MULTIPLE PERIODIC SOLUTIONS FOR AUTONOMOUS LAGRANGIAN SYSTEMS (*)

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SOMMARIO. - *In questo lavoro si studia il seguente sistema Lagrangiano autonomo*

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \xi}(q,\dot{q}) - \frac{\partial \mathcal{L}}{\partial q}(q,\dot{q}) = 0 \quad q \in C^2(\mathbf{R},\mathbf{R}^N)$$

dove

$$\mathcal{L}(q,\xi) = \frac{1}{2}\sum_{i,j=1}^{N} a_{ij}(q)\xi_i\xi_j - V(q) \quad q,\xi \in \mathbf{R}^N .$$

Con metodi variazionali, si stabilisce l'esistenza di soluzioni periodiche multiple di periodo prefissato, nei casi in cui $V(q) \to c$ *per* $|q| \to +\infty$ *e* V *non è limitato ed è sottoquadratico all'infinito.*

SUMMARY. - *This paper deals with the autonomous Lagrangian system*

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \xi}(q,\dot{q}) - \frac{\partial \mathcal{L}}{\partial q}(q,\dot{q}) = 0 \quad q \in C^2(\mathbf{R},\mathbf{R}^N)$$

where

$$\mathcal{L}(q,\xi) = \frac{1}{2}\sum_{i,j=1}^{N} a_{ij}(q)\xi_i\xi_j - V(q) \quad q,\xi \in \mathbf{R}^N .$$

Using variational methods the existence of multiple periodic solutions of prescribed period is established, when $V(q) \to c$ *as* $|q| \to +\infty$ *and when* V *is subquadratic and unbounded at infinity.*

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1. Introduction.

In this paper it will be studied the existence of periodic solutions $q = q(t)$ of the Lagrangian system of ordinary differential equations:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \xi}(q,\dot{q}) - \frac{\partial \mathcal{L}}{\partial q}(q,\dot{q}) = 0 \quad q \in C^2(\mathbf{R},\mathbf{R}^N) \tag{1.1}$$

where $\mathcal{L}$ denotes the Lagrangian function

$$\mathcal{L}(q,\xi) = \frac{1}{2}\sum_{i,j=1}^{N} a_{ij}(q)\xi_i\xi_j - V(q) \quad q,\xi \in \mathbf{R}^N .$$

the a_{ij}'s $(i,j = 1,\ldots,N)$ and V being C^1 real functions of $\mathbf{R}^N$.

As problem (1.1) is autonomous, it admits trivial solutions (i.e. the critical points of V) and thus one of the difficulties which arises when studying such a problem, lies in avoiding trivial solutions.

The present note is essentially devoted to the study of problem (1.1) when the potential V is bounded.

That problem has been already examined when the a_{ij} are constant and V depends on t (see [4],[6],[7],[8],[9] and [10]) or when symmetry hypotheses on V are assumed (see [5]).

Here it will be shown that when V is bounded and satisfies suitable technical assumptions, multiple nontrivial solutions of problem (1.1), depending on a prescribed period T, are found.

Actually, a number of solutions large as desidered can be obtained on condition that longer periods are prescribed.

Finally, it will be pointed out that an analogous result holds also when V is assumed to be unbounded and subquadratic at infinity, thus generalizing [2],[10],[11].

Even when the particular case of the a_{ij} constant is considered our results are not contained in the quoted papers.

2. Notations and preliminaries.

Some notations which will be used in the following sections, are now stated:

– $|\cdot|$ denotes the Euclidean norm of $\mathbf{R}^N$ and $(\cdot|\cdot)$ its usual inner product;

– if $1 \leq p < \infty$, the space

$$L^p = L^p(S^1, \mathbf{R}^N) = \{q : \mathbf{R} \to \mathbf{R}^N | q\ 2\pi - \text{periodic}, \int_0^{2\pi} |q(t)|^p < \infty\}$$

is meant to be endowed with the usual L^p norm, here denoted by $|\cdot|_p$, while $L^\infty = L^\infty(S^1, \mathbf{R}^N)$ indicates the space of the essentially bounded 2π-periodic $\mathbf{R}^N$-valued functions, endowed with the usual norm $|\cdot|_\infty$;

– $H^1 = H^1(S^1, \mathbf{R}^N)$ represents the Sobolev space obtained by the closure of the C^∞ 2π-periodic $\mathbf{R}^N$-valued functions $q = q(t)$, endowed with the norm

$$||q|| = \int_0^{2\pi} [(|\dot{q}|^2 + |q|^2)\, dt]^{1/2}$$

– B_R indicates the closed H^1-ball of radius R centered at the origin, while ∂B_R denotes its boundary;

– if f is a C^1 functional on H^1, $f'(q)$ denotes the Frechét derivative at $q \in H^1$.

