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Global Existence and Blow-up of the Classical Solutions to Systems of Semilinear Wave Equations in Three Space Dimensions

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Summary. - *We consider the Cauchy problem for a system of semilinear wave equations with small initial data whose propagation speeds may be different. As for a system of quasilinear wave equations, the discrepancy of the speeds makes the maximal existence time of solutions be longer, when we treat the critical nonlinearity. In contrast with the quasilinear case, we show that for the semilinear case, such phenomena does not occur, by establishing estimates of the lifespan from upper and lower.*

1. Introduction and Main Result

In this note, we consider the small data global existence and blowup for the Cauchy problem of the following system of semilinear wave equations with different propagation speeds in three space dimen-

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sions:

$$\begin{cases} \partial_t^2 u - c^2\Delta u = |v|^p, & (x,t) \in \mathbf{R}^3 \times [0,\infty), \\ \partial_t^2 u - \Delta v = |u|^q, & (x,t) \in \mathbf{R}^3 \times [0,\infty), \\ u(x,0) = \varepsilon f_1(x),\ \partial_t u(x,0) = \varepsilon g_1(x), & x \in \mathbf{R}^3, \\ v(x,0) = \varepsilon f_2(x),\ \partial_t v(x,0) = \varepsilon g_2(x), & x \in \mathbf{R}^3, \end{cases} \tag{1}$$

where $c > 0$, $0 < \varepsilon \le 1$, $1 < p \le q < \infty$ and $(u(x,t), v(x,t))$ is a real unknown. As for the initial data, we assume that $f_j \in C^3(\mathbf{R}^3)$ and $g_j \in C^2(\mathbf{R}^3)$ $(j = 1, 2)$.

The small data global existence and blowup problem for (1) has been recently studied by Del Santo, Georgiev and Mitidieri [21], Del Santo [20], Deng [5], Agemi, Kurokawa and Takamura [2] and the authors [16] when $c = 1$ in (1). To state the results obtained by [2], [5] and [16], [20], [21] here and hereafter, we put

$$\alpha = p(q-2) - 1, \quad \beta = q(p-2) - 1. \tag{2}$$

Then, the following results have been obtained for (1) with $c = 1$. When $\alpha + p\beta > 0$ and $p \ge 2$, the small data global existence holds (see [20] and [21]), and when $\alpha + p\beta \le 0$, the small data blowup occurs (see [21] and [5] for $\alpha + p\beta > 0$, and [2] and [16] for $\alpha + p\beta = 0$). Here, we say that the small data global existence holds for (1) if for any $f_j \in C_0^3(\mathbf{R}^3)$ and $g_j \in C_0^2(\mathbf{R}^3)$ $(j = 1, 2)$ there exists a positive constant ε_0 such that (1) has a global classical solution provided $0 < \varepsilon \le \varepsilon_0$. Otherwise, we say that the small data blowup occurs for (1). We also note that the condition $\alpha + p\beta = 0$ is equivalent to $\Gamma(p,q,3) = (q + 2 + p^{-1})/(pq-1) - 1 = 0$ in [2], [21] and [16], in fact we have $\Gamma(p,q,3) = (\alpha + p\beta)/p(pq-1)$.

The system (1) is closely related to the scalar equation

$$\partial_t^2 u - \Delta u = |u|^p, \quad (x,t) \in \mathbf{R}^n \times [0,\infty). \tag{3}$$

It is known that the critical power $p_0(n)$ for the small data global existence and blowup is given by

$$p_0(n) = \frac{n + 1 + \sqrt{n^2 + 10n - 7}}{2(n-1)},$$

which is the positive root of the quadratic equation

$$p\left(\frac{n-1}{2}p - \frac{n+1}{2}\right) - 1 = 0$$

(see, e.g., John [12], Strauss [26], Glassey [8], [7], Schaeffer [22], Sideris [24], Lindblad [18], Zhou [29], [30], Georgiev, Lindblad and Sogge [6], and the references cited therein). In particular, we note that for the critical case $p = p_0(n)$ in (3), the small data blowup is proved only for $n = 2$ and 3 (see [22], [29] and [30]), and it is still open for $n \geq 4$. We also note that if $p = q = p_0(3) = 1 + \sqrt{2}$ in (1), $\alpha = \beta = 0$ in (2), so $\alpha + p\beta = 0$.

