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Lectures on Rosenthal's l^1-Theorem

Ehrhard Behrends (*)

Summary. - *Let* (x_n) *be a bounded sequence in a Banach space* X*.* Rosenthal's l^1-theorem *states that there is essentially only one exceptional situation where it is* not *possible to extract a subsequence which is a weak Cauchy sequence: This happens if* (x_n) *is the sequence of unit vectors in* l^1*.*

The aim of these lectures is twofold: On the one hand results from the last few years centering around this theorem are presented, and on the other hand the opportunity is taken to introduce the audience to a number of techniques which are of importance in modern Banach space theory (Ramsey theory, Martin's axiom, ...).

1. Introduction

Let X be a Banach space and (x_n) a sequence in X. What assertions concerning *convergence* can be made in such a general situation? Since – with respect to every reasonable definition of convergence – convergent sequences are *bounded* it is natural to assume throughout that the sequence (x_n) under consideration is a bounded sequence.

Will there be a *norm convergent* subsequence? In general this will surely not be be the case: Every bounded sequence of X contains a (norm) convergent subsequence iff X is finite-dimensional. This well-known and elementary fact forces one to investigate less restrictive

(*) Author's adress: I. Mathematisches Institut, Freie Universität Berlin, Arnimallee 2–6, D-14 195 Berlin, Germany, e–mail: behrends@math.fu-berlin.de.
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forms of convergence. A natural candidate is *weak convergence*. Recall that, for $x, x_1, x_2, \ldots$ in X one says that (x_n) *converges weakly to* x provided that the sequence $(x'(x_n))$ (of scalars) converges to $x'(x)$ for every continuous linear functional x' on X (as usual, the collection of these x' , i. e. the *dual of* X, will be denoted by X').

It has to be emphasized that this type of convergence is very important in applications. In a sense, the calculation of $x'(x)$ can be considered as a measurement of a certain "aspect" of x: A position, an energy, ... Thus, if (x_n) tends to x weakly, the x_n for "large" n are as good as x if only certain aspects of x are under consideration.

Examples where this type of convergence is of crucial importance are mathematical physics (quantum mechanics), the theory of differential equations (weak solutions), probability theory (convergence in distribution).

In the sequel we will make free use of some properties of weak convergence, also the connection with reflexivity should be known.

Returning to the problem we started with we now ask: Does every bounded sequence of X has a weakly convergent subsequence? The answer: This holds iff X is reflexive. The proof of this assertion is usually beyond the scope of a lecture in functional analysis. The reason is the necessity to know some topological subtleties. (Usually sequences do not suffice to characterize compactness in topological spaces, but in the case of the weak topology this holds by the theorem of Eberlein-Šmulian. With this theorem and the well-known facts on reflexivity it is easy to derive the characterization; see e. g. [8, theorem V.6.1]).

As a third (and last) step we weaken our question once more. We will say that (x_n) is a *weak Cauchy sequence* if $(x'(x_n))$ is convergent for every $x' \in X'$.

Its plain that this is formally weaker than weak convergence: If (x_n) is weakly Cauchy the "aspect" x' of this sequence will converge to something which will not necessarily have the form $x'(x)$ for a suitable vector x.

But the generalization is not only formal: Denote by e_n the n-th unit vector in c_0 (= the space of null suquences) and set

$x_n := e_1 + \cdots + e_n$. Then (x_n) is a weak Cauchy sequence which is not weakly convergent.

The question "Does, in any Banach space, an arbitrary bounded sequence have a subsequence which is a weak Cauchy sequence?" cannot have a general positive answer as the following example shows: There is no such subsequence in the sequence of unit vectors in l^1. The surprising fact, however, is the assertion that this is essentially the only counterexample. It was proved by H. Rosenthal in [22] more than twenty years ago:

THEOREM 1.1. *Let X be a real or complex Banach space and (x_n) a bounded sequence in X. If there exists no subsequence which is a weak Cauchy sequence then one can find a subsequence (x_{n_k}) which is equivalent with the unit vector basis of l^1 (i. e. $(t_k) \mapsto \sum t_x x_{n_k}$, from l^1 to X, is an isomorphism).*

In particular one has: If X does not *contain an isomorphic copy of l^1, then every bounded sequence admits a subsequence which is a weak Cauchy sequence.*

The appealing feature of this theorem is that it can be stated after a few minutes of introduction, that it is comprehensible even with a small background in functional analysis, that it is important, and that it is really a deep theorem: No proof exists which could reasonably presented to students in a general functional analysis course.

Let us try, nevertheless, to understand at least some special cases. Since weakly convergent sequences are weakly Cauchy it follows immediately by the preceding remarks that Rosenthal's theorem holds in reflexive spaces.

Further, let (x_n) be bounded and x' be a fixed functional. If we apply the Bolzano-Weierstraß theorem to the scalar sequence $(x'(x_n))$ we easily get a subsequence (x_{n_k}) such that $x'(x_{n_k})$ converges. Applying the same idea to (x_{n_k}) with a second functional, say y', we get a subsequence of this subsequence such that the application of y' produces something which is convergent. x', applied to this new subsequence, also gives rise to convergence (since subsequences of convergent sequences are also convergent). Thus we have a subsequence of the original sequence where x' and y' both converge, and similarly one can achieve this for any prescribed finite number of functionals. Even countably many functionals are manageable, one only has to remember the diagonal sequence trick. And since we are dealing with bounded sequences (y_n) (typically subsequences of the original sequence) the collection of x' where $(x'(y_n))$ converges is a norm closed subspace of X'. *Summing up* these elementary remarks *we conclude that weak Cauchy subsequences can always be found, i. e. in particular that Rosenthal's theorem holds, whenever* X *is such that* X' *is separable.*

There is also another way to look at the problem. Start as before with the bounded sequence (x_n), but this time we regard each particular x_n as a functional on X', i. e. as an element of the bidual X''. In this space bounded sets are relatively weak*-compact by the Alaoglu theorem so that there is a weak*-accumulation point x'' of the (x_n). If we would be able to find a subsequence (x_{n_k}) which converges to x'' then this subsequence would be a weak Cauchy sequence.[1] This indicates that the difficulty with the proof will be a consequence of properties of the weak*-topology on X'' where accumulation points of a countable set not necessarily are limits of suitable subsequences. We hasten to add that this observation will by no means facilitate what has to be done in the sequel.

Here is *a brief sketch* of that what follows. Section 2 and section3 have a *preparatory character*. We will discuss properties of a *special type of ordered spaces* and give a brief survey of *Ramsey theory*. Then, in section 4, we prove Rosenthal's theorem; in fact we show a little bit more, namely a quantitative variant of the theorem due to the author. This quantitative setting has a reformulation where blockings of sequences occur, and this reformulation will motivate further investigations: Finally it will turn out how relative compactness can be characterized by blockings. In section 5 we aim to present a non-trivial application, we will treat the *Josefson-Nissenzweig theorem*. The situation with this theorem is as in the case of Rosenthal's theorem: It is basic and it can easily be formulated (for any infinite-dimensional Banach space there is a sequence of normalized vectors in the dual which converges to zero with respect

[1] And vice versa: If (x_{n_k}) is weakly Cauchy, then it has a weak*-limit in X''.

to the weak*-topology), some special cases can be treated by using the usual framework of a functional analysis course, but the proof of the general assertion is extremely difficult. In the end, in section 6 we turn to a *topological setting*. Since Rosenthal's theorem can be derived from approximation properties of continuous functions (one only has to identify the space X with a space of such functions) it is natural to investigate the general version of the theorem in the language of functions on Polish spaces.

There are *also some appendices* which provide additional and preparatory material. Appendix 1 contains the *complete proof of the Ramsey theorem* which is used here, Appendix 2 deals with *Banach limits* (to prepare the proof of the Josefson-Nissenzweig theorem), Appendix 3 contains some information on *ultrafilters on the integers*, and in Appendix 4 one learns what everybody should know about *Martin's axiom.*

2. Monotone functions on σ-grounded ordered spaces

This section – as is also the next one – is devoted to prepare the proof of Rosenthal's theorem. We will see that in the seemingly innocent framework of subsequences of a sequence surprising order theoretical phenomena may occur.

DEFINITION 2.1. *A nonvoid ordered set $(M, \leq)$ will be said to be called σ–*grounded *if every decreasing sequence in M has a lower bound.*

We are mainly interested in the example $(\tilde{\mathbb{N}}, \leq)$; here $\tilde{\mathbb{N}}$ stands for the set $\mathcal{P}_\infty(\mathbb{N})/\sim$ (with $\mathcal{P}_\infty(\mathbb{N})$ = the infinite subsets of $\mathbb{N}$, the equivalence relation " $\sim$ " is defined by $\Omega_1 \sim \Omega_2$ iff $\{k \mid k \geq k_0, k \in \Omega_1\} = \{k \mid k \geq k_0, k \in \Omega_2\}$ for a suitable k_0), and the order on $\mathbb{N}$ is given by $[\Omega_1] \leq [\Omega_2]$ iff $\{k \mid k \geq k_0, k \in \Omega_1\} \subset \{k \mid k \geq k_0, k \in \Omega_2\}$ for sufficiently large k_0.

It is not hard to see that $(\tilde{\mathbb{N}}, \leq)$ is σ–grounded.

PROPOSITION 2.2. *Let $(M, \leq)$ be σ–grounded, $(N, \leq)$ another ordered space and $\varphi : M \longrightarrow N$ a monotone map. Suppose that $(N, \leq)$ is countably determined in the following sense:*

> *There are monotone mappings $f_1, f_2, \ldots$ from N to $\mathbb{R}$ such that $f_i(x) = f_i(y)$ for $i = 1, 2, \ldots$ always implies that $x = y$.*

Then φ is eventually constant, i.e. there is an $\tilde{m} \in M$ such that $\varphi(m) = \varphi(\tilde{m})$ for all $m \leq \tilde{m}$.

Proof. Since M is σ–grounded it surely suffices to consider the case $(N, \leq) = (\mathbb{R}, \leq)$.

First of all we note that there is an m_0 such that φ is bounded from below on $\{m | m \leq m_0\}$. Otherwise we would get a sequence $m_1 \geq m_2 \geq \ldots$ with $\varphi(m_k) \leq -k$, and there could be no lower bound for the m_k.

Choose such an m_0 and define $\eta_0 := \inf\{\varphi(m) | m \leq m_0\} \in \mathbb{R}$. Now let $m_1 \leq m_0$ be such that $\eta_0 \leq \varphi(m_1) \leq \eta_0 + 1/1$ and define $\eta_1 := \inf\{\varphi(m) | m \leq m_1\}$. Select $m_2 \leq m_1$ such that $\eta_1 \leq \varphi(m_2) \leq \eta_1 + 1/2$. Continuing this way we provide $m_0 \geq m_1 \geq m_2 \cdots$ and $\eta_k = \inf\{\varphi(m) | m \leq m_k\}$ such that $\eta_0 \leq \eta_1 \leq \cdots$ and $\varphi(m) \leq \eta_k + 1/k$ whenever $m \leq m_k$. Clearly φ is constant (with value sup η_k) on $\{m \mid m \leq \tilde{m}\}$ if $\tilde{m}$ is a lower bound of the m_k. □

Note A similar technique has been used in the proof of the Hagler–Johnson theorem in [7, p. 230].

