo and Axhausen (2006).

### **3. VTTS case study**

In this section, we present the findings of an analysis making use of real world data. We first give a brief description of the data in Section 1, before looking at model specification in Section 2. The estimation results are presented in Section 3.

#### *3.1 Data*

The study presented here makes use of Stated Preference (SP) data collected as part of a recent value of time study undertaken in Denmark (Burge and Rohr, 2004).

Specifically, we make use of data describing a binary choice process for car travellers, with alternatives described only in terms of travel cost and travel time. This may be seen as a simple design for a SP survey. However, this is now a standard SP format for VTTS studies in Europe (e.g. Mackie et al., 2003). In any case, the design of the survey should have no direct impact on the comparison between DM and MMNL models in this analysis.

Each respondent was presented with 9 choice situations, including one with a dominating alternative. After eliminating the observations with a dominating alternative, as well as additional data cleaning (removing non-traders and respondents who did not choose the dominating alternative<sup>3</sup>), a sample of 13,386 observations from 1,723 respondents was obtained. This equates to 3,037 observations from 392 commuters, 1,081 observations from 142 respondents travelling for education purposes, 1,767 observations from 230 people on shopping trips, 3,155 observations from 404 people travelling to visit friends or relatives, 1,752 observations from 224 general leisure travellers and 2,594 observations from 331 respondents travelling for other purposes.

To allow us to gauge the stability of the results, multiple random subsamples of around 80% of the original sample size were generated for each of the above listed six purpose segments<sup>4</sup>.

### *3.2 Model specification*

The models used in this paper were estimated in log-WTP (willingness to pay) space, avoiding the effect of heterogenous scale (cf. Fosgerau and Bierlaire, 2006), while allowing us to represent random variations in the VTTS without the issue of calculating

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<sup>3</sup>The dominating alternative was both cheaper and faster. Respondents observed to choose the slower and more expensive alternative were deemed not to have correctly understood the survey, and were removed from the analysis. The number of non-traders in the data was fairly low, but their removal was in line with previous studies using the same data (cf. Fosgerau, 2006).

<sup>4</sup>The selection was performed at the individual-specific level, rather than the observation-specific level.

the VTTS on the basis of separate randomly distributed coefficients for travel time and travel cost. This was found to be the best specification for this dataset by Fosgerau (2006).

Details of the specification are given by Fosgerau (2006). In summary, we let  $T_i$  and  $C_i$  define the time and cost attributes of alternative  $i$ , and rearrange the data such that  $T_1 > T_2$  and  $C_1 < C_2$ , i.e., the first alternative is slower but cheaper than the second alternative. By further setting  $\alpha_{LV} = \ln(VTTS)$ , we get the following utility functions:

$$U_1 = \lambda \ln \left( -\frac{\Delta_C}{\Delta_T} \right) + \varepsilon_1 \quad (5)$$

and

$$U_2 = \lambda \alpha_{LV} + \varepsilon_2, \quad (6)$$

where  $\Delta_C = C_1 - C_2$  and  $\Delta_T = T_1 - T_2$ , while  $\varepsilon_1$  and  $\varepsilon_2$  give the usual type I *iid* extreme value terms. The scale  $\lambda$  is estimated in addition to  $\alpha_{LV}$ , and, with travel costs given in Danish Krona (DKK) and travel times given in minutes, the actual VTTS in DKK per hour is obtained by  $60 \cdot \exp(\alpha_{LV})$ . The specification set out above can now be used in a standard discrete choice framework, with either a fixed estimate for  $\alpha_{LV}$ , or with random variation across respondents.

### 3.3 Model results

During the analysis, four different types of model were estimated on the data; a simple MNL model, a MMNL model using a Normal distribution, and two DM specifications, one with two support points, DM(2), and one with three support points, DM(3)<sup>5</sup>. In the MMNL and DM models, the repeated choice nature of the data was taken into account by specifying the likelihood function with the integration

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<sup>5</sup>Models with more than three support points collapsed back to the more basic specifications.



(respectively summation in the DM models) outside the product over replications for the same respondent.

Each of these models was estimated across the six population segments, with 10 different random subsamples for each segment. Given this wealth of results, we presented detailed results only for a single subsample for shopping trips (Section 1), and give summary results for the remaining five population segments (Section 2). It should be said that, across segments and models, the results were very stable across subsamples, where a similar observation was also made with slightly smaller subsamples, allowing for smaller overlap.

### 3.3.1 Detailed results for shopping trips

The results for the various models estimated on the data for shopping trips are summarised in Table 1. Several differences arise across models in the presentation of the results. As such, for the MNL model, only  $\alpha_{LV}$  and  $\lambda$  are estimated. For the MMNL model,  $\alpha_{LV}$  follows a Normal distribution, with mean  $\alpha_{LV,\mu}$  and standard deviation  $\alpha_{LV,\sigma}$ . For the two DM models, the value of  $\alpha_{LV}$  is spread across several support points  $\alpha_{LV,k}$  with associated probabilities  $0 \leq \pi_k \leq 1$ , such that  $\sum_{k=1}^K \pi_k = 1$ , with  $K = 2$  and  $K = 3$  in DM(2) and DM(3) respectively.

The table also shows the calculated VTTS. For the MNL model, the mean VTTS is simply obtained through  $60 \cdot \exp(\alpha_{LV})$ . However, for the three mixture models, the non-linearity in the exponential means that a different approach is required. With  $\alpha_{LV} \sim N(\mu_\alpha, \sigma_\alpha)$  in the MMNL model, the actual VTTS follows a log-normal distribution with mean  $\mu_{VTTS} = \exp\left(\mu_\alpha + \frac{\sigma_\alpha^2}{2}\right)$  and standard deviation  $\sigma_{VTTS} = \mu \sqrt{\exp(\sigma_\alpha^2) - 1}$ . Both  $\mu_{VTTS}$  and  $\sigma_{VTTS}$  can then be multiplied by 60 to obtain hourly values. For the DM models, a slightly different approach was used. As such, with  $K$  support points  $\alpha_{LV,k}$  and associated probabilities  $\pi_k$ , a sequence of draws was generated that contained  $\pi_k \cdot N$  points with a value equal to  $\exp(\alpha_{LV,k})$ , with  $k = 1, \dots, K$ . The sample mean and standard deviation from this sequence were then used

as estimates of the mean and standard deviation for the actual VTTS. For the results presented here, the value of  $N$  was set to 100,000, beyond which no visible differences were observed for  $\sigma_{VTTS}$ . Finally, along with the results for individual subsamples, the table also shows some overall measures, namely the average of the adjusted  $\rho^2$  measure, the average estimation time, and the average for  $\mu_{VTTS}$  and  $\sigma_{VTTS}$  (together with a standard deviation of this mean across subsamples).