In the present note, we study the small data global existence and blowup for (1) when the propagation speeds are different from each other, i.e., $c \neq 1$ in (1). This work is motivated by the recent results established by Kovalyov [14], Agemi and Yokoyama [3], Hoshiga and Kubo [10] and Yokoyama [28]. In those papers, the small data global existence for systems of nonlinear wave equations with different propagation speeds has been well developed when the nonlinear terms depend only on the derivatives of unknown functions but not on unknown functions themselves (see also [25] and [1] for related results on nonlinear elastic wave equations, and [19] on Klein-Gordon-Zakharov equations). We explain some of their results by examples such as the scalar equation

$$\partial_t^2 u - \Delta u = |\partial_t u|^p, \quad (x,t) \in \mathbf{R}^3 \times [0, \infty), \tag{4}$$

and the corresponding systems

$$\begin{cases} \partial_t^2 u - c^2 \Delta u = F_1(\partial_t u, \partial_t v), & (x,t) \in \mathbf{R}^3 \times [0,\infty), \\ \partial_t^2 u - \Delta v = F_2(\partial_t u, \partial_t v), & (x,t) \in \mathbf{R}^3 \times [0,\infty). \end{cases} \tag{5}$$

For (4), it is known that the small data blowup occurs when $1 < p \leq 2$, and the small data global existence holds when $p > 2$ (see John [13], Sideris [23], Hidano and Tsutaya [9] and Tzvetkov [27]). So, we may think that the quadratic nonlinearity is also critical for the small data global existence and blowup for (5). First, we consider the case when $F_1 = F_2 = \partial_t u \partial_t v$ in (5). In this case, if $c = 1$, it is trivial that the small data blowup occurs for (5), by the small data blowup result for (4) in the critical case $p = 2$ by John [13]. However, when $c \neq 1$, Kovalyov [14] proved that the small data global existence holds for (5) with $F_1 = F_2 = \partial_t u \partial_t v$. Next, we consider the case when $F_1 = (\partial_t v)^2$ and $F_2 = (\partial_t u)^2$ in (5). In this case, it is more difficult to prove the small data global existence, but it is shown in

[28] that the small data global existence holds if $c \neq 1$ (see also [3] and [10] for two space dimensional case).

Therefore, it is interesting to ask whether the discrepancy between the propagation speeds in (1) yields the small data global existence or not, especially in the critical case $\alpha + p\beta = 0$. In this note, we show that the small data blowup occurs for (1) in the critical case $\alpha + p\beta = 0$ even if $c \neq 1$. To our knowledge, this is the first result on the critical small data blowup for systems of semilinear wave equations with different propagation speeds, although, as stated above, the critical small data global existence has been recently well studied by [3], [10], [14] and [28]. More precisely, we obtain in [16] and [15]

THEOREM 1.1. *Assume that* $\alpha + p\beta \leq 0$ *with* (2),

$$f_j(x) = 0, \quad g_j(x) \geq 0 \quad (x \in \mathbf{R}^3, \ j = 1, 2), \tag{6}$$

and $g_2(0) > 0$. *Then the classical solution of* (1) *does not exist globally in* $\mathbf{R}^3 \times [0, \infty)$. *Moreover, there exists a positive constant* C_0, *independent of* ε, *such that the life span* $T^*(\varepsilon)$ *of the classical solution of* (1) *satisfies*

$$T^*(\varepsilon) \leq \exp\left(C_0 \varepsilon^{-p(pq-1)}\right) \qquad \text{if} \qquad \alpha + p\beta = 0, \tag{7}$$

$$T^*(\varepsilon) \leq \exp\left(C_0 \varepsilon^{-p(p-1)}\right) \qquad \text{if} \qquad \alpha + p\beta = 0, \ p = q, \ c = 1, \tag{8}$$

$$T^*(\varepsilon) \leq C_0 \varepsilon^{-p(pq-1)/(\alpha+p\beta)} \qquad \text{if} \qquad \alpha + p\beta < 0. \tag{9}$$