There are many situations where the conditions of the preceding proposition are satisfied. For us the case $M = (\tilde{\mathbb{N}}, \leq)$, $N :=$ the compact nonvoid subsets of K, " $\leq$ " = " $\subset$ " will be of importance, where K is a compact metric space. Define the $f_n : N \longrightarrow \mathbb{R}$ by $f_n(L) := \sup h_n|_L$, where (h_n) is a dense sequence in $C_{\mathbb{R}}(K)$.[2,3]

> In non–technical terms the moral of the story is as follows: Whenever one has a definition which associates with every subsequence of a given sequence a nonvoid closed subset of a compact metric space in such a way that the definition is not affected when changing finitely many terms, then one may pass to a subsequence such that all its subsequences are associated with the same set.

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[2] Recall that a compact space K is metrizable iff $C_{\mathbb{R}}(K)$ is separable.

[3] A participant of the Grado conference, Mr. Gelbrich, noted that a more elementary approach is possible: Choose a dense sequence $x_1, \ldots$ in K and define $f_n(x) :=$"the distance between x and x_n".

3. Ramsey theory

Ramsey theory can be thought of as a general strategy of "condensation of good properties": If one has a sufficiently large set which behaves "not too bad" then there is a subset with "very good" properties. Sometimes typical theorems have the form of a dichotomy: Either every (large) subset has a "very nice" subset or there is a "very bad" subset. Examples to illustrate such facts can be taken from daily life ("in every set containing three people there are two having the same sex") or from several places in known-to-everybody mathematics (remember, e. g., the Bolzano-Weierstraß theorem). A moment's reflection reveals that Rosenthal's theorem is also a theorem of this kind, and therefore it is not too surprising that Ramsey theory plays a crucial role in our proof.

We aim here to survey some facts from this branch of combinatorics and to establish what is needed for our proof, the material we present can be found in greater detail in [7, section X], [20], [21] and [24].

A. The finite case

This generalizes the assertion "in every set containing three people there are two having the same sex". Consider the set $[n] = \{1, \ldots, n\}$, where n is a natural number. Suppose further that, for some natural numbers r, k, the set $[n]^{[r]}$ of all families $(n_1, \ldots, n_r)$ of integers with $1 \leq n_1 < \cdots < n_r \leq n$ is written as the disjoint union of k subsets (it is a tradition in this theory to use term "apply k colours to sign the elements of $[n]^{[r]}$" instead of prescribing a partition into k subsets). For any $N \subset [n]$ the set $N^{[r]}$ (defined in the obvious way) lies in $[n]^{[r]}$, and one might ask how large such an N can be in order to be able to guarantee that all elements of $N^{[r]}$ are of the same colour. Let us write $n \to (l; r, k)$ if – regardless of the colouring – one always can find an N with l elements and this property.

Then our introducing example is just the statement that $3 \to (2; 1, 2)$. Also $6 \to (3; 2, 2)$ holds (which could be given an elementary proof; see [20, section 1]). A daily-life interpretation could look like this: Whenever six people come together one can select three of them which mutually know each other or one finds three which mutually don't know each other. Or: If one colours the edges and diagonals of a hexagon arbitrarily with two colours, then there will be a monochromatic triangle.

The very surprising and deep fact is that for every choice of l, r, k there is a finite n such that $n \to (l; r, k)$. This is *Ramsey's theorem for the finite case* ([21]), the smallest possible n which can be chosen for l, r, k and which has this property is called the associated *Ramsey number*.

B. The infinite situation

Here $[n]$ is replaced with $\mathbb{N}$. By the results of the preceding subsection a k-colouring of $\mathbb{N}^{[r]}$ (that is a disjoint partition of $\mathbb{N}^{[r]}$ into k subsets) admits subsets $N \subset \mathbb{N}$ with arbitrarily many (but finitely many) elements such that all elements of $N^{[r]}$ are of the same colour. Is there also an infinite N? The answer is in the affirmative, and this is *Ramsey's theorem for the infinite case* ([21]).

Consider a special case, namely $r = k = 2$. Then the theorem states the following: Suppose that C is a subset of $\mathbb{N}^{[2]}$. Then there is an infinite set N of integers such that either $N^{[2]}$ is contained in C or in the complement of C. Often one knows that for some reason the second alternative cannot occur, here is a well-known illustration:

> Suppose that in a certain city there are countably many inhabitants. We assume that in every infinite subcollection there are at least two persons which know each other. Then it is possible to find an infinite part of the population such that in this part every two people are acquainted with each other.
>
> (In order to get a feeling for the difficulty of such statements try to find *three* persons which know each other.)

The next theorem is a generalization of this infinite Ramsey theorem, and it is this generalization which will be needed in the proof of Rosenthal's theorem. (The preceding case corresponds to $T_r := \mathbb{N}^{[r]}$ for $r = 1, 3, 4, 5, \ldots$ and $T_2 \subset \mathbb{N}^{[2]}$ such that every infinite set M contains two elements $i < j$ with $(i, j) \in T_2$.)

THEOREM 3.1. *For $r \in \mathbb{N}$ let T_r be a family of r–tupels of increasing* *integers. Suppose that*

(1) $$\forall_{\substack{M\subset\mathbb{N}\\ M \text{ infinite}}} \ \exists_{\substack{i_1,i_2,\ldots\in M\\ i_1<i_2<\ldots}} \ \forall_r \ (i_1,\ldots,i_r)\in T_r.$$

Then it follows that

(2) $$\exists_{\substack{M_0\subset\mathbb{N}\\ M_0 \text{ infinite}}} \ \forall_{\substack{r\\ i_1,i_2,\ldots,i_r\in M_0\\ i_1<\ldots<i_r}} \ (i_1,\ldots,i_r)\in T_r.$$

The *proof* will be given appendix 1 below, here we only add a non-technical interpretation: Suppose that one assigns in some way families of increasing r-tupels of integers for every r such that these r-tupels reflect what can happen in a "typical" infinte set. Then there is an infinite M such that for every r and every $N\subset M$ with at least r elements one has $N^{[r]}\subset T_r$.

Everybody is invited to check some examples.

What about, e. g., with

$$T_r := \{(n_1,\ldots,n_r) | a_r \le n_1 < \cdots < n_r\},$$

where $a_1 < a_2 < \cdots$ is a prescribed sequence of integers? Are the conditions of the theorem satisfied, and if yes, what are the possible sets M_0?

And what about

$$T_r := \{(n_1,\ldots,n_r) | x_{n_1} < 0, x_{n_2} > 0, \ldots\},$$

where (x_n) is a sequence which is dense in $\mathbb{R}$?

C. The topological approach

This is the most ambitious variant of Ramsey theorems, we start with some definitions.

For an infinite $A\subset\mathbb{N}$ we denote by $A^{[\infty]}$ the collection of all infinite subsets of A. A subset $\mathcal{A}$ of $\mathbb{N}^{[\infty]}$ is called a *Ramsey collection* if there is an infinite A such that either $A^{[\infty]}$ lies in $\mathcal{A}$ or in the complement of $\mathcal{A}$. For example, theorem 3.1 is nothing but the statement that

$$\mathcal{T} := \{M = (i_1, i_2, \ldots) | \text{ for all } r \text{ one has } (i_1, \ldots, i_r) \in T_r\}$$

is a Ramsey collection. The idea is now to characterize Ramsey collections by suitable topological properties.

A comparatively easy way to formulate typical results is a follows. $\mathbb{N}^{[\infty]}$ is a subset of the power set of $\mathbb{N}$ which in turn can be identified with the product $\{0,1\}^{\mathbb{N}}$, hence carries the product topology which will be called the *natural topology* in the sequel. (Roughly speaking, a "typical simple" set $\mathcal{A}$ is open if the assertion $A \in \mathcal{A}$ can be checked by having a look to a fixed finite set of integers: some of these integers have to belong to A, others must not.) For example, the above collection $\mathcal{T}$ is easily seen to be closed, and hence its Ramsey property follows from a result of Nash-Williams [17] by which open and closed collections are always Ramsey.

> The complete truth is a little bit more complicated. The appropriate topology is finer than the natural topology. This topology – which we will call τ here - has as a base of the open sets in $\mathbb{N}^{[\infty}]$ collections $(i_1, \ldots, i_r; M)$ with M infinite and $i_1 < \cdots < i_r$, where $(i_1, \ldots, i_r; M)$ is the collection of all infinite A which "start with" $(i_1, \ldots, i_r)$ and are contained in M. (We note that one gets the natural topology if one restricts the M to the set $\mathbb{N}$.)
>
> Also we need a refinement of the notion of Ramsey collection. A subset $\mathcal{A}$ of $\mathbb{N}^{[\infty]}$ is said to be a *complete Ramsey collection* if for arbitrary $i_1 < \cdots < i_r$ and infinite A there is an infinite $M \subset A$ such that $(i_1, \ldots, i_r; M)$ is either contained in $\mathcal{A}$ or in the complement of $\mathcal{A}$.[4]
>
> And here are the main results:
>
> - Open sets and closed sets (with respect to τ) are completely Ramsey ([17]).
> - Every Borel set with respect to τ (in particular every Borel set with respect to the natural topology) of $\mathbb{N}^{[\infty]}$ is completely Ramsey([10]).

[4]Note that complete Ramsey collections are Ramsey collections.

- An $\mathcal{A}$ is completely Ramsey iff it is the symmetric difference of an open set and a set of first category; here "open" and "first category" refer to τ ([9]).
- Suppose that a set $\mathcal{A}$ can be written as the continuous image of the irrational numbers (alternatively: of $\mathbb{N}^{\mathbb{N}}$; alternatively: of any Polish space). Then $\mathcal{A}$ is called an *analytic set*, and every such set is completely Ramsey ([9]).

4. The proof of Rosenthal's theorem

DEFINITION 4.1. *Let (x_n) be a bounded sequence in a Banach space X, and $\varepsilon > 0$. We say that (x_n)* admits ε–ℓ^1–blocks *if for every infinite $M \subset \mathbb{N}$ there are $a_1, \ldots, a_r \in \mathbb{K}$ with $\sum |a_r| = 1$ and $i_1 < \ldots < i_r$ in M such that $\| \sum a_\rho x_{i_\rho} \| \leq \varepsilon$.*

Clearly there will be no subsequence of (x_n) equivalent to the ℓ^1–basis iff (x_n) admits ε–ℓ^1–blocks for arbitrarily small $\varepsilon > 0$.