Table 1: Estimation results on Danish shopping data.

	<i>Model</i>	<i>MNL</i>	<i>MMNL</i>	<i>DM(2)</i>	<i>DM(3)</i>
	Final LL	-880.96	-880.96	-849.65	-845.40
	adj. $\rho^2$	0.1036	0.1036	0.13433	0.136613
	Estimation time (s)	1	1	75	1
$\alpha_{LV,\mu}$	est.	-1.1100	-1.0800	-	-
	asy. t-ratio	-14.20	-11.30	-	-
$\alpha_{LV,\sigma}$	est.	-	0.8950	-	-
	asy. t-ratio	-	8.75	-	-
$\alpha_{LV,1}$	est.	-	-	0.5410	0.7770
	asy. t-ratio	-	-	1.89	2.26
$\pi_1$	est.	-	-	0.2130	0.1560
	asy. t-ratio <sup>(i)</sup>	-	-	3.73	2.69
$\alpha_{LV,2}$	est.	-	-	-1.4700	-1.7800
	asy. t-ratio	-	-	-13.40	-5.40
$\pi_2$	est.	-	-	0.7870	0.4550
	asy. t-ratio <sup>(i)</sup>	-	-	13.80	1.53
$\alpha_{LV,3}$	est.	-	-	-	-0.9100
	asy. t-ratio	-	-	-	-2.09
$\pi_3$	est.	-	-	-	0.3890
	asy. t-ratio <sup>(i)</sup>	-	-	-	1.35
$\lambda$	est.	0.8380	1.0300	1.0100	1.0300
	asy. t-ratio	11.50	12.10	12.30	12.10
	Mean VTTS (DKK/hour)	19.77	30.41	32.81	34.29
	VTTS standard deviation	-	33.70	36.55	41.86

The first observation that can be made from Table 1 is that all three mixture models offer significant improvements in model fit over the base MNL model. Given the structural differences between the continuous and discrete mixture models, the comparison between these models is carried out using the adjusted  $\rho^2$  measure rather than the log-likelihood function. Here, we can see that DM(2) offers the best performance, ahead of DM(3) and the MMNL model. While the model with three support points obtains slightly better model fit than the model with two support points, the gains are not large enough to be significant when taking into account the additional cost in terms of the number of parameters. In other words, the model with three support points is not able to retrieve significant amounts of additional heterogeneity when compared to the model with two support points. This can partly be seen as a reflection of the success of the model with two support points, but is also an illustration of the difficulties of estimating models with more than two support points, as alluded to in Section 2.

The next observation relates to the much lower estimation cost for the DM(2) model, with an average estimation time of one second, compared to seventy-five with the MMNL model. This much lower estimation cost would give the DM models a significant advantage in the case of larger datasets, where the absolute estimation times would be more substantial. Furthermore, the estimation time for the MMNL model was in this case kept low through the use of only 250 Halton draws in the estimation.

In terms of substantive results, the mean VTTS measures obtained by the three mixture models are significantly higher than the point estimate obtained with the MNL model. This is at least partly a result of the asymmetrical distribution of the VTTS in the mixture models. While there are also some differences between the three mixture models in the estimates for  $\mu_{VTTS}$ , these are much smaller than the difference when compared to the MNL estimates. Finally, the estimate for  $\sigma_{VTTS}$  is much higher in the DM(3) model, while the estimate for the DM(2) model and the MMNL model are very similar.

3.3.2 Other results

Table 2 summarises the results for the various models estimated on the remaining five purpose segments. The results are very similar to those obtained on the data for shopping trips. As such, all three mixture models outperform the MNL model, where the best performance is consistently obtained by the DM(2) model. Again, the DM(3) model is not able to retrieve significant levels of additional taste heterogeneity to warrant the estimation of two additional parameters. In fact, the estimates for  $\mu_{VTTS}$  and  $\sigma_{VTTS}$  are almost universally equivalent across the two models<sup>6</sup>. As in the case of shopping trips, the advantages of the DM models in terms of estimation time are again very significant, across all five purpose segments. Finally, while there are almost no differences in the estimates for  $\mu_{VTTS}$  between the three different mixture models (where the estimates are again significantly higher than those for the MNL models), the estimates for  $\sigma_{VTTS}$  are now lower in the DM models, something that was not the case in the shopping segment.

Table 2: Summary of results for commuters, education trips, leisure trips, other purposes and visits.

		<i>Commuters</i>	<i>Education</i>	<i>Leisure</i>	<i>Other</i>	<i>Visit</i>
MNL	adj. $\rho^2$ :	0.1017	0.1282	0.1102	0.0888	0.1007
	estimation time (s):	1	1	1	1	1
	Mean VTTS (DKK/hour):	29.08	29.32	26.40	22.73	23.82
MMNL	adj. $\rho^2$ :	0.1263	0.1599	0.1395	0.1127	0.1294
	estimation time (s):	131	51	74	107	127
	Mean VTTS (DKK/hour):	39.51	37.28	37.62	34.76	35.83
	Std.dev. VTTS	35.90	29.24	38.43	39.85	39.61
DM(2)	adj. $\rho^2$ :	0.1291	0.1609	0.1433	0.1156	0.1337
	estimation time (s):	2	1	1	2	2
	Mean VTTS (DKK/hour):	39.78	36.96	37.03	34.43	37.36
	Std.dev. VTTS	30.50	24.01	28.03	29.17	36.28
DM(3)	adj. $\rho^2$ :	0.1279	0.1576	0.1412	0.1142	0.1326
	estimation time (s):	4	1	2	3	4
	Mean VTTS (DKK/hour):	39.78	37.04	37.03	34.43	37.18
	Std.dev. VTTS	30.50	24.36	28.03	29.17	36.16

<sup>6</sup>It is worth noting that, with the exception of the education segment, the adjusted  $\rho^2$  measure is higher for the DM(3) model than for the MMNL model.