Some shortened matrix notations are now further established for the functions $a_{ij}(i,j = 1,\ldots,N)$:

$$a(q) = \{a_{ij}(q)\} \quad i,j = 1,\ldots,N \tag{2.1}$$

$$a'(q)v = \{(\nabla a_{ij}(q)|v)\} \quad i,j = 1,\ldots,N \tag{2.2}$$

In all the theorems which will be set in the following sections the matrix $a(q)$ is assumed to be symmetric.

Moreover, given a Hilbert space H and a functional $f \in C^1(H, \mathbf{R})$, f is said to satisfy the Palais-Smale condition, here recalled in its weaker version, iff:

(P.S.) Any sequence $\{q_n\}$ in H such that $\{f(q_n)\}$ is bounded and $||f'(q_n)||\ ||q_n|| \to 0$, possesses a convergent subsequence in H.

3. The main results.

The research of T-periodic solutions of problem (1.1) will be carried on using variational methods.

Actually, consider the following action functional

$$(3.1) \qquad f(q) = \frac{1}{2}\int_0^{2\pi} (a(q)\dot{q}|\dot{q})\,dt - \omega^2 \int_0^{2\pi} V(q)\,dt$$

where $\omega = T/2\pi$.

It is well known that f is C^1 in the space H^1 and that its critical points corresponds to the T-periodic solutions of problem (1.1).

The main theorem of this paper is the following:

THEOREM 3.1. *Let us assume that:*

(3.2) *there exists* $\mu > 0$ *such that*

$$(a(q)\xi|\xi) \geq \mu|\xi|^2 \quad \textit{for each } q,\xi \in \mathbf{R}^N$$

(3.3) $\lim_{|q|\to+\infty} a'_{ij}(q) = 0$ *for any* $i,j = 1,\ldots,N$

(3.4) *there exists* $c \in \mathbf{R}$ *such that*

$$\lim_{|q|\to+\infty} V(q) = c$$

(3.5) $V(q) < c$, *for any* $q \in \mathbf{R}^N$

(3.6) $\lim_{|q|\to+\infty} V'(q) = 0$

(3.7) *the set of the critical points of* V *is bounded.*

Then, for any $k \in \mathbf{N}$, $k \neq 0$, *there exists* $\bar{T}(K) \in \mathbf{R}_+$ *such that, for any* $T \geq \bar{T}(k)$, *problem (1.1) admits at least* kN T*-periodic geometrically distinct solutions.*

We want to point out that the thesis of theorem 3.1 still holds when the potential V is unbounded and subquadratic at infinity.

Indeed, the following theorem holds:

THEOREM 3.2. *Suppose (3.2), (3.7) hold in addition with the following further hypotheses:*

(3.8) *there exist* $\alpha \in]0,2[$ *and* $R \in \mathbf{R}_+$ *such that*

$$(\nabla V(q)|q) - \alpha V(q) \leq 0 \quad \textit{for any } q \in \mathbf{R}^N,\ |q| > R$$

(3.9) $V(q) \to +\infty,\ as\ |q| \to +\infty$

(3.10) *there exists* $\beta \in]0, 2 - \alpha[$ *such that*

$$a'(q)q + \beta a(q) \quad \textit{is positive semidefinite}\ .$$

Then, for any $k \in \mathbf{N}$, $k \neq 0$, *there exists* $\bar{T}(k) \in \mathbf{R}_+$ *such that for any* $T \geq \bar{T}(k)$ *problem (1.1) admits at least* kN T*-periodic geometrically distinct solutions.*

4. Proofs of the theorems.

In order to prove the theorems of the above section, first we need to recall an abstract theorem, a lemma and then state a particular case of theorem 3.1.

THEOREM 4.1. *Let* H *be a real Hilbert space on which a unitary representation* G *of the* S^1 *group acts.*

Suppose that $f \in C^1(H, \mathbf{R})$ *verifies the following assumptions:*

(4.1) f *is invariant under the action of* G;

(4.2) f *satisfies the (P.S.) condition;*

(4.3) *there exist two closed subspaces* V *and* W *of* H *with* $\operatorname{codim} W < +\infty$ *and there exist two real constants* $c_0 > c_\infty$ *and* $\rho \in \mathbf{R}_+$ *such that*

(i) $f(q) < c_0$ *for each* $q \in \partial B_\rho \cap V$;

(ii) $f(q) \geq c_\infty$ *for each* $q \in W$;

(4.4) *$f(q) > c_0$ for each $q \in Fix\ S^1$ such that $f'(q) = 0$. Then there exist at least*

$$\frac{1}{2}(\dim V - \operatorname{codim} W)$$

orbits of critical points with critical values in $[c_\infty, c_0]$.