> **Thus Rosenthal's theorem is the assertion that (x_n) has a weak Cauchy subsequence provided it admits ε–ℓ^1–blocks for all ε.**

Here is our quantitative version of this fact, for simplicity we restrict ourselves to the case of real spaces:

THEOREM 4.2. *Let X be a real Banach space and (x_n) a bounded sequence. Suppose that, for some $\varepsilon > 0, (x_n)$ admits small ε–ℓ^1–blocks. Then there is a subsequence (x_{n_k}) of (x_n) such that (x_{n_k}) is "close to being a weak Cauchy sequence" in the following sense:*

$$\limsup x'(x_{n_k}) - \liminf x'(x_{n_k}) \leq 2\varepsilon$$

for every x' with $\|x'\| = 1$.

Remark. It is simple to derive the original theorem from 4.2. (If (x_n) and thus every subsequence has ε–ℓ^1–blocks for all ε, apply th:4.2 successively with ε running through a sequence tending to zero. The diagonal sequence which is obtained from this construction will be a weak Cauchy sequence.)

Proof. Suppose the theorem were not true. We claim that without loss of generality we may assume that there is a $\delta > 0$ such that

$$(4)\quad \varphi((x_{n_k})) := \sup_{\|x'\|=1} \Big(\limsup_k x'(x_{n_k}) - \liminf_k x'(x_{n_k})\Big) > 2\varepsilon + \delta$$

for all subsequences (x_{n_k}). In fact, if every subsequence contained another subsequence with a φ–value arbitrarily close to 2ε, an argument as in the preceding remark would even provide one where $\varphi((x_{n_k})) \le 2\varepsilon$ in contrast to our assumption.

Fix a $\tau > 0$ which will be specified later. The essential tool in order to get a contradiction will be the

LEMMA 4.3. *After passing to a subsequence we may assume that (x_n) satisfies the following conditions:*

(i) *If C and D are finite disjoint subsets of $\mathbb{N}$ there are a $\lambda_0 \in \mathbb{R}$ and an $x' \in X'$ with $\|x'\| = 1$ such that $x'(x_n) < \lambda_0$ for $n \in C$ and $x'(x_n) > \lambda_0 + 2\varepsilon + \delta$ for $n \in D$.*

(ii) *There are $i_1 < \ldots < i_r$ in $\mathbb{N}$, $a_1, \ldots, a_r \in \mathbb{R}$ with*

$$\sum |a_\rho| = 1,\ |\sum a_\rho| \le \tau,\ \|\sum a_\rho x_{i_\rho}\| \le \varepsilon.$$

Proof of the lemma. (i) Define, for $r \in \mathbb{N}, T_r$ to be the collection of all $(i_1, \ldots, i_r)$ (with $i_1 < \ldots < i_r$) such that there are a $\lambda_0 \in \mathbb{R}$ and a normalized x' such that $x'(x_{i_\rho}) < \lambda_0$ if ρ is even and $> \lambda_0 + 2\varepsilon + \delta$ otherwise. (4) implies that (1) of 3.1 is valid. Thus there is an M_0 for which all $(i_1, \ldots, i_r)$ are in T_r for $i_1 < \ldots < i_r$ in M_0. Let us assume that $M_0 = \mathbb{N}$.

Let C and D be finite disjoint subsets of $2\mathbb{N} = \{2, 4, \ldots, \}$. We may select $i_1 < \ldots < i_r$ in $\mathbb{N}$ such that $C \subset \{i_\rho \mid \rho \text{ even}\}$ and $D \subset \{i_\rho \mid \rho \text{ odd}\}$. Because of $(i_1, \ldots, i_r) \in T_r$ we have settled (i) provided C and D are in $2\mathbb{N}$, and all what's left to do is to consider (x_{2n}) instead of (x_n).

(ii) By assumption we find $i_1 < \ldots < i_r$, $a_1, \ldots, a_r \in \mathbb{R}$ such that $\sum |a_\rho| = 1$ and $\|\sum a_\rho x_{i_\rho}\| \le \varepsilon$ with arbitrarily large i_1. Therefore we obtain $i_1^1 < \ldots < i_{r_1}^1 < i_1^2 < \ldots < i_{r_2}^2 < i_1^3 < \ldots < i_{r_3}^3 < \ldots$

and associated a^i_ρ. The numbers $\eta_j := \sum_{\rho=1}^{r_j} a^j_\rho$ all lie in $[-1,+1]$ so that we find $j < k$ with $|\eta_j - \eta_k| \le 2\tau$. Let $i_1 < \ldots < i_r$ be the family $i^j_1 < \ldots < i^j_{r_j} < i^k_1 < \ldots < i^k_{r_k}$, and define the $a_1, \ldots, a_r$ by $\frac{1}{2}a^j_1, \ldots, \frac{1}{2}a^j_{r_j}, -\frac{1}{2}a^k_1, \ldots, -\frac{1}{2}a^k_{r_k}$. □

We are now ready to derive a contradiction. On the one hand, by (ii) of the lemma, we find $i_1 < \ldots < i_r$ in $\mathbb{N}$, $a_1, \ldots, a_r \in \mathbb{R}$, $\sum |a_\rho| = 1$, $|\sum a_\rho| \le \tau$ with $\|\sum a_\rho x_{i_\rho}\| \le \varepsilon$. On the other hand we may apply (i) with $C := \{i_\rho \mid a_\rho < 0\}$, $D := \{i_\rho \mid a_\rho > 0\}$. We put $\alpha := -\sum_{\rho \in C} a_\rho$, $\beta := \sum_{\rho \in D} a_\rho$, and we note that $|\alpha - \beta| \le \tau$, $\alpha + \beta = 1$ so that $|\beta - \frac{1}{2}| \le \tau$; hence

$$
\begin{aligned}
\varepsilon &\ge \left\| \sum a_\rho x_{i_\rho} \right\| \ge \sum a_\rho x'(x_{i_\rho}) \ge -\lambda_0 \alpha + (\lambda_0 + 2\varepsilon + \delta)\beta \\
&\ge -|\lambda_0|\tau + \varepsilon + \frac{\delta}{2} - \tau\delta.
\end{aligned}
$$

This expression can be made larger than ε if τ has been chosen sufficiently small (note that the numbers $|\lambda_0|$ are bounded by $\sup \|x_n\|$), a contradiction which proves the theorem. □

Note. Since for the unit vector basis (x_n) of real ℓ^1 the assumption of the theorem holds with $\varepsilon = 1$ and since for every subsequence (x_{n_k}) one may find $\|x'\| = 1$ with $\limsup x'(x_{n_k}) - \liminf x'(x_{n_k}) = 2$ there can be no better constant than that given in our theorem.

We close this section by investigating the case when the blockings can be chosen such that they have a uniformly bounded length. The results appeared first in [1], here we follow the lines of [2].

THEOREM 4.4. *Let (x_n) be a bounded sequence in a real or complex Banach space X such that there is an $r \in \mathbb{N}$ with the following property: Whenever $M \subset \mathbb{N}$ is infinite and $\varepsilon > 0$, there are $i_1 < \ldots < i_r$ in M and $a_1, \ldots, a_r \in \mathbb{K}$ with $\sum |a_\rho| = 1$ such that $\|\sum a_i x_{i\rho}\| \le \varepsilon$. Then (x_n) has a convergent subsequence.*

Proof. Fix $\varepsilon > 0$. The first step is as in [1], we refer the reader to this paper: With the help of proposition 2.2 one can choose the same a's for all M. Fix $a_1, \ldots, a_r$ and suppose that $a_r \ne 0$. Define sets of $\widetilde{r}$–tupels $T_{\widetilde{r}}$ as follows. $T_{\widetilde{r}}$ is the collection of *all* $\widetilde{r}$–tupels if $\widetilde{r} \ne r$, and the set of those $(i_1, \ldots, i_r)$ with $i_1 < \ldots < i_r$ and $\| \sum a_\rho x_{i\rho} \| \leq \varepsilon$ if $\widetilde{r} = r$. Then theorem 3.1 may be applied, and we get (x_{n_k}) such that $\| \sum a_\rho x_{n_{k_\rho}} \| \leq \varepsilon$ for arbitrary $k_1 < \ldots < k_r$.

Now let r_0 be an index such that $|a_{r_0}| \geq 1/r$ and $n_k < n_m$ such that $k > r_0$. Then $||a_{r_0} x_{n_k} + y||, ||a_{r_0} x_{n_m} + y|| \leq \varepsilon$, where $y := a_1 x_{n_1} + \cdots + a_{r_0 - 1} x_{n_{r_0 - 1}} + a_{r_0+1} x_{n_{m+1}} + \cdots + a_r x_{n_{m+r-r_0+1}}$. Consequently $x_{n_{r+1}}, x_{n_{r+2}}, \ldots$ lie in a ball with radius $2r\varepsilon$.

Starting this construction with $\varepsilon = 1$ and applying it repeatedly to $\varepsilon = 1/2$, $\varepsilon = 1/3$, ... one gets a descending family of subsequences for which the diagonal sequence surely is convergent. □

Note. In this situation a quantitative version is not to be expected in general. Consider e.g. the unit vector basis (e_n) in c_0. For $\varepsilon > 0$ fixed one can find the $a_1, \ldots, a_r$ such that $\sum a_\rho e_\rho$ with the same r for all sets M (e. g. $a_1 = \cdots = a_r = 1/r$ with $1/r \leq \varepsilon$. But $\|e_n - e_m\| = 2$ for $n \neq m$.

5. The Josefson-Nissenzweig theorem

Here we want to discuss another deep theorem from Banach space theory. In the proof we will use Rosenthal's l^1–theorem, also the material of appendix 2 will come into play.

The **Josefson–Nissenzweig theorem** ([16],[18]) is the following assertion, it first was used to study some examples and counter–examples in the theory of holomorphic mappings on infinite–dimensional spaces:

THEOREM 5.1. *For every infinite–dimensional Banach space there are $x'_1, x'_2, \ldots$ in X' such that $\|x'_n\| = 1$ for all n, but $x'_n \to 0$ w.r.t. the weak*–topology.*

> One should pause for a moment to check the theorem at some instances where one has explicit access to the dual space: c_0, l^1, CK (with K compact), ...

Also it should be observed that – as in the case of Rosenthal's theorem – the difficulties stem from topological subtleties. Since, as is well–known, the dual unit ball is in the infinite–dimensional case the weak*-closure of the sphere, there always is a *net* of normalized vectors in X' which tends to zero with respect to the weak*-topology. And it "only" has to be shown that one may replace nets with sequences. This is possible, e. g., if X is separable since then the weak*-topology on the dual unit ball is metrizable. Also, in reflexive spaces, one may work with sequences by the Eberlein-Šmulian theorem. Thus we already have a proof for separable and for reflexive spaces, but how to treat the general case?

We prepare the proof with two lemmas, one concerns trees of numbers, the other deals with a general property of Banach limits.