#### **4. Simulated data case studies**

The application presented in Section 3 has shown the potential advantages of using a discrete mixture approach. However, it is clearly impossible to generalise these results, which could well be specific to the data at hand. For this, a systematic comparison between discrete and continuous mixture models is required; this is the topic of this section, which presents the findings of four case studies making use of simulated data.

In each of the four case studies, the generation of the data is based on the Danish VOT data used in the case study described in Section 3. Specifically, we use 10,776 observations from 1,347 respondents, and generate choices based on the attributes used in the original survey data. For each of the four different true models, ten sets of choices are generated for each observation, allowing us to gauge the stability of results across different samples. With the exception of the first case study, where the MMNL model had slightly higher variation across samples, the results were relatively stable, such that we only present results for the first subsample in each case<sup>7</sup>.

Unlike in the case study described in Section 3, we now work in preference space, with separate coefficients for travel time and travel cost. In each case, the travel cost coefficient is kept fixed while some random distribution is used for the travel time coefficient, with distributions chosen so as to give realistic ranges for the VTTS distribution. Finally, the data generation was in each case carried out under the assumption of constant tastes across replications for the same individual, and the same approach was later used in model estimation.

In the first two case studies, the true model is a discrete mixture, while in the final two case studies, the true model is a continuous mixture. This allows us to gauge the relative difficulties of the two types of model in dealing with data for which the other model type is more appropriate.

Before proceeding to the discussion of the results, it should be noted that all MMNL models presented here make use of a Normal distribution. Attempts to use alternative continuous distribution functions, such as Johnson's  $S_B$ , did not lead to consistent

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<sup>7</sup>Detailed results are available from the first author on request.

results on the data used here. While the findings from this analysis are thus limited to a comparison between a discrete mixture and a normal mixture, it should be remembered that the vast majority of MMNL studies make use specifically of this Normal distribution, such that the results are still relevant.

#### 4.1 Case study 1: discrete mixture with two support points

The first case study makes use of data generated with the help of a discrete mixture model with two mass points for  $\beta_T$ , at  $-1$  and  $-0.5$ , with probabilities of  $0.25$  and  $0.75$  respectively. The travel cost coefficient is fixed at a value of  $-1$ , such that we obtain a mean VTTS of  $37.5$  DKK per hour with a standard deviation of  $13.33$  DKK per hour.

Table 3 presents detailed results for the first of the ten subsamples generated for this case study. In addition to a basic MNL model, we estimated a MMNL model using a Normal distribution and a discrete mixture model with two support points on this dataset<sup>8</sup>. In both cases, we allowed for random variations in  $\beta_C$  as well as  $\beta_T$ . Consistent with the true model, no variations were observed for  $\beta_C$  in the discrete mixture model, labelled  $DM(2)_A$ , such that a second model,  $DM(2)_B$ , was estimated, in which  $\beta_C$  was kept fixed.

In a comparison between the three remaining models, MNL, MMNL and  $DM(2)_B$ , we observe that the discrete mixture model outperforms the continuous mixture model, which in turn outperforms the MNL model. In terms of estimation time,  $DM(2)_B$  has clear advantages over the MMNL model, and the higher estimation cost when compared to MNL is well justified on the basis of the improvements in model performance. All three models offer very good performance in retrieving the mean VTTS, while the two mixture models additionally offer good performance in the estimation of the standard deviation.

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<sup>8</sup>No further gains in model performance were obtained by allowing for more than two support points.

Table 3. Estimation results for first simulated dataset.

	<i>MNL</i>		<i>MMNL</i>		<i>DM(2)<sub>A</sub></i>		<i>DM(2)<sub>B</sub></i>	
Final LL	-4565.42		-4122.22		-4007.05		-4007.05	
par.	2		4		8		5	
adj. $\rho^2$	0.3885		0.4476		0.4625		0.4629	
est.time (s)	2		234		17		6	
	est.	asy.t-rat.	est.	asy.t-rat.	est.	asy.t-rat.	est.	asy.t-rat.
$\beta_T$	-0.4081	-36.16	-	-	-	-	-	-
$\beta_{T,\mu}$	-	-	-0.6409	-36.28	-	-	-	-
$\beta_{T,\sigma}$	-	-	0.1553	10.31	-	-	-	-
$\beta_{T,1}$	-	-	-	-	-0.5050	-40.99	-0.5050	-40.99
$\pi_{T,1}$	-	-	-	-	0.7258	50.22	0.7258	50.22
$\beta_{T,2}$	-	-	-	-	-1.0231	-40.81	-1.0231	-40.81
$\pi_{T,2}$	-	-	-	-	0.2742	18.97	0.2742	18.97
$\beta_C$	-0.6424	-34.09	-	-	-	-	-1.0083	-42.20
$\beta_{C,\mu}$	-	-	-1.0613	-36.64	-	-	-	-
$\beta_{C,\sigma}$	-	-	0.2071	9.66	-	-	-	-
$\beta_{C,1}$	-	-	-	-	-1.0083	-12.20	-	-
$\pi_{C,1}$	-	-	-	-	0.3035	0.00	-	-
$\beta_{C,2}$	-	-	-	-	-1.0083	-24.03	-	-
$\pi_{C,2}$	-	-	-	-	0.6965	0.00	-	-
$\mu_{VTTS}$	38.11		37.81		38.50		38.50	
$\sigma_{VTTS}$	-		12.75		13.75		13.75	

A final point deserves some special attention. As mentioned above, we initially allowed for random variation in  $\beta_C$  as well as  $\beta_T$ . The estimation of the first discrete mixture model,  $DM(2)_A$ , offered no evidence of such heterogeneity, such that the model was replaced by  $DM(2)_B$ . However, for the continuous mixture model, *MMNL*, we retrieved significant heterogeneity for  $\beta_C$  as well as for  $\beta_T$ , despite the fact that  $\beta_C$  was kept fixed in the generation of the data. This offers clear evidence of confounding; by being unable to retrieve the correct patterns of heterogeneity for  $\beta_T$ , the *MMNL* model explains part of the remaining error in the model through heterogeneity in  $\beta_C$ . As such, while the model is able to correctly retrieve the mean and standard deviation of the VTTS, it does so by incorrectly indicating a variation across respondents in the sensitivity to changes in travel cost.