Proof. The claim follows from theorem 2.4 of [1], by suitable modifications contained in theorem 1.4 of [3].

LEMMA 4.2. *Suppose the hypotheses of theorem 3.1 hold; then the (P.S.) condition is satisfied by f in $\mathbf{R} - \{-2\pi\omega^2 c\}$.*

Proof. See lemma 4.3 of [10].

THEOREM 4.3. *Assume that (3.2), (3.3), (3.4), (3.5), (3.6) hold and that*

(4.5) *$V(0) = 0$ is the minimum of V and $V'(q) \neq 0$ for any $q \neq 0$.*

Then for any $k \in \mathbf{N}$, $k \neq 0$, there exists $\bar{T}(k) \in \mathbf{R}_+$ such that for any $T \geq \bar{T}(k)$, problem (1.1) admits at least kN T-periodic distinct solutions.

Proof. Let us consider the following subspace of H^1

$$W = \bigoplus_{n \geq 1} M_{\lambda_n}$$

$$W_k = \bigoplus_{n \leq k} M_{\lambda_n} \quad k \in \mathbf{N},\ k \neq 0$$

where M_{λ_n} denotes the eigenspace corresponding to the eigenvalue λ_n of the operator $q \to -\ddot{q}$ in H^1.

We want to show that, for any fixed $k \in \mathbf{N}$, $k \neq 0$, there exist $c_\infty < c_0$ and $R \in \mathbf{R}_+$ such that

(4.6) $$f(q) < c_0 \quad \text{for each } q \in W_k,\ ||q|| = R;$$

(4.7) $$f(q) \geq c_\infty \quad \text{for each } q \in W;$$

(4.8) $$f(q) > c_0 \quad \text{for each } q \in \mathbf{R}^N s.t. f'(q) = 0\ .$$

Let $c_m = V(0)$ the minimum of V, then

(4.9) there exist $r \in \mathbf{R}_+$ and $\varepsilon(r) \in \mathbf{R}_+$ such that:

$$|q| > r \Rightarrow V(q) > c_m + \varepsilon(r) .$$

Moreover, if $q \in W_k$, denote q_i the component of q in the space M_{λ_i}. Then, for any $R \in \mathbf{R}_+$, $R \geq 1$, set

$$2\pi c_m + M_R = \inf \{ \int_0^{2\pi} V(q)\,dt | q \in W_k,\ \exists i \in \{1,\ldots,k\} \ni' \\ 1 \leq ||q_i|| \leq R\} \tag{4.10}$$

Thence it easily follows that

$$M_R > 0 . \tag{4.11}$$

In order to prove (4.6), let us fix $q \in W_k$, $||q|| = R$.
i) If $||q_i|| < 1$ for any $i \in \{1,\ldots,k\}$, then

$$||q_0||^2 = R^2 - \sum_{i=1}^{k} ||q_i||^2 > R^2 - k .$$

Hence, it can be found R large enough and $c_1, c_2 \in \mathbf{R}_+$ such that:

$$|q|_\infty \geq \left| |q_0|_\infty - |\sum_{i=1}^{k} q_i|_\infty \right| \geq c_1\sqrt{R^2 - k} - c_2 k > r . \tag{4.12}$$

Let us denote

$$D = \{t \in [0,2\pi] \big| |q(t)| > r\}$$

then, by (4.9), there exists $\varepsilon'(R) \in \mathbf{R}_+$ such that

$$\int_0^{2\pi} V(q)\,dt = \int_D V(q)\,dt + \int_{[0,2\pi]-D} V(q)\,dt > \\ > (c_m + \varepsilon(r))\operatorname{mis} D + c_m(2\pi - \operatorname{mis} D) = \\ = 2\pi c_m + \varepsilon'(R) . \tag{4.13}$$

Moreover there exists $c_3 \in \mathbf{R}_+$ such that

$$f(q) \leq c_3 \sup_{\substack{\|q_i\| \leq 1 \\ i=1,\dots,k}} |a(q)|_\infty R^2 - \omega^2 \int_0^{2\pi} V(q)\,dt . \tag{4.14}$$