LEMMA 5.2. *For $\ell = 1, 2, \ldots$ and $\varepsilon_1, \ldots, \varepsilon_\ell \in \{0, 1\}$ let $r_{\varepsilon_1 \ldots \varepsilon_\ell}$ be a number such that the family $((r_{\varepsilon_1 \ldots \varepsilon_\ell})_{\varepsilon_1 \ldots \varepsilon_\ell})_{\ell=1,2,\ldots}$ satisfies $|r_{\varepsilon_1 \ldots \varepsilon_\ell}| \leq 2^{-\ell}$ and $r_{\varepsilon_1 \ldots \varepsilon_\ell} = r_{\varepsilon_1 \ldots \varepsilon_\ell 0} + r_{\varepsilon_1 \ldots \varepsilon_\ell 1}$ for all $\ell, \varepsilon_1 \ldots \varepsilon_\ell$. Define*

$$\eta_\ell := \sum_{\varepsilon_1 \ldots \varepsilon_\ell} (r_{\varepsilon_1 \ldots \varepsilon_\ell 0} - r_{\varepsilon_1 \ldots \varepsilon_\ell 1}).$$

Then $\sum |\eta_\ell|^2 \leq 1$ so that in particular $\eta_\ell \to 0$.

Proof. An elegant proof could be given using *martingales*: the $r_{\varepsilon_1 \ldots \varepsilon_\ell}$ give rise to a bounded martingale, the martingale convergence theorem guarantees the existence of a limit f in L^1, and the η_ℓ are the integrals over f multiplied by suitable Rademacher functions.

However, a much simpler approach is possible. Set

$$\begin{aligned} a_\ell &:= 2^\ell \sum_{\varepsilon_1 \ldots \varepsilon_\ell} |r_{\varepsilon_1 \ldots \varepsilon_\ell}|^2 \\ b_\ell &:= 2^\ell \sum_{\varepsilon_1 \ldots \varepsilon_\ell} |r_{\varepsilon_1 \ldots \varepsilon_\ell 0} - r_{\varepsilon_1 \ldots \varepsilon_\ell 1}|^2 \\ c_\ell &:= \sum_{\varepsilon_1 \ldots \varepsilon_\ell} |r_{\varepsilon_1 \ldots \ell 0} - r_{\varepsilon_1 \ldots \varepsilon_\ell 1}|. \end{aligned}$$

Then $a_{\ell+1} - a_\ell = b_\ell$ (by the parallelogram law $|\alpha + \beta|^2 + |\alpha - \beta|^2 = 2(|\alpha|^2 + |\beta|^2)$) so that the a_ℓ are increasing. Surely $a_\ell \leq 1$, and we get $\sum b_\ell \leq 1$. Finally note that $|\eta_\ell| \leq c_\ell$ and that $c_\ell^2 \leq b_\ell$ since

$$(|\alpha_1| + \cdots + |\alpha_k|)^2 \leq k(|\alpha_1|^2 + \cdots + |\alpha_k|^2) \quad \text{for all families } \alpha_1, \ldots, \alpha_k.$$

□

Now Banach limits come into play, we need the facts prepared in appendix 2. For the rest of this section we assume that L is a fixed Banach limit.

LEMMA 5.3. *Define* λ_ℓ, $\mu_{\varepsilon_1\ldots\varepsilon_\ell} \in \ell^\infty$ *for* $\ell = 1, 2, \ldots$ *and* $\varepsilon_1, \ldots, \varepsilon_\ell \in \{0, 1\}$ *as follows:*

$$\begin{aligned}
\lambda_1 &= (1, -1, 1, -1, \ldots)\\
\lambda_2 &= (1, 1, -1, -1, 1, 1, \ldots)\\
\lambda_3 &= (1, 1, 1, 1, -1, -1, -1, -1, \ldots)\\
&\vdots\\
\mu_0 &= (1, 0, 1, 0, \ldots), \quad \mu_1 = (0, 1, 0, 1, \ldots)\\
\mu_{00} &= (1, 0, 0, 0, 1, 0, 0, 0, 1, \ldots), \quad \mu_{01} = (0, 0, 1, 0, 0, 0, 1, 0, \ldots)\\
\mu_{10} &= (0, 1, 0, 0, 0, 1, 0, 0, 0, \ldots), \quad \mu_{11} = (0, 0, 0, 1, 0, 0, 0, 1, \ldots);
\end{aligned}$$

in general: $\mu_{\varepsilon_1\ldots\varepsilon_\ell}$ *is* 1 *at positions of the form* $k \cdot 2^\ell + 1 + \varepsilon_1 2^0 + \cdots + \varepsilon_\ell 2^{\ell-1}$ $(k = 0, 1, \ldots)$ *and* 0 *otherwise, and*

$$\lambda_\ell = \sum_{\varepsilon_1,\ldots,\varepsilon_{\ell-1}} (\mu_{\varepsilon_1,\ldots,\varepsilon_{\ell-1},0} - \mu_{\varepsilon_1,\ldots,\varepsilon_{\ell-1},1}).$$

Then $|L(\mu_{\varepsilon_1\ldots\varepsilon_\ell} x)| \leq 2^{-\ell}$ *for every* $x \in \ell^\infty$ *with* $\|x\| \leq 1$.

Proof. Let $T : \ell^\infty \to \ell^\infty$ be the shift operator $(y_1, y_2, \ldots) \mapsto (0, y_1, y_2, \ldots)$ and x_0 the pointwise product of $\mu_{\varepsilon_1\ldots\varepsilon_\ell}$ with x. Then $L(T\widetilde{x}) = L(\widetilde{x})$ for every $\widetilde{x}$, and $\|x_0 + Tx_0 + T^2x_0 + \cdots + T^{2^\ell-1}x_0\| \leq 1$. Hence $2^\ell|L(x_0)| \leq 1$. □

We now turn to the

Proof of the Josefson-Nissenzweig theorem.

CASE 1. ℓ^1 *is not contained in* X'. *Suppose that every weak*–convergent sequence is already norm convergent; we will show that* X' *is finite-dimensional.*

Let (x'_n) *be a bounded sequence. By Rosenthal's theorem and since* ℓ^1 *does not embed into* X' *we find a subsequence which is weakly Cauchy and thus weak*–convergent. By our assumption it is convergent, and thus* X' *is finite–dimensional.*

CASE 2. *ℓ^1 embeds into X', i.e., there are x_n' in X' and $A, B > 0$ such that*

$$A \sum_1^r |t_i| \geq \left\| \sum_1^r t_i x_i' \right\| \geq B \sum_1^r |t_i|$$

for arbitrary r and $t_1, \ldots, t_r \in \mathbb{K}$.

In order to continue we remind the reader of the following notion: A sequence (y_n') is said to be obtained from the (x_n') by *blocking* if there are disjoint finite sets $A_1, A_2, \ldots$ in $\mathbb{N}$ with $A_1 \leq A_2 \leq \ldots$ and numbers (a_k) with $\sum_{k \in A_n} |a_k| = 1$ for every n such that $y_n' = \sum_{k \in A_n} a_k x_k'$. Note in particular that all subsequences arise in this way.

CASE 2.1. *It is possible to get (y_n') by blocking (x_n) such that $y_n' \to 0$ with respect to the weak*–topology. Then we are done since $\|y_n'\| \geq B$ so that the $y_n'/\|y_n'\|$ have the desired properties.*

CASE 2.2. *For no blocking (y_n') we have $y_n' \to 0$ (w.r.t. the weak*–topology). In order to measure the property of being a weak*–null sequence we introduce the number*

$$\varphi((y_n')) := \sup_{\|x\|=1} \limsup |y_n'(x)|$$

for the (y_n') constructed as before. In the case under consideration we know that always $\varphi((y_n')) > 0$. We claim that even more is true.

CLAIM 1. *There are a $\delta > 0$ and a block sequence (y_n') such that $\varphi((z_n')) = \delta$ for every (z_n') which is obtained from (y_n') by blocking.*

Proof of claim 1. Let $\delta_0 \geq 0$ be the infimum of the numbers $\varphi((y_n'))$, where the infimum runs over all block sequences (y_n'). Choose $(y_n^{[1]})$, a block sequence of (x_n'), such that $\varphi((y_n^{[1]})) \leq \delta_0 + 1/2^0$. Let δ_1 be the infimum of the $\varphi((y_n'))$, where this time only blockings of $(y_n^{[1]})$ are under consideration. Since a block sequence of a block sequence is a block sequence we have $\delta_0 \le \delta_1$. Choose $(y_n^{[2]})$, a block sequence of $(y_n^{[1]})$, with $\varphi((y_n^{[2]})) \le \delta_1 + 1/2^1$. In this way we get successively $(y_n^{[1]}), (y_n^{[2]}), \ldots$, and $\delta_1 \le \delta_2 \le \ldots$, where $(y_n^{[k+1]})$ is obtained from $(y_n^{(k)})$ by blocking and where $\delta_k \le \varphi((y_n')) \le \delta_k + 1/2^k$ for all block sequences (y_n') of $(y_n^{[k+1]})$. Our candidate is the diagonal sequence (y_n') containing the n'th element of the $(y_n^{[n]})$ for every n. For every k, (y_n') is – after possibly finitely many exceptions – a block sequence of $(y_n^{[k]}$; therefore the φ–value lies between δ_{k+1} and $\delta_k + 1/2^k$. It follows that $\delta := \sup \delta_k$ has the claimed properties; note that we also know that $\delta > 0$ since $\delta = \varphi((y_n'))$. □

Claim 2. *Fix (y_n') and $\delta > 0$ as in claim 1. Further let $\tau > 0$ be arbitrary. There is a subsequence which we will denote by (z_n') with the following property: It is possible to find normalized $x_1, x_2, \ldots$ in X such that:*

$$\begin{aligned}
|z_n'(x_1) - \delta| &\le \tau \text{ for all } n;\\
|z_n'(x_2) - \delta| &\le \tau \text{ for } n = 3, 5, \ldots\\
|z_n'(x_2) + \delta| &\le \tau \text{ for } n = 4, 6, \ldots\\
|z_n'(x_3) - \delta| &\le \tau \text{ for } n = 5, 6, 9, 10, \ldots\\
|z_n'(x_3) + \delta| &\le \tau \text{ for } n = 7, 8, 11, 12, \ldots\\
&\vdots
\end{aligned}$$

(In general $z_n'(x_k)$ is τ–close to δ on segments of length 2^{k-2} beginning at $2^{k-1} + 1, 2 \cdot 2^{k-1} + 1, 3 \cdot 2^{k-1} + 1, \ldots$ and τ–close to $-\delta$ at segments of the same length, beginning at $2^{k-1} + 2^{k-2} + 1$, $2 \cdot 2^{k-1} + 2^{k-2} + 1$, $3 \cdot 2^{k-1} + 2^{k-1} + 1$; thus, if we regard the x_k as functions on the set $\{z_n' \mid n \in \mathbb{N}\}$ they behave like the Rademacher functions, at least for large n.)

proof of claim 2. By assumption we know that $\varphi((y_n')) = \delta$, and this makes it easy to find x_1 with $\|x_1\| = 1$ and a subsequence $(w_n^{[1]})$ of (y_n') with $|w_n^{[1]}(x_1) - \delta| \le \tau$ for every n. Our final (z_n') will be a subsequence of $(w_n^{[1]})$ so that we will have no problems with x_1. Put

$z_1' := w_1^{[1]}$, $z_2' := w_2^{[2]}$. Now consider

$$(u_n) := \left(\frac{w_3^{[1]} - w_4^{[1]}}{2}, \frac{w_5^{[1]} - w_6^{[1]}}{2}, \dots \right).$$

This is a block sequence of (y_n') so that $\varphi((u_n)) = \delta$. Hence we find a normalized x_2 such that for infinitely many n, say $n \in N$, we have $|u_n(x_2) - \delta| \le \tau'$; here τ' denotes any positive number such that

$$|\alpha|, |\beta| \le \delta + \tau', \ |\frac{1}{2}(\alpha + \beta) - \delta| \le \tau' \quad \Rightarrow \quad |\alpha - \delta| \le \tau, \ |\beta - \delta| \le \tau.$$

Since $\limsup |w_n^{[1]}(x_2)| \le \delta$ we may also assume that for $n \in N$ and $u_n = (w_{2n-1}^{[1]} - w_{2n}^{[1]})/2$, the w's satisfy $|w_{2n-1}^{[1]}(x_2)|$, $|w_{2n}^{[1]}(x_2)| \le \delta + \tau'$. Thus $|w_{2n-1}^{[1]}(x_2) - \delta| \le \tau$ and $|w_{2n}^{[1]}(x_2) + \delta| \le \tau$.