The findings from Table 3 are confirmed by a graphical analysis of the shape for the distribution of  $\beta_T$  in Figure 1, where this comparison is made possible by the fact that the mean estimate for  $\beta_C$  is essentially equal to  $-1$  in all models.

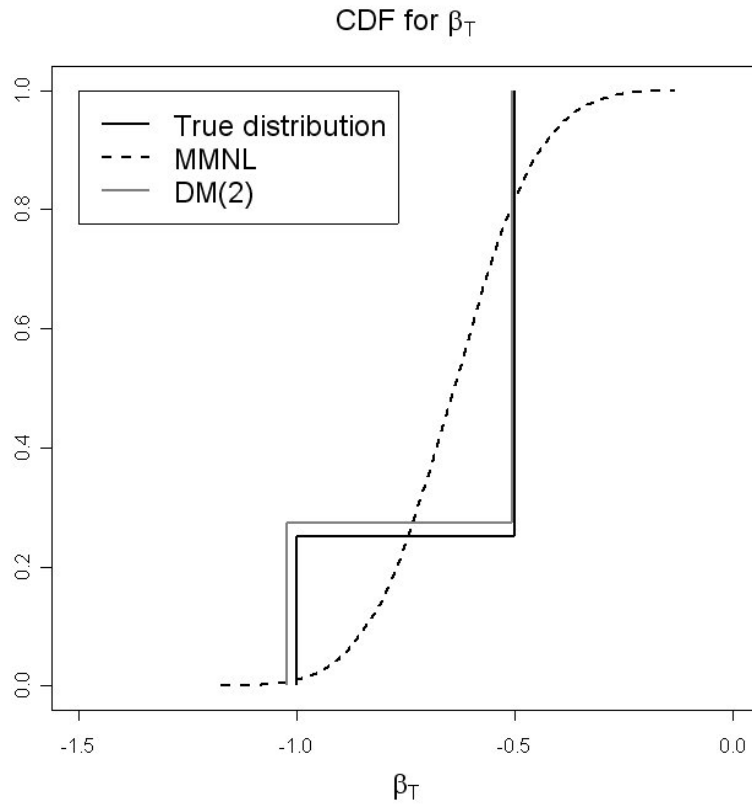


Figure 1: Cumulative distribution function for  $\beta_T$  for first simulated dataset.

#### 4.2 Case study 2: discrete mixture with three support points

In the second case study, the true model is again a discrete mixture of a MNL model, where this time, three support points are used for  $\beta_T$ , at  $-1$ ,  $-0.7$  and  $-0.4$ , with probabilities of 0.3, 0.35 and 0.35. This leads to a true mean VTTS of 41.1 DKK per hour, with a standard deviation of 14.48 DKK per hour. Four different models were estimated on these data; along with the usual MNL and MMNL models, we estimated a



DM with two support points, and a DM with three support points<sup>9</sup>. Again, the DM models were estimated with two different specifications, using a randomly distributed  $\beta_C$  coefficient in DM(2)<sub>A</sub> and DM(3)<sub>A</sub>, and a fixed  $\beta_C$  coefficient in DM(2)<sub>B</sub> and DM(3)<sub>B</sub>. The detailed results for the first sample are presented in Table 4.

Table 4: Estimation results for second simulated dataset.

	MNL		MMNL		DM(2) <sub>A</sub>		DM(2) <sub>B</sub>		DM(3) <sub>A</sub>		DM(3) <sub>B</sub>	
Final LL	-4721.69		-4155.65		-4126.23		-4227.43		-4120.96		-4120.99	
par.	2		4		8		5		12		7	
adj. $\rho^2$	0.3676		0.4431		0.4465		0.4334		0.4467		0.4473	
est.time (s)	1		346		16		6		151		13	
	est.	asy.t-rat	est.	asy.t-rat	est.	asy.t-rat	est.	asy.t-rat	est.	asy.t-rat	est.	asy.t-rat
$\beta_T$	-0.3925	-33.72	-	-	-	-	-	-	-	-	-	-
$\beta_{T,\mu}$	-	-	-0.6817	-34.78	-	-	-	-	-	-	-	-
$\beta_{T,\sigma}$	-	-	0.2423	25.15	-	-	-	-	-	-	-	-
$\beta_{T,1}$	-	-	-	-	-0.8561	-36.56	-0.4005	-36.62	-0.3930	-29.72	-0.7015	-34.17
$\pi_{T,1}$	-	-	-	-	0.6210	27.38	0.5028	27.72	0.3185	14.88	0.4069	14.02
$\beta_{T,2}$	-	-	-	-	-0.4221	-28.99	-0.8084	-39.33	-0.7031	-32.76	-0.3927	-29.84
$\pi_{T,2}$	-	-	-	-	0.3790	16.71	0.4972	27.41	0.4093	13.50	0.3187	14.89
$\beta_{T,3}$	-	-	-	-	-	-	-	-	-1.0262	-32.13	-1.0234	-34.26
$\pi_{T,3}$	-	-	-	-	-	-	-	-	0.2723	10.94	0.2744	11.64
$\beta_C$	-0.5732	-33.39	-	-	-	-	-0.8783	-40.98	-	-	-1.0084	-39.31
$\beta_{C,\mu}$	-	-	-0.9965	-37.48	-	-	-	-	-	-	-	-
$\beta_{C,\sigma}$	-	-	0.0591	4.51	-	-	-	-	-	-	-	-
$\beta_{C,1}$	-	-	-	-	-1.2023	-35.79	-	-	-1.0015	-23.66	-	-
$\pi_{C,1}$	-	-	-	-	0.5357	13.28	-	-	0.8114	0.69	-	-
$\beta_{C,2}$	-	-	-	-	-0.8469	-33.57	-	-	-1.0454	-6.67	-	-
$\pi_{C,2}$	-	-	-	-	0.4643	11.51	-	-	0.1886	0.16	-	-
$\beta_{C,3}$	-	-	-	-	-	-	-	-	-1.2583	0.00	-	-
$\pi_{C,3}$	-	-	-	-	-	-	-	-	0.0000	0.00	-	-
$\mu_{VTTS}$	41.08		41.18		41.22		41.25		41.18		41.09	
$\sigma_{VTTS}$	-		14.86		14.68		13.94		14.41		14.44	

<sup>9</sup>No further gains could be made by using more than three support points.