Here we denoted by $T^*(\varepsilon)$ *the supremum of all* $T > 0$ *such that the classical solution* (u, v) *of* (1) *exists in* $\mathbf{R}^3 \times [0, T)$ *for given* c, p, q, f_j *and* g_j.

REMARK 1.2. *When* $c = 1$, *a similar result to Theorem 1.1 holds in two space dimensional case (see [16]). However, there are some difficulties to treat the case* $c \neq 1$ *in two space dimensional case, so it is an open problem whether a similar result to Theorem 1.1 holds or not when* $c \neq 1$ *in two space dimensional case.*

On the contrary to Theorem 1.1, we obtain in [17] the following lower bounds of the life span of the solution to (1) with different propagation speeds.

THEOREM 1.3. *Assume that* f_j *and* g_j $(j=1,2)$ *satisfy*

$$f_j(x) = g_j(x) = 0 \quad \textit{for } |x| \geq R, \tag{10}$$

where $0 < \varepsilon \leq 1$ *and* $R > 0$, *and that* $2 < p \leq q$. *Then there is a positive number* $\varepsilon_0 = \varepsilon_0(c, p, q, R)$ *such that for any* ε *with* $0 < \varepsilon \leq \varepsilon_0$ *we have*

$$T^*(\varepsilon) = \infty \qquad \textit{if} \qquad \alpha + p\beta > 0, \tag{11}$$

$$T^*(\varepsilon) \geq \exp\left(C^* \varepsilon^{-p(pq-1)}\right) \qquad \textit{if} \qquad \alpha + p\beta = 0,\ p \neq q, \tag{12}$$

$$T^*(\varepsilon) \geq \exp\left(C^* \varepsilon^{-p(p-1)}\right) \qquad \textit{if} \qquad \alpha + p\beta = 0,\ p = q, \tag{13}$$

$$T^*(\varepsilon) \geq C^* \varepsilon^{p(pq-1)/\Gamma(p,q)} \qquad \textit{if} \qquad \alpha + p\beta < 0, \tag{14}$$

where C^* *is a positive constant independent of* ε.

REMARK 1.4. *Having in mind the estimates (7)–(9), we see that the estimates (11)–(14) are optimal concerning the order of* ε, *except for the case where* $\alpha + p\beta = 0$, $p = q$ *and* $c \neq 1$. *In other words, it is an open problem whether the estimate (7) is optimal or not for the case when* $c \neq 1$ *and* $p = q = 1 + \sqrt{2}$. *And when* $c = 1$, *the statement of Theorem 1.3 was also proved in [2].*

In what follows, we denote a positive constant in the estimates by C, which will change from step to step.

2. Proof of Theorem 1.1

To prove Theorem 1.1, following the argument as in Zhou [29], we reduce the blowup problem for the system (1) to that for a system of integral equations with one variable (see Subsection 2.3), and prove the small data blowup for the reduced system with one variable (see Subsection 2.2).

2.1. Preliminaries

We denote the spherical mean of a function $f(x)$ of $x \in \mathbf{R}^3$ at the origin with radius r by

$$\widetilde{f}(r) = \frac{1}{4\pi} \int_{|\omega|=1} f(r\omega) dS_\omega.$$

Let $u_0(x,t)$ be the solution of the Cauchy problem to the homogeneous wave equation

$$\begin{cases} \partial_t^2 u - c^2 \Delta u = 0, & (x,t) \in \mathbf{R}^3 \times [0,\infty), \\ u(x,0) = 0, \quad u_t(x,0) = g(x), & x \in \mathbf{R}^3. \end{cases} \tag{15}$$