Let $(w_n^{[2]})$ be the sequence of the $w_{2n-1}^{[1]}, w_{2n}^{[1]}$ with $n \in N$. Set $z_3' := w_2^{[2]}$, $z_4' := w_3^{[2]}$.

We consider now

$$\left(\frac{w_1^{[2]} + w_2^{[2]} - w_3^{[2]} - w_4^{[2]}}{4}, \frac{w_5^{[2]} + w_6^{[2]} - w_7^{[2]} - w_8^{[2]}}{4}, \dots \right) =: (u_n).$$

Again we find an infinite N and an x_3 such that $|u_n(x_3) - \delta| \le \tau'$, where this time $\tau' > 0$ is such that

$$|\alpha_1|, |\alpha_2|, |\alpha_3|, |\alpha_4| \le \delta + \tau' \text{ and } \left| \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{4} - \delta \right| \le \tau'$$

$$\Rightarrow \quad |\alpha_j - \delta| \le \tau \text{ for } j = 1, 2, 3, 4.$$

Also we can assume that the $|w_{4n-j}^{[2]}(x_3)| \le \delta + \tau'$ for $n \in N$, $j = 0, 1, 2, 3$. Let $(w^{[3]})$ consist of the $w_{4n-3}^{[2]}, \dots, w_{4n}^{[2]}$ with $n \in N$, and define $z_5', \dots, z_8'$ to be $w_1^{[3]}, \dots, w_4^{[3]}$ respectively.

It should be clear how this construction (which is similar to that in [3]) has to be continued and that (z_n') has the claimed properties. □

Now it is fairly easy to finish the proof of the Josefson–Nissenzweig theorem. With (z_n') as in the second claim we denote by $T : X \to \ell^\infty$

the operator $x \mapsto (z'_1(x), z'_2(x), \ldots)$, and we define w'_n to be the functional $x \mapsto L(\lambda_n Tx)$ (notation as in section 5) where L is a fixed Banach limit. Every w'_n has a norm not smaller than $\delta - \tau$ since $\lambda_n Tx_n$ is by construction a sequence which – up to finitely many exceptions – is τ–close to the sequence $(\delta, \delta, \delta, \ldots)$, and the Banach limit property of L implies that $L(\lambda_n Tx_n)$ is – up to τ – $L(\delta, \delta, \ldots) = \delta$.

The w'_n also tend to zero w.r.t. the weak* topology since by 5.2 for every $x \in \ell^\infty$ with $\|x\| = 1$ the numbers $r_{\varepsilon_1 \ldots \varepsilon_\ell} := L(\mu_{\varepsilon_1 \ldots \varepsilon_\ell} x)$ satisfy the hypothesis of th:5.1 and since the η_ℓ of th:5.1 are just the $L(\lambda_\ell \cdot x)$ in this case. □

Note. In fact we have shown more than required. When case 2.2 leads to a Josefson–Nissenzweig sequence then the sequence (w'_n) is not only weak* null but in fact a weak*–ℓ^2–sequence (i.e. $\sum |w'_n(x)|^2 < \infty$ for every $x \in X$).

6. Approximation of Borel functions on Polish spaces

Rosenthal's theorem can be thought of as a search for the possibility to find sequences which converge to an accumulation point of a given countable set. There is a topological approach which precisely investigates such questions in the framework of continuous functions, a celebrated result ([23],[15]) reads as follows:

> Let $(T, \mathcal{T})$ be a Polish space, i.e., $\mathcal{T}$ is generated by a metric with respect to which T is separable and complete. Further, let $(f_n)_{n \in \mathbb{N}}$ be a pointwise bounded sequence of continuous real-valued functions on T and f an accumulation point with respect to pointwise convergence of the (f_n). Then f is in fact the limit of a subsequence provided that every accumulation point of the (f_n) is Borel measurable.

How this topological fact is connected with the characterization of Banach spaces which contain l^1 is thoroughly discussed in chapter III of [12], here we aim to investigate a quantitative variant of this Borel-versus-Baire approximation theorem.

Here is our main result:

THEOREM 6.1. *Let* $(T, \mathcal{T})$ *be a Polish space and* (f_n) *a pointwise bounded sequence of continuous real-valued functions on* T. *Suppose that* $\epsilon_0 \geq 0$ *is such that every accumulation point of the* (f_n) *is* $\epsilon_0/2$-*close to a Borel function.*

Then for every accumulation point f *there is a subsequence* (f_{n_k}) *such that* $\limsup_k |f_{n_k}(x) - f(x)| \leq \epsilon_0$ *for* $x \in T$.

The proof will make use of the results of 3, also we will need some further preparations.

DEFINITION 6.2. *Let* $(T, \mathcal{T})$ *be a topological space and* $\mathcal{A}$ *the collection of closed subsets. A function* $\Phi : \mathcal{A} \longrightarrow \mathcal{A}$ *is called a* subset assignment *if* $\Phi(A)$ *is a proper subset of* A *for all nonempty* A, *and* $\Phi(\emptyset) = \emptyset$.

PROPOSITION 6.3. *Let* Φ *be a subset assignment on a topological space* $(T, \mathcal{T})$. *Further, let* $\mathcal{V}$ *be a collection of open sets which separates points from closed sets: For every closed* A *and every* $x \notin A$ *there is* $V \in \mathcal{V}$ *with* $x \in V$ *but* $V \cap A = \emptyset$.

Then there is a collection $\mathcal{S}$ *of subsets of* T *with* $\text{card}(\mathcal{S}) \leq \text{card}(\mathcal{V})$ *such that* T *is the union of the* $S \in \mathcal{S}$ *and each of these* S *is of the form* $A \setminus \Phi(A)$.

In particular there is a countable $\mathcal{S}$ *if* T *is a first countability space.*

Proof. Call a collection $\mathcal{B}$ of closed sets a Φ-system if the following hold:

(i) $T \in \mathcal{B}$; (ii) if $A, B \in \mathcal{B}$ then $A \subset B$ or $B \subset A$, and if A is a proper subset of B then $A \subset \Phi(B)$; (iii) with B also $\Phi(B)$ is in $\mathcal{B}$; (iv) for every x which is not in $\bigcap\{B \mid B \in \mathcal{B}\}$ there exists $B \in \mathcal{B}$ with $x \in B \setminus \Phi(B)$.

For example, $\{T, \Phi(T), \Phi^2(T), \ldots\}$ is a Φ-system, and a routine calculation shows that the collection of all Φ-systems is inductively ordered with respect to inclusion.

Let $\mathcal{B}_0$ be a maximal Φ-system. Necessarily $B_0 := \bigcap\{B \mid B \in \mathcal{B}_0\} = \emptyset$ since otherwise $\mathcal{B}_0 \cup \{B_0, \Phi(B_0), \ldots\}$ would be a Φ-system strictly larger than $\mathcal{B}_0$. By property (iv) T is the union over the $B \setminus \Phi(B)$, $B \in \mathcal{B}_0$. Also $\mathcal{B}_0$ has at most $\text{card}(\mathcal{V})$ elements: Choose, for $B \in \mathcal{B}_0$ with $B \neq \emptyset$, a V_B in $\mathcal{V}$ such that $B \cap V_B \neq \emptyset = \Phi(B) \cap V_B$; $B \mapsto V_B$ is clearly one-to-one. $\square$

Now suppose that the assumptions of the theorem are satisfied and that f is a fixed accumulation point of the (f_n).

LEMMA 6.4. *Choose any ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that $f_n(x) \xrightarrow{\mathcal{U}} f(x)$ for all x. Let $a < b$ be real numbers such that $U_V := \{n | \sup f_n|_V > b, \inf f_n|_V < a\}$ belongs to $\mathcal{U}$ for any open nonvoid V. Then (f_n) admits an accumulation point h such that $\|g - h\|_\infty \geq (b-a)/2$ for every Borel function g on T.*

Proof. We will suppose that $[a, b] = [0, 1]$. We start with a nonvoid open V such that diam(V), the diameter of V, is ≤ 1. Fix any $n_1 \in U_V$ and choose open $V_0, V_1 \subset V$ with diam$(V_0) \leq 1/2$, diam$(V_1) \leq 1/2$, $f_{n_1}|_{V_0} < 0$, $f_{n_1}|_{V_1} > 1$.

Now select an $n_2 \in U_{V_0} \cap U_{V_1}$ with $n_2 > n_1$, open V_{00}, V_{01} in V_0 as well as open $V_{10}, V_{11} \subset V_1$ such that the diameters of the V_{ij} are $\leq 1/2^2$ and $f_{n_1}|_{V_{j0}} < 0, f_{n_1}|_{V_{j1}} > 1$ for $j = 0, 1$. It should now be clear how to continue: We construct functions $f_{n_1}, f_{n_2}, \ldots$ and nonvoid open subsets $V_{j_1 \ldots j_r}$ for $r \in \mathbb{N}$, $j_1, \ldots, j_r \in \mathbb{N}$ such that diam$(V_{j_1, \ldots, j_r}) \leq 1/2^r$, and f_{n_r} is < 0 (resp. > 1) on $V_{j_1 \ldots j_{r-1} 0}$ (resp. $V_{j_1 \ldots j_{r-1} 1}$).

By construction and the completeness assumption on T there is, for any $(j_1, j_2, \ldots) \in \{0, 1\}^{\mathbb{N}}$, precisely one point $x_{j_1 j_2 \ldots}$ in $\bigcap_r V^-_{j_1 \ldots j_r}$, and this gives rise to a mapping $\phi : T \longrightarrow \{0, 1\}^{\mathbb{N}}$. It is routine to show that ϕ is continuous.