The results show major improvements for the MMNL and various DM models when compared to the MNL model. All six models perform very well in terms of retrieving the mean VTTS, while the five mixture models also obtain a good approximation to the true standard deviation of the VTTS. We now look in more detail at the differences between the various mixture models. As was the case in the case study discussed in Section 1, the MMNL model again falsely recovers some random variation for  $\beta_C$ , where the level of variation is however much lower than was the case in the first case study. When only allowing for two support points, the DM models also retrieve significant variation for  $\beta_C$ , as reflected in the drop in model fit observed from DM(2)<sub>A</sub> to DM(2)<sub>B</sub> when constraining  $\beta_C$  to a fixed value. This is no longer the case when using three support points. Finally, as was the case in Section 1, the DM models again have a significant advantage over the MMNL model in terms of estimation cost.

Figure 2 shows the cumulative distribution functions for  $\beta_T$  in the MMNL model, as well as in DM(2)<sub>A</sub> and DM(3)<sub>B</sub>. The advantages of the DM models are again very obvious, especially in the case of the model with three support points.

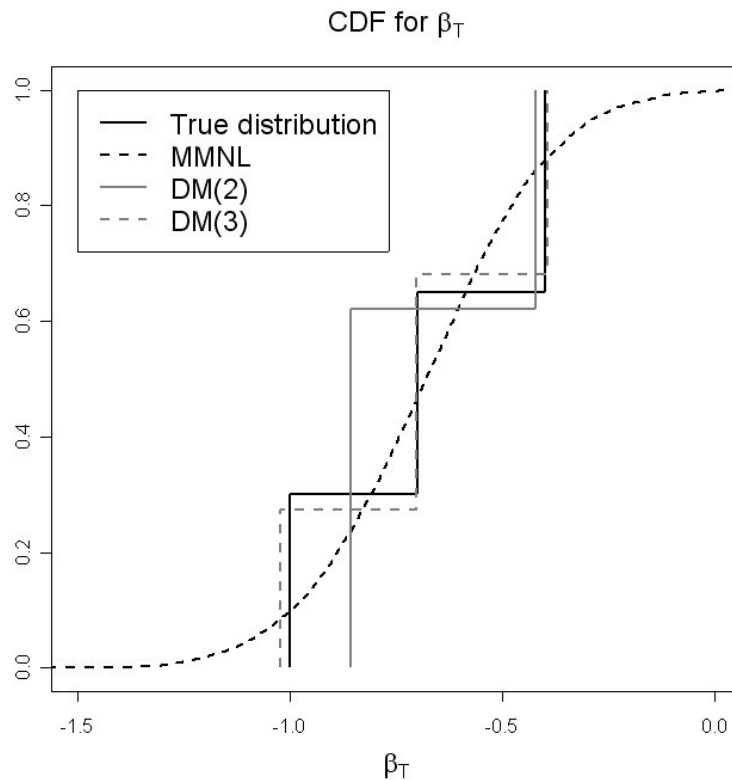


Figure 2: Cumulative distribution function for  $\beta_T$  for second simulated dataset.

### 4.3 Case study 3: normal mixture

For the third case study, a MMNL model with a normally distributed travel time coefficient was chosen as the true model. Specifically,  $\beta_C$  is still fixed to a value of  $-1$ , while  $\beta_T$  now follows a Normal distribution with mean of  $-0.8$  and a standard deviation of  $0.3$ , leading to a mean VTTS of  $48$  DKK/hour, with a standard deviation of  $18$  DKK.

The results for the first subsample of the third simulated dataset are summarised in Table 5. A slightly different strategy was employed in the model estimation in this case study. From the experience of the first two case studies, it had to be assumed that some of the distribution of  $\beta_T$  would erroneously be picked up as heterogeneity in  $\beta_C$ . This would apply especially in the discrete mixture models with a low number of support points. As such, alongside the MNL model, two different MMNL models were estimated, one with  $\beta_C$  kept fixed, and one with a randomly distributed  $\beta_C$ . In the discrete mixture models, 2 support points were used for  $\beta_C$ , while the number of support points for  $\beta_T$  was gradually increased up to the point where no heterogeneity was retrieved for  $\beta_C$ , i.e. the random taste heterogeneity in the data is captured correctly by  $\beta_T$  on its own. It was found that this point was reached between five and six support points for  $\beta_T$ . No further gains in model performance could be obtained by increasing the number of support points for  $\beta_T$  any further, independently of the treatment of  $\beta_C$ .

Again, all the different models offer good performance in retrieving the true mean value of the VTTS, while the various mixture models additionally offer a good approximation to the true standard deviation. The six mixture models offer significant improvements in model performance when compared to the MNL model. As in the other examples, the DM models again have computational advantages over the MNL model. Given the results from the other case studies, it is of interest to look at the issue of confounding between the heterogeneity for  $\beta_T$  and  $\beta_C$ . In the MMNL model and the DM model with six support points, the reductions in model fit resulting from using a fixed  $\beta_C$  coefficient are not significant. With only five support points, the drop in

model fit is slightly more visible ( $DM(5)_A$  vs  $DM(5)_B$ ), yet still not significant when taking into account the cost of estimating three additional parameters. However, in earlier models, using fewer than five support points for  $\beta_T$ , this was not the case, and there were significant amounts of confounding<sup>10</sup>.