Then, as is well known, $u_0(x,t)$ is given by $u_0(x,t) = I_c(g)(x,t)$, where

$$I_c(g)(x,t) = \frac{t}{4\pi} \int_{|\omega|=1} g(x+ct\omega) dS_\omega. \tag{16}$$

We also introduce for $(r,t) \in (0,\infty) \times (0,\infty)$

$$J_c(g)(r,t) = \frac{1}{2cr} \int_{|ct-r|}^{ct+r} \rho g(\rho) d\rho. \tag{17}$$

Then we have

LEMMA 2.1. *Let $g \in C(\mathbf{R}^3)$ and $g(x) \geq 0$. Then we have for $(r,t) \in (0,\infty) \times (0,\infty)$*

$$\widetilde{I_c(g)}(r,t) \geq J_c(\widetilde{g})(r,t). \tag{18}$$

Proof. We will use the identity for iterated spherical means (see [11] at page 81):

$$\begin{aligned} &\frac{1}{\omega_n^2} \int_{|\zeta|=1} \int_{|\omega|=1} g(r\zeta + \rho\omega) dS_\omega dS_\zeta \\ &= \frac{2\omega_{n-1}}{\omega_n (2r\rho)^{n-2}} \int_{|\rho-r|}^{\rho+r} \lambda \{h(\rho,\lambda;r)\}^{\frac{n-3}{2}} \widetilde{g}(\lambda) d\lambda, \end{aligned} \tag{19}$$

where we have set

$$h(\rho,\lambda;r) = ((\lambda+r)^2 - \rho^2)(\rho^2 - (\lambda-r)^2).$$

Then we see from (16), (19) with $n=3$ and (17) that (18) is valid. □

From this lemma, one can derive the following.

LEMMA 2.2. *Assume* (6) *and let* $(\widetilde{u}(r,t), \widetilde{v}(r,t))$ *be the spherical means of the classical solution* $(u(x,t), v(x,t))$ *of* (1). *Then*

$$\begin{cases} \widetilde{u}(r,t) \geq \varepsilon J_c(\widetilde{g_1})(r,t) + L_c(|\widetilde{v}|^p)(r,t), \\ \widetilde{v}(r,t) \geq \varepsilon J_1(\widetilde{g_2})(r,t) + L_1(|\widetilde{u}|^q)(r,t) \end{cases} \tag{20}$$

holds for $(r,t) \in \mathbf{R}^2_+ = (0,\infty) \times (0,\infty)$, *where*

$$L_c(G)(r,t) = \frac{1}{2cr} \int_0^t \int_{|c(t-s)-r|}^{c(t-s)+r} \lambda G(\lambda, s) d\lambda ds.$$

2.2. Reduced systems with one variable

We begin by showing the following proposition, from which we shall derive a result on the blowup for the reduced systems with one variable (36) at the end of this subsection.

PROPOSITION 2.3. *Let* a, b, p, q, α, β, γ *and* λ *be constants such that*

$$1 < p \leq q < \infty,\ \beta \leq 0,\ \gamma > 0,\ \lambda \geq 1,\ a + p(b-1) \geq 0,\ \alpha + p\beta \leq 0 \tag{21}$$

and suppose that $K(z)$ *is a continuous function such that*

$$m := \min\{K(z) :\ 0 \leq z \leq 1\} > 0. \tag{22}$$

Then, the life span of solution $(\varphi(z), \phi(z))$ *of*

$$\begin{cases} \varphi(z) \geq 1 + \gamma\lambda^a \displaystyle\int_0^z \left(1 - e^{-\lambda(z-\zeta)}\right) e^{-\alpha\lambda\zeta} |\phi(\zeta)|^p d\zeta, \\ \phi(z) \geq \gamma\lambda^b \displaystyle\int_0^z \left(1 - e^{-\lambda(z-\zeta)}\right) K\left(\lambda(z-\zeta)\right) e^{-\beta\lambda\zeta} |\varphi(\zeta)|^q d\zeta, \end{cases} \tag{23}$$

for $z \leq 0$ *is bounded from above by a positive constant depending only on* p, q, β, γ *and* m.