Now let h be any accumulation point of the (f_{n_k}), we will show that h has the claimed properties. To this end note that the $f_{n_k} \circ \phi$ behave like the functions h_k from lemma 2.1. More precisely, let τ on $\mathbb{R}$ be defined by $\tau(x) := \min\{\max\{x, 0\}, 1\}$; then $\tau \circ f_{n_k} \circ \phi = h_k$. By the continuity of τ, the function $\tau \circ h \circ \phi$ is an accumulation point of the (h_k). Now let g be a Borel function on T and suppose that $\|g - h\|_\infty < 0.5$. Then, since τ is a contraction, $\|\tau \circ g \circ \phi - \tau \circ h \circ \phi\|_\infty < 0.5$. But $\tau \circ g \circ \phi$ is Borel, and this contradicts lemma A3.2. $\square$

We now turn to the

proof of theorem 6.1. Fix f and choose an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that $f_n(x) \xrightarrow{\mathcal{U}} f(x)$ for all x. For rational $a < b$ with $b - a > \epsilon_0$ and closed subsets $A \subset T$ define $\Phi_{a,b}(A)$ to be the collection of all $x \in A$ such that $\{n | \sup f_n|_{V\cap A} > b, \inf f_n|_{V\cap A} < a\}$ belongs to $\mathcal{U}$ for all open sets V with $x \in V$. Clearly $\Phi_{a,b}(A)$ is a closed subset of A, and it is – for nonempty A – a proper subset by lemma 6.4. Thus $\Phi_{a,b}$ is a subset assignment, and since T is Polish we get from proposition 6.5 a countable cover $\mathcal{S}^*_{a,b}$ of T with each member being of the form $A \setminus \Phi_{a,b}(A)$.

Fix a countable base $\mathcal{T}^*$ of the topology. By the definition of $\Phi_{a,b}$ there is, for A closed and $x \in A \setminus \Phi_{a,b}(A)$, a $V \in \mathcal{T}^*$ such that $x \in V$ and $\{n \mid \sup f_n|_{V\cap A} > b, \inf f_n|_{V\cap A} < a\}$ does *not* belong to $\mathcal{U}$.

Next wellknown properties of ultrafilters apply: If $M \subset \mathbb{N}$ and $M \notin \mathcal{U}$, then $\mathbb{N} \setminus M \in \mathcal{U}$, and $M \cup N \in \mathcal{U}$ yields $M \in \mathcal{U}$ or $N \in \mathcal{U}$. In our situation this gives $\{n \mid \sup f_n|_{V\cap A} \leq b\} \in \mathcal{U}$ or $\{n \mid f_n|_{V\cap A} \geq a\} \in \mathcal{U}$.

We are now going to construct (f_{n_k}) such that $\limsup_k |f_{n_k}(x) - f(x)| \leq \epsilon_0$ for $x \in T$. To this end we consider the families $\mathcal{S}_{a,b} := \{S^* \cap V \mid S^* \in \mathcal{S}^*_{a,b}, V \in \mathcal{T}^*\}$ for rational $a < b$ such that $b - a > \epsilon_0$. We also define a family $\mathcal{U}_{a,b}$ of subsets of $\mathcal{U}$ by $U \in \mathcal{U}_{a,b}$ iff there exists $S \in \mathcal{S}_{a,b}$ such that $U = U_{S,\geq a} := \{n | \sup f_n|_S \geq a\}$ or $U = U_{S,\leq b} := \{n | \sup f_n|_S \leq b\}$. Then $\mathcal{U}^* := \bigcup \mathcal{U}_{a,b}$, where a, b run over all admissible values, is a countable subset of $\mathcal{U}$. Enumerate $\mathcal{U}^*$ as $\{U_1, U_2, \ldots\}$ and choose $n_1 < n_2 < \cdots$ such that $n_k \in U_1 \cap \cdots \cap U_k$.

We claim that $\limsup_k |f_{n_k}(x) - f(x)| \leq \epsilon_0$ for $x \in T$. Let x and $\epsilon > \epsilon_0$ be given. Choose rational a, b, a', b' such that $a < b < f(x) < a' < b'$, $b - a, b' - a' > \epsilon_0$, and $b' - f(x), f(x) - a < \epsilon$. By our construction we find $S \in \mathcal{S}_{a,b}$, $S' \in \mathcal{S}_{a',b'}$ with $x \in S \cap S'$, and $U_{S,\geq a}$ or $U_{S,\leq b}$ as well as $U_{S',\geq a'}$ or $U_{S',\leq b'}$ belong to $\mathcal{U}$. Since $\{n \mid b < f(x) < a'\}$ is in $\mathcal{U}$ it is not possible, however, that $U_{S',\geq a'}$ or $U_{S,\leq b}$ lie in this ultrafilter (otherwise $\emptyset \in \mathcal{U}$ would follow). Hence $U_{S,\geq a}, U_{S',\leq b'} \in \mathcal{U}^*$ so that, for sufficiently large k, $n_k \in U_{S,\geq a} \cap U_{S,\leq b'}$ which implies $|f_{n_k}(x) - f(x)| \leq \epsilon$. □

(We note that further material on the questions we dealt in this section is contained in appendix 4).

Appendix 1. Ramsey theory–The proof of theorem 3.1

This appendix contains the complete proof of theorem 3.1. The proof appeared in [2], for a variant see [7, section X]. The assertion is repeated here for the reader's convenience:

THEOREM A1.1. *For $r \in \mathbb{N}$ let T_r be a family of r–tupels of increasing integers. Suppose that*

$$\forall_{\substack{M\subset\mathbb{N}\\ M \text{ infinite}}} \ \exists_{\substack{i_1,i_2,\ldots\in M\\ i_1<i_2<\ldots}} \ \forall_{r} \ (i_1,\ldots,i_r) \in T_r. \tag{1}$$

Then it follows that

$$\exists_{\substack{M_0\subset\mathbb{N}\\ M_0 \text{ infinite}}} \ \forall_{\substack{r\\ i_1,i_2,\ldots,i_r\in M_0\\ i_1<\ldots<i_r}} \ (i_1,\ldots,i_r) \in T_r. \tag{2}$$

The key of the proof will be the following

DEFINITION A1.2. *Let $M \subset \mathbb{N}$ be infinite, $i_1,\ldots,i_k \in \mathbb{N}$, $i_1 < \ldots < i_k$. Moreover let $(T_r)_{r=1,2,\ldots}$ be as in the theorem.*

(i) *$i_1,\ldots,i_k \downarrow M$ abbreviates the following fact:*
If $i_{k+1} < i_{k+2} < \ldots$ are points in M with $i_{k+1} > i_k$, then there is an r such that $(i_1,\ldots,i_r) \notin T_r$. The case $k = 0$ will also be admissible, we will then write $\emptyset \downarrow M$.

(ii) *$i_1,\ldots,i_k \uparrow M$ stands for the following:*
Whenever N is an infinite subset of M, there are $i_{k+1} < i_{k+2} < \ldots$ in N with $i_{k+1} > i_k$ such that $(i_1,\ldots,i_r) \in T_r$ for every r. Again the definition is meant to contain the case $k = 0$.

Note. Our " $\downarrow$ " and " $\uparrow$ " are closely related with "acceptance" and "rejection" in Diestel's book ([7, p.192]). Our approach, however, is more direct since we only have in mind a special version of a Ramsey theorem.

The following facts are immediate consequences of the definitions, they are stated only for the sake of easy reference.

OBSERVATION A1.3. (i) *(1) of the theorem just means* $\emptyset \uparrow \mathbb{N}$, *and (2) follows as soon as one has found an* M_0 *such that* $i_1, \ldots, i_k \uparrow M_0$ *for arbitrary* $i_1 < \ldots < i_k$ *in* M_0.

(ii) *Let* $i_1 < \ldots < i_k$ *be given and* $M \subset \mathbb{N}$ *be infinite. If* $i_1, \ldots, i_k \downarrow M$ *does not hold, then there are* $i_{k+1} < i_{k+2} < \ldots$ *(with* $i_{k+1} > i_k$*) in* M *such that* $(i_1, \ldots, i_r) \in T_r$ *for every* r.

(iii) *If* $i_1, \ldots, i_k \downarrow M$ *and* $N \subset M$ *is infinite, then* $i_1, \ldots, i_k \downarrow N$. *The same holds if the "* $\downarrow$ *" are replaced by "* $\uparrow$ *".*

Here is the first step of our construction:

LEMMA A1.4. *There is an infinite* $\widetilde{M_0} \subset \mathbb{N}$ *such that*

$$i_1, \ldots, i_k \downarrow \widetilde{M_0} \quad \text{or} \quad i_1, \ldots, i_k \uparrow \widetilde{M_0} \tag{3}$$

for each choice of $i_1 < \ldots < i_k$ *in* $\widetilde{M_0}$ *(including the case* $k = 0$*).*

Proof. It will be convenient to say that an infinite $\widetilde{M_0}$ satisfies $(3)_s$ (where $s \in \mathbb{N}_0$) if the assertion (3) holds under the additional assumption that $\{i_1, \ldots, i_k\}$ is contained in the set of the first s elements of $\widetilde{M_0}$. We combine the following observations:

- We are looking for an infinite $\widetilde{M_0}$ such that $\widetilde{M_0}$ satisfies $(3)_s$ for every s.
- Suppose we are able to make the following induction work: Given an infinite $\widetilde{M}^{(s)}$ (which we write in increasing order as $\widetilde{M}^{(s)} = \{i_1, i_2, \ldots, i_s, \ldots\}$) such that $(3)_s$ holds for $\widetilde{M}^{(s)}$, there is an infinite subset N of $\{i_{s+1}, i_{s+2}, \ldots\}$ such that $\widetilde{M}^{(s+1)} := \{i_1, \ldots, i_s\} \cup N$ satisfies $(3)_{s+1}$.
  This would suffice: We start our construction by setting $\widetilde{M}^{(0)} := \mathbb{N}$ (note that $\widetilde{M}^{(0)}$ satisfies $(3)_0$ by 2.3(i)), use the induction to construct the $\widetilde{M}^{(0)} \supset \widetilde{M}^{(1)} \supset \widetilde{M}^{(2)} \ldots$ and set $\widetilde{M_0}$ = "the collection of the s'th elements of $\widetilde{M}^{(s)}$, $s \in \mathbb{N}$." For fixed s, $\widetilde{M_0}$ has the same first s elements as $\widetilde{M}^{(s)}$, so that in view of 2.3(iii) $(3)_s$ necessarily holds for $\widetilde{M_0}$.

Therefore let's concentrate on the induction step. Let $s \geq 0$ and $\widetilde{M}^{(s)}$ with $(3)_s$ be given. Denote by $\Delta_1, \ldots, \Delta_{2^s}$ the 2^s different subsets of $\{i_1, \ldots, i_s\}$.