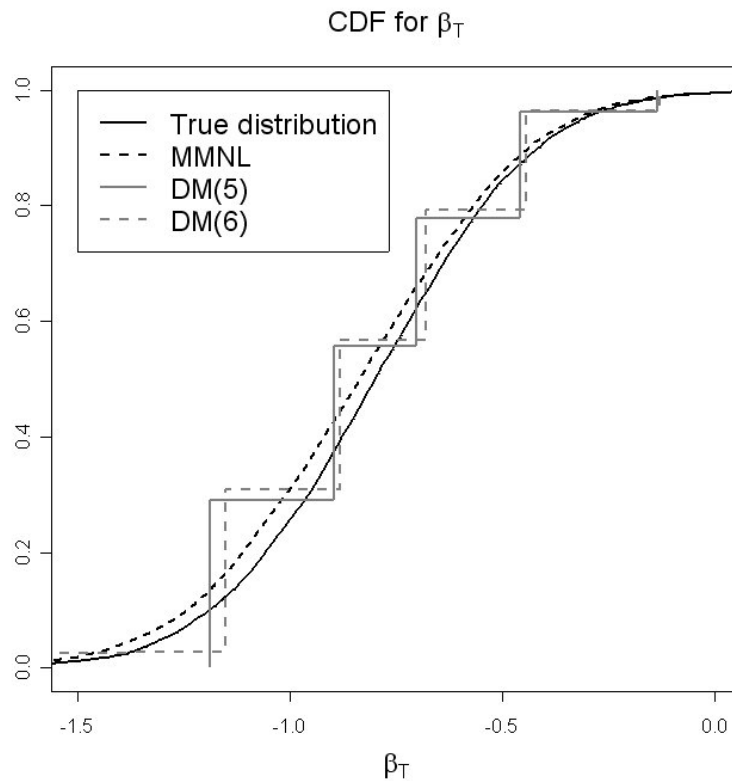


Figure 3: Cumulative distribution function for  $\beta_T$  for third simulated dataset.

Finally, it is of interest to look at the specific patterns of heterogeneity retrieved by the discrete mixture models, where we focus on  $MMNL_B$ ,  $DM(5)_A$  and  $DM(6)_B$ . Here, it can be seen from Figure 3 that the two DM models offer a very good approximation to the Normal distribution.

<sup>10</sup>Detailed results available on request.

Table 5: Estimation results for third simulated dataset.

	MNL		MMNL <sub>A</sub>		MMNL <sub>B</sub>		DM(S) <sub>A</sub>		DM(S) <sub>B</sub>		DM(6) <sub>A</sub>		DM(6) <sub>B</sub>	
Final LL par.	-4742.06		-3912.57		-3913.9		-3910.2		-3913.54		-3908.43		-3908.61	
adj. $\rho^2$	0.3649		0.4756		0.4756		0.4746		0.4746		0.4746		0.4750	
est. time (s)	1		341		233		143		41		141		59	
	est.	asy.t-rat	est.	asy.t-rat	est.	asy.t-rat	est.	asy.t-rat	est.	asy.t-rat	est.	asy.t-rat	est.	asy.t-rat
$\beta_T$	-0.4008	-32.11	-	-	-	-	-	-	-	-	-	-	-	-
$\beta_{T,\mu}$	-	-	-0.8359	-36.67	-0.8329	-36.69	-	-	-	-	-	-	-	-
$\beta_{T,\sigma}$	-	-	0.3134	25.27	0.3113	25.03	-	-	-	-	-	-	-	-
$\beta_{T,1}$	-	-	-	-	-	-	-0.1343	-1.99	-0.1867	-4.82	-0.0859	-1.55	-0.1293	-1.88
$\pi_{T,1}$	-	-	-	-	-	-	0.0372	2.34	0.0566	3.82	0.0245	2.55	0.0362	2.32
$\beta_{T,2}$	-	-	-	-	-	-	-0.4585	-13.87	-0.5071	-17.37	-0.3621	-8.79	-0.4449	-14.26
$\pi_{T,2}$	-	-	-	-	-	-	0.1838	6.55	0.2326	9.14	0.0742	1.90	0.1710	6.66
$\beta_{T,3}$	-	-	-	-	-	-	-0.7021	-22.49	-0.7904	-28.13	-0.7102	-20.65	-0.6800	-24.56
$\pi_{T,3}$	-	-	-	-	-	-	0.2231	3.96	0.3529	9.63	0.2026	3.28	0.2247	4.71
$\beta_{T,4}$	-	-	-	-	-	-	-1.1872	-29.66	-1.1208	-26.75	-0.9006	-19.61	-0.8830	-20.63
$\pi_{T,4}$	-	-	-	-	-	-	0.2905	7.70	0.3296	9.67	0.2653	5.11	0.2597	5.79
$\beta_{T,5}$	-	-	-	-	-	-	-0.8964	-19.79	-1.6253	-20.78	-0.5177	-10.19	-1.1502	-29.93
$\pi_{T,5}$	-	-	-	-	-	-	0.2654	5.25	0.0283	2.85	0.1441	3.01	0.2818	7.47
$\beta_{T,6}$	-	-	-	-	-	-	-	-	-	-	-1.1902	-29.58	-1.6423	-21.24
$\pi_{T,6}$	-	-	-	-	-	-	-	-	-	-	0.2893	7.61	0.0267	3.10
$\beta_C$	-0.4999	-30.13	-	-	-1.0267	-38.53	-	-	-1.0135	-37.67	-	-	-1.0213	-37.66
$\beta_{C,\mu}$	-	-	-1.0254	-38.58	-	-	-	-	-	-	-	-	-	-
$\beta_{C,\sigma}$	-	-	0.0080	0.47	-	-	-	-	-	-	-	-	-	-
$\beta_{C,1}$	-	-	-	-	-	-	-0.7467	-18.57	-	-	-0.7485	-18.50	-	-
$\pi_{C,1}$	-	-	-	-	-	-	0.0862	2.67	-	-	0.0861	2.68	-	-
$\beta_{C,2}$	-	-	-	-	-	-	-1.0545	-34.46	-	-	-1.0569	-34.41	-	-
$\pi_{C,2}$	-	-	-	-	-	-	0.9138	28.31	-	-	0.9139	28.42	-	-
$\mu_{FTTS}$	48.10		48.93		48.68		48.96		48.72		48.77		48.81	
$\sigma_{FTTS}$	-		18.34		18.20		18.08		18.15		18.15		18.15	