To prove the proposition, we prepare a couple of lemmas. In what follows, we assume that $(\varphi(z), \phi(z))$ is a solution of (23).

LEMMA 2.4. *Assume that* (21) *and* (22) *hold. Let* $A > 0$, $0 < h \leq 1$ *and* $Z \geq 0$. *Suppose that*

$$\varphi(z) \geq A \qquad (z \geq Z). \tag{24}$$

Then there exists a positive constant C *depending only on* p, β *and* γ *such that*

$$\varphi(z) \geq CA^{pq}(z - Z - 2) \qquad (z \geq Z + 2), \tag{25}$$

$$\varphi(z) \geq CA^{pq}h^{2p+2} \qquad (z \geq Z + 2h). \tag{26}$$

Proof. First we shall show that there exists a constant $C > 0$ depending only on β, γ and m such that

$$\phi(z) \geq CB\lambda^{b-1}e^{-\beta\lambda z} \qquad (z \geq Z + h), \tag{27}$$

where we have set $B = A^q h^2$. Since $\lambda \geq 1$ and $0 < h \leq 1$, we see that $z - h/\lambda \geq Z$ for $z \geq Z + h$. Therefore we have from (23) and (24)

$$\phi(z) \geq \gamma e^{\beta} A^q \lambda^b e^{-\beta\lambda z} \int_{z-h/\lambda}^{z} \left(1 - e^{-\lambda(z-\zeta)}\right) K(\lambda(z-\zeta))d\zeta$$

for $z \geq Z+h$. Since we have $0 \leq \lambda(z-\zeta) \leq h \leq 1$ for $z-h/\lambda \leq \zeta \leq z$, we see from (22) that $K(\lambda(z-\zeta)) \geq m$ for $z - h/\lambda \leq \zeta \leq z$. Hence, (27) follows from the fact that

$$\int_{z-h/\lambda}^{z} \left(1 - e^{-\lambda(z-\zeta)}\right) d\zeta = \frac{e^{-h} - 1 + h}{\lambda} \geq \frac{h^2}{e\lambda} \qquad (0 < h \leq 1). \tag{28}$$

Next it follows from (23) and (27) that for $z \geq Z + h$

$$\begin{aligned} \varphi(z) &\geq \gamma B^p \lambda^{a+p(b-1)} \int_{Z+h}^{z} \left(1 - e^{-\lambda(z-\zeta)}\right) e^{-(\alpha+p\beta)\lambda\zeta} d\zeta \qquad (29) \\ &\geq \gamma B^p \int_{Z+h}^{z} \left(1 - e^{-\lambda(z-\zeta)}\right) d\zeta, \end{aligned}$$

by (21). Thus, (25) follows from (29) and the fact that for $z \geq Z+2$ we have

$$\int_{Z+h}^{z} \left(1 - e^{-\lambda(z-\zeta)}\right) d\zeta = z - Z - h - \frac{1}{\lambda} + \frac{1}{\lambda}e^{-\lambda(z-Z-h)} \geq z - Z - 2.$$

While (26) follows from (29) and (28), since $z - h/\lambda \geq Z + h$ for $z \geq Z + 2h$. □

LEMMA 2.5. *For any $L > 0$ there exists a constant $Z_0 = Z_0(L) > 0$ such that*

$$\varphi(z) \geq L \qquad (z \geq Z_0). \tag{30}$$

Proof. From (23) we have $\varphi(z) \geq 1$ for all $z \geq 0$. Thus it follows from Lemma 2.4 (we take $A = 1$ and $Z = 0$) that

$$\varphi(z) \geq C(z-2) \qquad (z \geq 2), \tag{31}$$

from which we see that the conclusion of the lemma is valid. □

LEMMA 2.6. *Let j be a nonnegative integer. Suppose that there exist constants A_j and Z_j such that*

$$\varphi(z) \geq A_j \qquad (z \geq Z_j). \tag{32}$$

Then there exists a constant $M > 1$ such that

$$\varphi(z) \geq A_{j+1} \qquad (z \geq Z_{j+1}), \tag{33}$$

where

$$A_{j+1} = \frac{A_j^{pq}}{M(j+1)^{4p+4}}, \quad Z_{j+1} = Z_j + \frac{2}{(j+1)^2}. \tag{34}$$