We will construct infinite subsets $\widetilde{N}^{[1]} \supset \widetilde{N}^{[2]} \supset \ldots \supset \widetilde{N}^{[2^s]}$ of $\{i_{s+2}, \ldots\}$ such that either $\Delta_j, i_{s+1} \uparrow \widetilde{N}^{[j]}$ or $\Delta_j, i_{s+1} \downarrow \widetilde{N}^{[j]}$ for every j. In view of 2.3(iii) it is then clear that $\widetilde{M}^{(s+1)} := \{i_1, \ldots, i_{s+1}\} \cup \widetilde{N}^{[2^s]}$ has $(3)_{s+1}$.

First consider Δ_1, i_{s+1}. Either we have $\Delta_1, i_{s+1} \uparrow \widetilde{M}^{(s)}$ (in which case we put $\widetilde{N}^{[1]} := \{i_{s+2}, \ldots\}$) or there is an infinite subset $\widetilde{N}^{[1]}$ of $\{i_{s+2}, \ldots\}$ such that $\Delta_1, i_{s+1} \downarrow \widetilde{N}^{[1]}$ (see 2.3(ii)).

Secondly, we investigate Δ_2, i_{s+1}. Either $\Delta_2, i_{s+1} \uparrow \widetilde{N}^{[1]}$ (we will put $\widetilde{N}^{[2]} := \widetilde{N}^{[1]}$ in this case) or there is an infinite subset $\widetilde{N}^{[2]}$ of $\widetilde{N}^{[1]}$ such that $\Delta_2, i_{s+1} \downarrow \widetilde{N}^{[2]}$. It should be clear how to construct the remaining $\widetilde{N}^{[3]} \supset \ldots \supset \widetilde{N}^{[2^s]}$. □

In order to get an M_0 with (2) from $\widetilde{M}_0$ we need

LEMMA A1.5. *Let $i_1 < \ldots < i_k$ in $\widetilde{M}_0$ be given and suppose that $i_1, \ldots, i_k \uparrow \widetilde{M}_0$. Then there are only finitely many $i > i_k$ in $\widetilde{M}_0$ such that $i_1, \ldots, i_k, i \downarrow \widetilde{M}_0$.*

Proof. Suppose that this were not the case. Put $N =$ the collection of these i. N is infinite, and by $i_1, \ldots, i_k \uparrow \widetilde{M}_0$ there would be $i_{k+1} < i_{k+2} < \ldots$ in N (with $i_{k+1} > i_k$) such that $(i_1, \ldots, i_r) \in T_r$ for every r. Note that this would contradict $i_1, \ldots, i_{k+1} \downarrow \widetilde{M}_0$. □

Finally, we are ready for the

Proof of the Ramsey theorem A1.1. We have already noted that (2) just means $i_1, \ldots, i_k \uparrow M_0$ for $i_1 < \ldots < i_k$ in M_0. Similarly to the proof of 2.4 we introduce $(2)_s$ for $s \geq 0$: This is (2) with the same additional assumption as in $(3)_s$.

The construction parallels that of 2.4: We need an M_0 with $(2)_s$ for all s, we know that $M^{(0)} := \mathbb{N}$ satisfies $(2)_0$, and, given an $M^{(s)}$ with $(2)_s$, we only have to construct an $M^{(s+1)}$ with $(2)_{s+1}$. In this construction the first s elements of $M^{(s)}$ and $M^{(s+1)}$ should be identical; $M_0 =$ "the collection of the s'th elements of $M^{(s)}, s \in \mathbb{N}$" then will have the desired properties.

Here is the induction. Write $M^{(s)} = \{i_1, \ldots, i_s, i_{s+1}, \ldots\}$ and put $N := \{i_{s+1}, \ldots\}$. Since $\Delta \uparrow N$ for every $\Delta \subset \{i_1, \ldots, i_s\}$ by assumption we conclude from 2.5 that there are only finitely many i in N such that $\Delta, i \downarrow N$ for any Δ. Choose $\widetilde{N} \subset N$ such that $\widetilde{N}$ does not contain such i. Then $M^{(s+1)} := \{i_1, \ldots, i_s\} \cup \widetilde{N}$ satisfies $(2)_{s+1}$, and this completes the proof. □

Appendix 2. Banach limits

Our proof of the Josefson-Nissenzweig theorem uses the existence of *Banach limits*, this existence will be proved here.

The natural place to treat Banach limits is the theory of *generalized limits*. Roughly speaking a generalized limit consists of a vector space V of sequences in $\mathbb{R}$ such that V contains the space c of convergent sequences together with a map $L : V \longrightarrow \mathbb{R}$, such that L generalizes "in a natural way" what is known about the map $(x_n) \mapsto \lim x_n$ for $(x_n) \in c$. A familiar example is the *Cesaro limit*: For sequences for which it can be defined one puts

$$C - \lim x_n := \lim(x_1 + \cdots + x_n)/n.$$

$L = C - \lim$ shares the typical properties of generaliized limits: It is defined on a linear space of sequences, the domain properly contains the space c, it extends the limit operation, it is monotone, ... There is a great variety of similar extensions of lim, the interested reader is referred to [14] or [27].

Here we will discuss a special case:

Definition A2.1. *A* Banach limit *is a linear map* $L : l^\infty \longrightarrow \mathbb{R}$ *such that*

- L *extends* lim*: For* $(x_n) \in c$ *one has* $L((x_n)) = \lim x_n$.
- L *is monotone:* $(x_n) \geq 0$ *implies* $L((x_n)) \geq 0$.
- L *is invariant:* $L((x_n)) = L((x_{n+1}))$.

Proposition A2.2. Banach limits exist.

Proof. We will present two proofs. Both heavily rely on Zorn's lemma.
First proof: This proof is a standard exercise in many courses of funcional analysis. The properties L should have are encoded in properties of a sublinear functional σ: a linear L dominated by σ will have the desired properties provided that σ is suitably defined. The difficulty, of course, is the proper definition of σ.

The following possibility is chosen on [26, p. 113]:

$$\sigma((x_n)) := \limsup(x_1 + \cdots + x_n)/n.$$

Then any $L \leq \sigma$ will be a Banach limit. Even more is true: In the Hahn-Banach theorem one may prescribe a linear functional on a subspace (provided it is dominated on this subspace by σ). Since there are one-dimensional subspaces where one can choose infinitely many such functionals – e. g. the subspace generated by $(1, 0, 1, 1, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, \ldots)$ – it follows that there are infinitely many Banach limits.

A variant, where another form of the Hahn-Banach theorem is used, can be found in [8] (p. 73): Any L such that $L((1, 1, \ldots)) = 1, L((x_1 - x_2, x_2 - x_3, \ldots)) = 0$ for all bounded (x_n) is a Banach limit, and the existence follows from a separation argument.

Second proof: In this variant we will apply the *fixed point theorem of Markov-Kakutani* : Whenever K is a compact convex subset of a locally convex vector space and $\mathcal{T}$ a family of affine continuous mappings from K to K such that, for $S, T \in \mathcal{T}$, the composition $S \circ T$ is in $\mathcal{T}$ and $S \circ T = T \circ S$ holds[5], then there exists a common fixed point, i. e. a $k \in K$ with $T(k) = k$ for all T. (A rather elementary proof which uses only the Hahn-Banach theorem has appeared in [25]).

The Markov-Kakutani theorem is applied as follows: Let K be the collection of all continuous linear functionals on the real l^∞ which are monotone and map the sequence (1) to one. K is easily seen to be a nonvoid convex set of the dual of l^∞ which is compact when this dual is provided with the weak*-topology.

[5]Such a family is called a commutative semigroup of affine continuous transformations

For $k \in \mathbb{N}$ let T_k be the following mapping from K to K: $(T_k(x'))((x_n)) := x'((x_{n+1}))$. Then the family $\mathcal{T}$ of these T_k satisfies the assumptions of the Markov-Kakutani theorem, and a common fixed point has the properties of a Banach limit. □

In order to understand a little bit the properties of Banach limits we finish this appendix by collecting *some general properties* (L denotes any Banach limit):

- L is continuous with $||L|| = 1$.
- $L(1, 0, 1, 0, 1, 0, \ldots) = 0.5$.

  Similarly $L(0, 1, 0, 1, \ldots) = 0.5$ holds, and this has the interesting consequence that Banach limits never can be multiplicative.
- $L(0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots) = 1/3$.
- $L(-1, +1, -1, +1, \ldots) = 0$.

Appendix 3. Ultrafilters and functions which are not Borel measurable

The material of this appendix will be of importance in section 6. At first we recall the definition of a *filter*. Let N be a nonvoid set and $\mathcal{F}$ a family of subsets (i. e. a subset of the power set). $\mathcal{F}$ is called a *filter*, if the following conditions are satisfied:

- N is contained in $\mathcal{F}$, but $\emptyset \notin \mathcal{F}$.
- With $F, G \in \mathcal{F}$ also $F \cap G \in \mathcal{F}$ holds.
- $F \in \mathcal{F}$ and $F \subset G$ imply $G \in \mathcal{F}$.

The collection of all filters on N together with the inclusion as an order relation is easily seen to be inductively ordered, hence – by Zorn's lemma – there are maximal filters which are called *ultrafilters*. Among the ultrafilters some are are of a particularly simple form: If $x \in N$, then the collection of all sets containing x is easily seen to be an ultrafilter. Such ultrafilters will be called *fixed ultrafilters*, the other – more interesting ones – are the *free ultrafilters*.

Here the free ultrafilters on $\mathbb{N}$ will be of interest. At first we derive an appropriate description of these objects.

Subsets of $\mathbb{N}$ are in one-to-one correspondence with $\{0,1\}$-valued functions on $\mathbb{N}$ so that the power set can be identified with the product $M := \{0,1\}^{\mathbb{N}}$. Consequently a collection of subsets of $\mathbb{N}$ is nothing but a $\{0,1\}$-valued function on this product. Which of these functions correspond to (free) ultrafilters?

For $n \in \mathbb{N}$, let $h_n : M \longrightarrow \{0,1\}$ be the natural projection onto the n'th coordinate. M can be identified either with the subsets of the integers or with the Cantor set, and according to the respective interpretation the h_n correspond to the Rademacher functions on the Cantor set or the Dirac measures on the subsets of the integers.

Consider now the functions h in the closure of the family (h_n) (closure with respect to pointwise convergence). Then one has

LEMMA A3.1. $h \mapsto h^{-1}(1)$ *defines a bijection between these* h *and the ultrafilters on* $\mathbb{N}$, *where the* h *which are not in* $\{h_n \mid n \in \mathbb{N}\}$, *i. e., the accumulation points, correspond to the free ultrafilters.*

Proof. A moment's reflection shows that h_n corresponds to the ultrafilter fixed at n, hence we only have to treat the free ultrafilters.

Here the following property of ultrafilters will be used (the proof of which is rather simple): A filter $\mathcal{F}$ on a set N is an ultrafilter iff for any partition of N into disjoint subsets F, G one has $F \in \mathcal{F}$ or $G \in \mathcal{F}$.