#### 4.4 Case study 4: mixture of two normals

For the fourth case study, a more complex mixture was used. As such, the true distribution is now a mixture of two Normal distributions, where  $\beta_T = \pi_1 \beta_{T_1} + \pi_2 \beta_{T_2}$ , with  $\pi_1 = \pi_2 = 0.5$ , and with  $\beta_{T_1} \sim N(-0.8, 0.2)$  and  $\beta_{T_2} \sim N(-0.3, 0.1)$ . The cost coefficient  $\beta_C$  was again kept fixed at  $-1$ . With this, we obtain a true mean VTTS of 33 DKK/hour, with a standard deviation of 17.76 DKK. In model estimation, the strategy from the third case study was again adopted, gradually increasing the number of support points for  $\beta_T$  in the DM models, while maintaining the number of support points for  $\beta_C$  fixed at 2. Again, the issue of confounding largely disappeared when using five or more support points.

The results for the first subsample are presented in Table 4. Along with the MNL model, two MMNL models were estimated, where  $MMNL_A$  and  $MMNL_B$  again differ by using a randomly distributed and fixed  $\beta_C$  coefficient respectively. Although the standard deviation for  $\beta_C$  is significantly different from zero in model  $MMNL_A$ , it is very small compared to the mean value, such that it is no surprise that the effect of using a fixed coefficient is very small, with very similar model performance for  $MMNL_B$ . In the DM models, we experience a very small, and insignificant drop in model fit when constraining  $\beta_C$  to a single value. Here, two further observations can be made. In model  $DM(5)_A$ , the difference between  $\beta_{C,1}$  and  $\beta_{C,2}$  is not significant beyond the 48% level of confidence, while, in model  $DM(6)_A$ , it is not significant beyond the 50% level of difference. It can also be seen that, on average, when moving from  $DM(5)_A$  to  $DM(5)_B$  and from  $DM(6)_A$  to  $DM(6)_B$ , the standard errors associated with the various  $\pi_{T,k}$  parameters decrease. Finally, model  $DM(6)_B$  can be seen to reduce to model  $DM(5)_B$ ; the additional support point, as well as its associated probability, are not significantly different from zero. All seven models again offer good performance in the retrieval of the true mean VTTS, where the six mixture models also perform well for the standard deviation. The DM models maintain their advantages in terms of estimation cost, where these are naturally smaller than before given the higher number of parameters. In terms

of model performance, the MMNL models clearly outperform the MNL model, while the various DM models have a small advantage over the MMNL models.

When looking at the retrieval of the true shape for the distribution of  $\beta_T$ , it can be seen that the MMNL models using a single Normal distribution produce a mean that is the weighted average of the mean of the two Normal distributions. The DM models on the other hand do recover the multi-modality of the true distribution<sup>11</sup>. These findings are reflected in the shape of the distributions for  $\beta_T$  in Figure 4, where the DM models (DM(5)<sub>A</sub> and DM(6)<sub>B</sub>) are better able to account for the multi-modality of the true distribution.

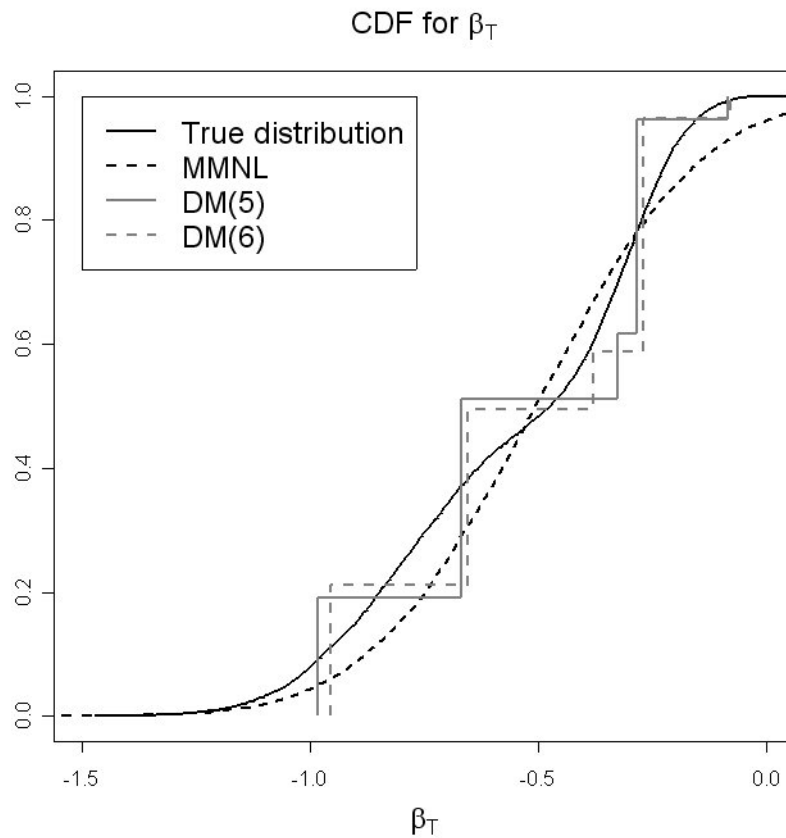


Figure 4: Cumulative distribution function for  $\beta_T$  for fourth simulated dataset.

<sup>11</sup>It should be noted that, in the retained DM model, DM(5)<sub>B</sub>, two of the probabilities for support points,  $\pi_{T,1}$  and  $\pi_{T,3}$ , are only significant at the 85% level of confidence.

Table 6: Estimation results for fourth simulated dataset.