Proof. From (32) and Lemma 2.4 (we take $A = A_j$, $h = 1/(j+1)^2$ and $Z = Z_j$), we have

$$\varphi(z) \geq \frac{CA_j^{pq}}{(j+1)^{4p+4}} \qquad \left(z \geq Z_j + \frac{2}{(j+1)^2}\right).$$

Thus we obtain (33) with (34). □

LEMMA 2.7. *Let $\{A_j\}_{j=0}^{\infty}$ be the sequence defined by (34). If $A_0 > L_0 := M^\nu e^{(4p+4)m}$, then we have $\lim_{j\to\infty} A_j = \infty$. Here $\nu = 1/(pq-1)$ and $m = \sum_{k=2}^{\infty}(pq)^{-k}\log k$.*

For the proof of the lemma, see Lemma 4.6 of [16].

Proof of Proposition 2.3. Put $A_0 = L_0 + 1$. If we take $Z_0 = Z_0(A_0)$ in Lemma 2.5, we have $\varphi(z) \geq A_0$ for $z \geq Z_0$. Moreover, it follows from (34) that $Z_j = Z_0 + \sum_{k=1}^{j} 2/k^2$ for $j \geq 1$. Thus if we put $Z^* := \sup_{j\geq 1} Z_j = Z_0 + \sum_{k=1}^{\infty} 2/k^2$, we have $Z^* < \infty$. From Lemma 2.6 for any $j \geq 1$ we have $\varphi(z) \geq A_j$ for all $z \geq Z^*$. Hence, from Lemma 2.7 we see that the life span of $(\varphi(z), \phi(z))$ is less than or equal to Z^*. Since the positive constant Z^* depends only on p, q, β and γ, this completes the proof. □

COROLLARY 2.8. *Let $1 < p \leq q < \infty$, $\alpha + p\beta \leq 0$ with (2) and $0 < \varepsilon \leq 1$. Suppose that C_1, C_2 and C_3 are positive constants, and $H(y)$ is a continuous function such that*

$$\min\{H(y) : \ e^{-1} \leq y \leq 1\} > 0. \tag{35}$$

Let $T(\varepsilon)$ be the life span of solution $(U(y), V(y))$ of

$$\begin{cases} U(y) = C_1\varepsilon^p + C_2 \displaystyle\int_1^y \frac{y-\eta}{y\eta^{p(q-2)}} |V(\eta)|^p d\eta, & y \geq 1, \\ V(y) = C_3 \displaystyle\int_1^y \frac{y-\eta}{y\eta^{q(p-2)}} H\left(\frac{\eta}{y}\right) |U(\eta)|^q d\eta, & y \geq 1. \end{cases} \tag{36}$$

Then, there exists a positive constant C_0, independent of ε, such that

$$T(\varepsilon) \leq \exp\left(C_0 \varepsilon^{-p(pq-1)}\right) \qquad \text{if} \qquad \alpha + p\beta \geq 0, \tag{37}$$

$$T(\varepsilon) \leq C_0 \varepsilon^{-p(pq-1)/(\alpha+p\beta)} \qquad \text{if} \qquad \alpha + p\beta > 0. \tag{38}$$

Proof. First, we show the estimate (37). For the solution $(U(y), V(y))$ of (36), we put

$$\varphi(z) = (C_1\varepsilon^p)^{-1} U(e^{\lambda z}), \ \phi(z) = (C_1\varepsilon^q)^{-1} V(e^{\lambda z}), \quad \lambda = \varepsilon^{-p(pq-1)}. \tag{39}$$