By (4.13) and (4.14) it follows that

$$f(q) \leq c_3 \sup_{\substack{\|q_i\| \leq 1 \\ i=1,\dots,k}} |a(q)|_\infty R^2 - 2\pi\omega^2 c_m - \omega^2 \varepsilon'(R) .$$

Then a $\omega_1 = \omega_1(R)$ large enough can be chosen and $c_4 \in \mathbf{R}$ exists such that, for each $q \in W_k$, $\|q\| = R$

$$f(q) \leq -2\pi\omega_1^2 c_m - c_4\omega_1^2 .$$

Hence

$$\sup_{\substack{q \in W_k \\ \|q\| = R}} f(q) < -2\pi\omega_1^2 c_m . \tag{4.15}$$

ii) If there exists $i \in \{1, \dots, k\}$ such that $1 \leq \|q_i\| \leq R$ then by (4.10)

$$\int_0^{2\pi} V(q)\,dt \geq 2\pi c_m + M_R$$

and hence

$$f(q) \leq c_3 \sup_{\|q\| \leq R} |a(q)|_\infty |\dot{q}|_2^2 - 2\pi\omega^2 c_m - M_R\omega^2 . \tag{4.16}$$

Then a $\omega_2 = \omega_2(R)$ large enough can be chosen such that

$$f(q) < -2\pi\omega_2^2 c_m - (M_R/2)\omega_2^2 \quad \text{for each } q \in W_k,\ \|q\| = R .$$

It follows that

$$\sup_{\substack{q \in W_k \\ \|q\| = R}} f(q) < -2\pi\omega_2^2 c_m . \tag{4.17}$$

As (4.15) and (4.17) hold, there exist $\omega = \omega(R)$ and $c_0 \in \mathbf{R}$ such that

$$\sup_{\substack{q \in W_k \\ \|q\|=R}} f(q) \leq c_0$$

and

$$c_0 < -2\pi\omega^2 c_m . \tag{4.18}$$

In order to prove (4.7), fix $q \in W$; to reach the claim it is enough to remark that (3.2) and (3.5) implies the existence of $c_\infty \in \mathbf{R}$ such that

$$f(q) \geq \frac{\mu}{2}|\dot{q}|_2^2 - 2\pi\omega^2 c \geq \lambda_1 \frac{\mu}{2}|q|_2^2 - 2\pi\omega^2 c \geq c_\infty .$$

Finally, by virtue of (4.5) and (4.18), $q = 0$ is the only element of $\mathbf{R}^N$ such that $f'(q) = 0$ and $f(0) = -2\pi\omega^2 c_m > c_0$; hence (4.8) holds too.

The functional f has been proved to satisfy (4.6), (4.7), (4.8) and the (P.S.) condition, hence theorem 4.1 holds and thus f has at least

$$\frac{1}{2}(\dim W_k - \operatorname{codim} W) = kN$$

orbits of critical points.

Proof of Theorem 3.1. Let us denote c_m the minimum of the potential V and

$$c_s = \sup\{y \in \mathbf{R} \,|\, \exists q \in \mathbf{R}^N \ni' V'(q) = 0 \wedge V(q) = y\} . \tag{4.19}$$

Remark that (3.4) implies that if $0 < \varepsilon < c - c_s$, there exists $r \in \mathbf{R}_+$ such that, for any $q \in \mathbf{R}^N$:

$$|q| > r \Rightarrow V(q) > c_s + \varepsilon . \tag{4.20}$$

For the sake of brevity, let us suppose $k = 1$; consider the following subspaces of H^1

$$W = \bigoplus_{n \geq 1} M_{\lambda_n} \quad W_1 = \mathbf{R}^N \oplus M_{\lambda_1}$$

where the spaces M_{λ_n} are the ones defined in theorem 4.3; let us prove that the hypotheses of theorem 3.2 are satisfied.

Take $r^* > r$, $R > r^*$ and

$$2\pi c_s + M_R = \inf\{\int_0^{2\pi} V(q)\,dt \mid q = q_0 + q_1 \in W_1, \quad r^* \leq \|q_1\| \leq R\} \tag{4.21}$$

then, it can be proved that

$$M_R > 0 \tag{4.22}$$

Indeed, denoting $c^* = c_s - c_m$ and

$$D = \{t \in [0,2\pi] \mid |q(t)| > r\}$$

by (4.20) it follows that

$$\begin{aligned}\int_0^{2\pi} V(q)\,dt &= \int_D V(q)\,dt + \int_{[0,2\pi]-D} V(q)\,dt > \\ &> (c_s+\varepsilon)\text{mis}\, D + c_m(2\pi - \text{mis}\, D) = 2\pi c_s + \\ &+ (\varepsilon + c^*)\text{mis}\, D - 2\pi c^* .\end{aligned} \tag{4.23}$$

As $M_{\lambda_i} = \text{span}\{\phi_i \sin t, \phi_i \cos t : i = 1,2,\ldots,N\}$ where $\{\phi_i\}$ is the standard basis in $\mathbf{R}^N$, it can be chosen $\varepsilon^* > r$ large enough such that $\|q\| > r^*$ implies that

$$\text{mis}\, D > \frac{2\pi c^*}{c^* + \varepsilon} . \tag{4.24}$$

By (4.23) and (4.24) it follows that (4.22) holds.