Let h be an accumulation point of the h_n, and $\mathcal{U} := h^{-1}(1)$. We show that $\mathcal{U}$ is a filter which satisfies the preceding property. Take for example the property $\mathbb{N} \in \mathcal{U}$. Since $n \in \mathbb{N}$ holds for every n this means that always $h_n(\mathbb{N}) = 1$ so that necessarily $h(\mathbb{N}) = 1$ as well, hence $\mathbb{N} \in \mathcal{U}$. Similarly one proves the other properties of a filter. As to the remaining claim we note that, for an arbitrary $F \subset \mathbb{N}$, $h_n(F) + h_n(\mathbb{N} \setminus F) = 1$ holds for all n. This equation passes to pointwise accumulation points, and this means that $F \in \mathcal{U}$ or $\mathbb{N} \setminus F \in \mathcal{U}$.

Conversely, let $\mathcal{U}$ be a free ultrafilter. We define a function h on M to be the characteristic function of $\mathcal{U}$, and we claim that h is an accumulation point of the h_n. It suffices to show that h lies in the closure of these functions ($\{n\}$ is *not* in $\mathcal{U}$, and thus h does not coincide with any h_n). We have to show that any neighbourhood of h contains an h_n. By the definition of pointwise convergence this just means that for arbitrary $F_1, \ldots, F_k \in \mathcal{U}$ and $G_1, \ldots, G_r \notin \mathcal{U}$ we have to provide an n such that $n \in F_1, \ldots, F_k, n \notin G_1, \ldots, G_r$. But by the above mentioned properties of ultrafilters we know that the complements of the G_ρ are in $\mathcal{U}$, and thus we have only to choose any n in the intersection of the sets $F_1, \ldots, F_k, \mathbb{N} \setminus G_1, \ldots, \mathbb{N} \setminus G_r$; note that this intersection is non-empty since filters are stable with respect to intersections, and $\emptyset$ is not in $\mathcal{U}$. □

We claim that the h which correspond to free ultrafilters correspond to non-Borel functions:

PROPOSITION A3.2. *Let h be an accumulation point of the (h_n). Then h is not a Borel function. In fact there is not even a Borel function g such that $\|g - h\|_\infty < 0.5$.*

Proof. At first we prove that h is not Borel: We assume the contrary and it will be shown that this leads to a contradiction.

We provide M with the product measure P with respect to the uniform distribution on the factors, and we regard M as a topological group. Since every h_n is a group homomorphism and satisfies $h_n(x) + h_n(1 - x) = 1$ for $x \in M$ the same properties hold for the pointwise accumulation point h.

By assumption h is Borel so that $A := h^{-1}(1)$ is measurable. Since $h(x) + h(1 - x) = 1$, the set M is the disjoint union of A and $1 - A$, and therefore $P(A) = 0.5$ (note that P is translation invariant).

Now consider the collection $\mathcal{E}$ of all Borel subsets B of M such that $P(A \cap B) = 0.5P(B)$. We already know that $M \in \mathcal{E}$, and it is straightforward to show that $\mathcal{E}$ is a Dynkin system. We aim at proving that A is also in $\mathcal{E}$ which would lead to the contradiction $P(A) = 0$, thus finishing the first part of the proof.

We define, for arbitrary m in $\mathbb{N}$ and $\epsilon_1, \ldots, \epsilon_m$ in $\{0, 1\}$, the set $A_{\epsilon_1,\ldots,\epsilon_m}$ by $\{(x_n) \in M \mid x_1 = \epsilon_1, \ldots, x_m = \epsilon_m\}$, and we claim that this set belongs to $\mathcal{E}$. To this end, consider

$$\phi_m : M \longrightarrow M, \quad (x_1, \ldots) \mapsto (x_1, \ldots, x_m, 1 - x_{m+1}, 1 - x_{m+2}, \ldots).$$

This mapping is measurable and measure-preserving, and ϕ_m maps A to $1-A$ (this follows from $h_n \circ \phi_m = 1 - h_n$ for $n > m$). Also $\phi_m(A_{\epsilon_1,\ldots,\epsilon_m}) = A_{\epsilon_1,\ldots,\epsilon_m}$ holds, and this yields $P(A \cap A_{\epsilon_1,\ldots,\epsilon_m}) = P(\phi_m(A \cap A_{\epsilon_1,\ldots,\epsilon_m})) = P((1-A) \cap A_{\epsilon_1,\ldots,\epsilon_m})$. Hence

$$\begin{aligned} P(A \cap A_{\epsilon_1,\ldots,\epsilon_m}) &= 0.5[P(A \cap A_{\epsilon_1,\ldots,\epsilon_m}) + P((1-A) \cap A_{\epsilon_1,\ldots,\epsilon_m})] \\ &= P(A_{\epsilon_1,\ldots,\epsilon_m}), \end{aligned}$$

i.e., $A_{\epsilon_1,\ldots,\epsilon_m} \in \mathcal{E}$.

Since the Borel sets in M are generated by the $A_{\epsilon_1,\ldots,\epsilon_m}$ and the collection of these sets is closed with respect to intersections it follows that $\mathcal{E}$ contains *every* Borel set, in particular A as claimed above.

Finally suppose that there exists a Borel function g such that $\|h - g\|_\infty < 0.5$. Then $\{g < 0.5\} = \{h = 0\}$ and $\{g > 0.5\} = \{h = 1\}$ would be Borel sets in contrast to that what has been shown before. □

Appendix 4. Martin's axiom

Theorem 6.1 can be rephrased as follows:

Let $(T, \mathcal{T})$ and (f_n) be as in that theorem and denote, for $\epsilon_0 \geq 0$, by $(\epsilon_0)_{\text{Borel}}$ and $(\epsilon_0)_{\text{Baire}}$ the properties

$(\epsilon_0)_{\text{Borel}}$: Every accumulation point of the (f_n) is ϵ_0-close to a Borel function;

$(\epsilon_0)_{\text{Baire}}$: For every accumulation point of the (f_n) there is a subsequence (f_{n_k}) with $\limsup_k |f_{n_k}(x) - f(x)| \leq \epsilon_0$, $x \in T$.

Then $(\epsilon_0/2)_{\text{Borel}}$ implies $(\epsilon_0)_{\text{Baire}}$.

Since the collection of all Borel functions is closed with respect to the supremum norm it is clear that $(\epsilon)_{\text{Borel}}$, all $\epsilon > 0$, yields $(0)_{\text{Borel}}$ and hence $(0)_{\text{Baire}}$, i.e., every accumulation point is the limit of a subsequence.

One might suspect that this last assertion could be established by a different argument: Given an f such that for $\epsilon > 0$ there is a subsequence (f_{n_k}) with $\limsup_k |f_{n_k}(x) - f(x)| \leq \epsilon$, $x \in T$, then there also should be a subsequence with pointwise limit f. We will concentrate on the set theoretical variant of this problem (which corresponds to the special case of discrete topological spaces). Also we may and will assume that the function to be approximated is the zero function.

PROPOSITION A4.1. *Let T be a set and (f_n) a sequence of real-valued functions on T such that for every $\epsilon > 0$ there is a subsequence (f_{n_k}) with* $\limsup |f_{n_k}(x)| \le \epsilon$.

(i) *If T is countable then there is a subsequence of (f_n) which converges pointwise to zero.*

(ii) *If* $\mathrm{card}(T) \ge 2^{\aleph_0}$ *then this is not necessarily true.*

Proof. (i) is obvious. For the proof of (ii) it suffices to treat the case $T = \mathbb{N}^{\mathbb{N}}$. We define, for integers m and n, a function $h_{m,n} : T \longrightarrow [0,1]$ by $h_{m,n}(\phi) := 1$ if $n \le \phi(m)$ and $:= 1/m$ otherwise. Let $f_1, f_2, \ldots$ be any enumeration of the family $(h_{n,m})$. Then (f_n) has subsequences with arbitrarily small lim sup, but there is no subsequence converging to zero: Let (h_{m_k,n_k}) be any subsequence; if the m_k are bounded, by m' say, then $(h_{m_k,n_k}) \ge 1/m'$, and if they are unbounded we can choose a ϕ_0 such that $\phi_0(m_k) > n_k$ infinitely often so that $(h_{m_k,n_k})(\phi_0)$ does not converge to zero. □

The proposition leaves open the case of sets with cardinality between $\aleph_0$ and $2^{\aleph_0}$. Under the assumption of the continuum hypothesis (CH) there are no such sets, but what about set theories without CH ? We refer the reader to [11] where a number of interesting set theoretical axioms and their implications for various branches of mathematics are discussed. For our problem *Martin's axiom (MA)* will be of importance.

> This is one of the rather few instances where people working in functional analysis come in touch with the set theoretical foundations. Usually one works as if the mathematical world would obey the familiar laws of logic, whenever it is needed one may apply Zorn's lemma or its equivalent formulations, nearly never one has to do with the continuum hypothesis, ...
>
> It has to be pointed out that this standpoint is naive, it resembles the standpoint of someone who thinks that the only natural geometry is the Euclidean one. Today we know that – similarly to the situation with geometry – there are several possibilities to choose the set theoretical foundations one works

with, and here we want to have a modest look to a world where Martin's axiom holds.

The precise meaning of MA is rather technical, fortunately it suffices for us to know

THEOREM A4.2. *Under the assumption of* MA *the following is true:* *Whenever* M *is a set with cardinality less than* $2^{\aleph_0}$ *and* $(C_x)_{x\in M}$ *is a collection of subsets of* $\mathbb{N}$ *such that all finite intersections* $C_{x_1}\cap \cdots \cap C_{x_r}$ *are infinite, then there exists an infinite* $I\subset\mathbb{N}$ *for which all* $I\setminus C_x$ *are finite.*

Proof. Let κ be the cardinality of M. Then, with the notation of [11], MA(κ) holds which implies, by theorem 11C of [11], that P(κ^+). But $\kappa<\kappa^+$, and our result follows from [11, 11B(c)]. □

It should be noted that MA is "nearly equivalent" with the statement of the theorem. The precise meaning of this can be found in [11, Bell's theorem, 14C].

THEOREM A4.3. *Under the assumption of Martin's axiom every set* T *with* card(T) $< 2^{\aleph_0}$ *has the following property: If the* (f_n) *are as in proposition 4.1, then there is a subsequence which converges pointwise to zero.*

Proof. Set $S := T\times\mathbb{N}$. For $(x,m)\in S$ define $C_{x,m} := \{k \mid f_k(x) < 1/m\}$. The assumption implies that finite intersections of such sets are infinite, and since card(S) $< 2^{\aleph_0}$ we may apply MA. Theorem 4.2 provides $I=\{n_1,n_2,\ldots\}$ such that all $C_{x,m}\setminus I$ are finite. It is then clear that (f_{n_k}) converges pointwise to zero. □

Note: It is left as an exercise to show that the assertions of the two preceding theorems are in fact equivalent.

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