Then, we have $\lambda \geq 1$ for $0 < \varepsilon \leq 1$, and a direct calculation shows that $(\varphi(z), \phi(z))$ satisfies (23) with (2), $K(z) = H(e^{-z})$ and

$$a = 1 - \frac{p(q-1)}{p(pq-1)}, \ b = 1 - \frac{q(p-1)}{p(pq-1)}, \ \gamma = \min\{C_1^{p-1}C_2, \ C_1^{q-1}C_3\}.$$

Then, we can easily check that the assumptions (21) and (22) in Proposition 2.3 are satisfied. Hence, from Proposition 2.3 and (39), we obtain the estimate (37).

Next, we show the estimate (38). For the solution $(U(y), V(y))$ of (36), we put

$$\varphi(z) = \varepsilon^{-p} U(\varepsilon^{-\lambda} z), \quad \phi(z) = \varepsilon^{-\mu} V(\varepsilon^{-\lambda} z), \tag{40}$$

$$\lambda = \frac{p(pq-1)}{\alpha + p\beta}, \ \mu = pq - \beta\lambda. \tag{41}$$

Then, we see that $(\varphi(z), \phi(z))$ satisfies

$$\begin{cases} \varphi(z) = C_1 + C_2 \varepsilon^{p(\mu-1)-\alpha\lambda} \displaystyle\int_{\varepsilon^\lambda}^{z} \frac{z-\zeta}{z\zeta^{p(q-2)}} |\phi(\zeta)|^p d\zeta, & z \geq \varepsilon^\lambda, \\ \phi(z) = C_3 \varepsilon^{pq-\mu-\beta\lambda} \displaystyle\int_{\varepsilon^\lambda}^{z} \frac{z-\zeta}{z\zeta^{q(p-2)}} H\left(\frac{\zeta}{z}\right) |\varphi(\zeta)|^q d\zeta, & z \geq \varepsilon^\lambda. \end{cases} \tag{42}$$

By (41), we have $p(\mu - 1) - \alpha\lambda = 0$ and $pq - \mu - \beta\lambda = 0$, and since $\lambda > 0$, we have $\varepsilon^\lambda \leq 1$ for $0 < \varepsilon \leq 1$. Therefore, by the estimate (37), we see that the life span of the solution $(\varphi(z), \phi(z))$ of (42) is dominated by a positive constant which is independent of ε. Hence, the estimate (38) follows from (40) and (41). □

2.3. An application

In this subsection, we prove (7) and (9) in Theorem 1.1 when $c \geq 1$, by applying Corollary 2.8 together with Proposition 2.12 below. For the case $0 < c < 1$, see Section 4 of [15], and for the proof of (8), see [16]. Throughout this subsection, we always assume that $c \geq 1$ and $1 < p \leq q < \infty$, and put

$$\Sigma_1 = \{(r,t) \in \mathbf{R}_+^2 : \ t - r \geq 1\}, \quad \mathbf{R}_+^2 = (0,\infty) \times (0,\infty).$$

We note that the continuity of $\widetilde{g_2}$ and the assumption $g_2(0) > 0$ in Theorem 1.1 imply that there exists a constant $\kappa \in (0,1]$ such that

$$\widetilde{g_2}(r) > 0 \ \text{ holds for } \ r \in [0,\kappa]. \tag{43}$$

For $(r,t) \in \Sigma_1$ and such κ as above, we define $D_j(r,t) \in \mathbf{R}^2_+$

$$D_1(r,t) = \{(\lambda,s):\ ct - r \le cs + \lambda \le ct + r,\ |s-\lambda| \le \kappa/2\},$$
$$D_2(r,t) = \{(\lambda,s):\ ct - r \le cs + \lambda \le ct + r,\ 1 \le s-\lambda \le t-r\},$$
$$D_3(r,t) = \{(\lambda,s):\ t - r \le s + \lambda \le t + r,\ 1 \le s-\lambda \le t-r\}.$$