We want to show now that there exist $c_0 \in \mathbf{R}$ and $R \in \mathbf{R}_+$ such that

$$f(q) \leq c_0 \quad \text{for each } q \in W_1,\ \|q\| = R . \tag{4.25}$$

In order to do so, let us consider $R > r^*$ and $q = q_0 + q_1 \in W_1$, $\|q\| = R$.

i) If $||q_1|| < r^*$, then

$$||q_0||^2 = ||q||^2 - ||q_1||^2 > R^2 - r^{*2}$$

and hence it can be chosen R large enough such that

$$|q(t)| \geq |\,|q_0| - |q_1(t)|\,| > r \quad \text{for each } t \in [0, 2\pi]$$

and thus, by (4.20)

$$\int_0^{2\pi} V(q)\,dt > 2\pi(c_s + \varepsilon)\ .$$

As $c_3 \in \mathbf{R}_+$ exists such that

$$f(q) \leq c_3 \sup_{||q_1||=r^*} |a(q)|_\infty R^2 - \omega^2 \int_0^{2\pi} V(q)\,dt$$

then it can be chosen $\omega_1 = \omega_1(R)$ large enough and $c_5 \in \mathbf{R}$ such that

$$(4.26) \qquad f(q) \leq -2\pi\omega_1^2 c_s - c_5\omega_1^2 \quad \text{for each } q \in W_1,\ ||q|| = R\,.$$

ii) If $r^* \leq ||q_1|| \leq R$, by (4.21) it follows that

$$\int_0^{2\pi} V(q)\,dt \geq 2\pi c_s + M_R$$

and then

$$f(q) \leq c_3 \sup_{||q|| \leq R} |a(q)|_\infty R^2 - 2\pi\omega^2 c_s - \omega^2 M_R\,.$$

If $\omega_2 = \omega_2(R)$ is chosen large enough, then

$$(4.27) \qquad f(q) \leq -2\pi\omega_2^2 c_s - M_R\omega_2^2 \quad \text{for each } q \in W_1,\ ||q|| = R\,.$$

Gathering (4.26) and (4.27) we obtain that there exists $\omega = \omega(R)$ and $c_0 \in \mathbf{R}$ such that

$$\sup_{\substack{q \in W_1 \\ ||q|| = R}} f(q) \leq c_0$$

and

$$c_0 < -2\pi\omega^2 c_s \,. \tag{4.28}$$

Moreover, in the same way as in theorem 4.3 it can be proved that there exists $c_\infty < c_0$ such that

$$f(q) \geq c_\infty \quad \text{for each } q \in W$$

Furthermore, if $q \in \mathbf{R}^N$ and $f'(q) = 0$ then $V'(q) = 0$ and therefore $V(q) \leq c_s$.

By virtue of (4.28), that implies

$$f(q) = -\omega^2 \int_0^{2\pi} V(q)\,dt \geq -2\pi\omega^2 c_s > c_0 \,.$$

Thus, the hypotheses of theorem 4.1 are satisfied and f admits at least

$$\frac{1}{2}(\dim W_1 - \operatorname{codim} W) = N$$

orbits of critical points.

If $k \in \mathbf{N}$, $k \geq 2$, let us consider the following subspaces of H^1:

$$W = \underset{n \geq 1}{\oplus} M_{\lambda_n}\,, \qquad W_k = \underset{n \leq k}{\oplus} M_{\lambda_n}\,.$$

The same arguments used above and the techniques of theorem 4.2 can be employed to prove that there exist

$$\frac{1}{2}(\dim W_k - \operatorname{codim} W) = kN$$

orbits of critical points.

Proof of Theorem 3.2. The claim of the theorem is easily reached arguing as in theorem 3.1.

We only need to remark that (4.20) still holds, for any $\varepsilon > 0$, because of (3.9) and that f satisfies the (P.S.) condition (see lemma 3.3 in [10]).

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