	MNL		MMNL <sub>A</sub>		MMNL <sub>B</sub>		DM(5) <sub>A</sub>		DM(5) <sub>B</sub>		DM(6) <sub>A</sub>		DM(6) <sub>B</sub>	
	est.	asy,t-rat	est.	asy,t-rat	est.	asy,t-rat	est.	asy,t-rat	est.	asy,t-rat	est.	asy,t-rat	est.	asy,t-rat
Final LL	-5296.84	-27.80	-4405.54	-4406.11	-4359.07	-4363.23	-4359.03	-4363.23	-4359.07	-4363.23	-4359.03	-4363.23	-4359.03	-4363.23
par.	2		4	3	14	11	16	11	14	11	16	13	13	
adj. $\rho^2$	0.2906		0.4096	0.4097	0.4145	0.4144	0.4143	0.4144	0.4145	0.4144	0.4143	0.4141	0.4141	
est.time (s)	7		341	213	197	33	174	33	197	33	174	75	75	
$\beta_T$	-0.2153	-	-	-	-	-	-	-	-	-	-	-	-	
$\beta_{T,\mu}$	-	-	-0.5115	-30.62	-	-	-	-	-	-	-	-	-	
$\beta_{T,\sigma}$	-	-	0.2930	29.23	0.2954	29.82	-	-	-	-	-	-	-	
$\beta_{T,1}$	-	-	-	-	-	-	-0.0844	-1.33	-0.0787	-1.27	-0.0855	-1.36	-0.0787	-1.27
$\pi_{T,1}$	-	-	-	-	0.0377	1.42	0.0377	1.42	0.0363	1.47	0.0385	1.41	0.0363	1.47
$\beta_{T,2}$	-	-	-	-	-0.2831	-7.90	-0.2831	-7.90	-0.2713	-16.19	-0.2835	-7.01	-0.2713	-16.19
$\pi_{T,2}$	-	-	-	-	0.3461	0.98	0.3461	0.98	0.3761	6.32	0.3395	0.77	0.3761	6.32
$\beta_{T,3}$	-	-	-	-	-0.9833	-20.76	-0.9833	-20.76	-0.3788	-11.94	-0.3216	-3.95	-0.3788	-11.94
$\pi_{T,3}$	-	-	-	-	0.1900	4.71	0.1900	4.71	0.0921	1.46	0.1102	0.25	0.0921	1.46
$\beta_{T,4}$	-	-	-	-	-0.3253	-4.58	-0.3253	-4.58	-0.6543	-28.15	-0.7247	-9.82	-0.6543	-28.15
$\pi_{T,4}$	-	-	-	-	0.1047	0.29	0.1047	0.29	0.2848	10.16	0.0749	0.33	0.2848	10.16
$\beta_{T,5}$	-	-	-	-	-0.6690	-20.61	-0.6690	-20.61	-0.9546	-30.00	-0.6555	-11.72	-0.6543	-28.15
$\pi_{T,5}$	-	-	-	-	0.3215	8.36	0.3215	8.36	0.2107	8.94	0.2508	1.13	0.2848	10.16
$\beta_{T,6}$	-	-	-	-	-	-	-	-	-	-	-0.9842	-21.79	-0.9546	-30.00
$\pi_{T,6}$	-	-	-	-	-	-	-	-	-	-	0.1860	4.77	0.2107	8.94
$\beta_C$	-0.4194	-31.89	-	-	-0.9721	-37.89	-	-	-0.9737	-37.40	-	-	-0.9737	-37.40
$\beta_{C,\mu}$	-	-	-0.9718	-38.17	-	-	-	-	-	-	-	-	-	-
$\beta_{C,\sigma}$	-	-	0.0352	1.97	-	-	-	-	-	-	-	-	-	-
$\beta_{C,1}$	-	-	-	-	-	-	-1.0781	-22.66	-	-	-0.8725	-15.68	-	-
$\pi_{C,1}$	-	-	-	-	0.5965	4.03	0.5965	4.03	-	-	0.3572	1.70	-	-
$\beta_{C,2}$	-	-	-	-	-0.8801	-20.83	-0.8801	-20.83	-	-	-1.0662	-18.58	-	-
$\pi_{C,2}$	-	-	-	-	0.4035	2.73	0.4035	2.73	-	-	0.6428	3.06	-	-
$\mu_{FTTS}$	30.80		31.60	31.82	32.60	32.49	32.64	32.49	32.60	32.49	32.64	32.49	32.49	
$\sigma_{FTTS}$	-		18.15	18.23	17.39	17.10	17.38	17.10	17.39	17.10	17.38	17.10	17.10	



In closing, it should be noted that, in this example, the uni-modal MMNL model still manages to retrieve the true mean and standard deviation of the multi-modal true distribution of the VTTS. This can be explained by the fact that the probabilities for the two Normal distributions were set evenly to 0.5, where the difference in the standard deviation for  $\beta_{T_1}$  and  $\beta_{T_2}$  was also rather small. Different patterns could be expected in a more asymmetrical scenario.

## 5. Summary and conclusions

With the availability of powerful computers and estimation tools, researchers and practitioners are increasingly making use of continuous mixture structures, such as Mixed Logit, in the representation of random taste heterogeneity across respondents. Despite the gains in estimation power, the cost of using such mixture models remains high, especially in large scale studies. Furthermore, several issues arise due to the models' reliance on specific distribution functions, whose shape is not necessarily consistent with that of the *true*, unobserved distribution.

In this paper, we have discussed an alternative approach for the representation of random taste heterogeneity, making use of discrete mixtures instead of continuous mixtures. Although several issues can also arise in the estimation of such models, they have the advantage of a closed form solution, and can hence be estimated and applied without relying on simulation processes. Furthermore, the models are free from a priori assumptions as to the shape of the *true* distribution.

The paper presents several case studies offering an in-depth comparison of the two modelling approaches, making use of real data as well as four separate simulated datasets. The results of these analyses clearly show the major advantage of the discrete mixture approach in terms of estimation cost. They also show that, across scenarios, the discrete mixture models are able to attain similar or indeed better performance than their

continuous counterparts. Finally, they are better able to deal with complicated *true* distributions, such as the presence of multiple modes.

Although further comparisons between the two modelling approaches are required, the results from this paper do suggest that discrete mixture models present a viable alternative, partly thanks to their lower cost in estimation and application, but also due to the absence of a priori shape assumptions, which is of great interest in the context of recent discussions of the issue of the specification of continuous heterogeneity by Hess et al. (2005).

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