Moreover, we define

$$L_{(j)}(G)(r,t) = \frac{1}{2cr}\iint_{D_j(r,t)} \lambda G(\lambda,s)\,d\lambda ds \qquad (j = 1,\ 2,\ 3).$$

Then we see from Lemma 2.2 and (6) that

$$\begin{cases} \widetilde{u}(r,t) \ge L_{(1)}(|\widetilde{v}|^p)(r,t) + L_{(2)}(|\widetilde{v}|^p)(r,t), \\ \widetilde{v}(r,t) \ge \varepsilon J_1(\widetilde{g_2})(r,t) + cL_{(3)}(|\widetilde{u}|^q)(r,t) \end{cases} \tag{44}$$

holds for $(r,t) \in \Sigma_1$, where $(\widetilde{u}(r,t), \widetilde{v}(r,t))$ be the spherical means of the classical solution of (1). We prepare a couple of lemmas. For the proof, see Section 3 in [15]

LEMMA 2.9. *There exists a constant $C_{11} > 0$ such that*

$$L_{(1)}(|\widetilde{v}|^p)(r,t) \ge \frac{2C_{11}\varepsilon^p}{(t+r)(ct-r)^{p-2}} \tag{45}$$

holds for $(r,t) \in \Sigma_1$.

For any continuous function $f(\eta)$ of $\eta \in [0,\infty)$, we put

$$R_q(f)(r,t) = \frac{f(t-r)}{(t+r)(t-r)^{q-2}}, \quad R_p(f)(r,t) = \frac{f(t-r)}{(t+r)(ct-r)^{p-2}}.$$

LEMMA 2.10. *There exists a constant $C_{12} > 0$ such that*

$$L_{(2)}(|R_q(f)|^p)(r,t)] \ge \frac{C_{12}}{(t+r)(ct-r)^{p-2}} \int_1^{t-r} \frac{t-r-\eta}{(t-r)\eta^{p(q-2)}} |f(\eta)|^p d\eta \tag{46}$$

holds for any continuous function f and $(r,t) \in \Sigma_1$.

LEMMA 2.11. *There exists a constant $C_{13} > 0$ such that*

$$cL_{(3)}(|R_p(f)|^q)(r,t) \tag{47}$$
$$\geq \frac{C_{13}}{(t+r)(t-r)^{q-2}} \int_1^{t-r} \frac{(t-r-\eta)|f(\eta)|^q}{(t-r)\{(c-1)(t-r)+\eta\}^{q(p-2)}} d\eta$$

holds for any continuous function f and $(r,t) \in \Sigma_1$.

From Lemmas 2.9–2.11, we obtain the following proposition.

PROPOSITION 2.12. *Suppose that $c \geq 1$ and $1 < p \leq q < \infty$. Let C_{11}, C_{12} and C_{13} be positive constants defined in Lemmas 2.9–2.11, and let $(\widetilde{u}(r,t), \widetilde{v}(r,t))$ be the spherical means of the classical solution of (1) and $(U(y), V(y))$ be the solution of*

$$\begin{cases} U(y) = C_{11}\varepsilon^p + C_{12} \displaystyle\int_1^y \frac{y-\eta}{y\eta^{p(q-2)}} |V(\eta)|^p d\eta, & y \geq 1, \\ V(y) = C_{13} \displaystyle\int_1^y \frac{y-\eta}{y\eta^{q(p-2)}} H_1\left(\frac{\eta}{y}\right) |U(\eta)|^q d\eta, & y \geq 1, \end{cases} \tag{48}$$

where

$$H_1(y) = \left(\frac{y}{y+(c-1)}\right)^{q(p-2)}.$$

Then,

$$\widetilde{u}(r,t) \geq R_p(U)(r,t) = \frac{U(t-r)}{(t+r)(ct-r)^{p-2}} \tag{49}$$

holds for $(r,t) \in \Sigma_1$ as long as $(\widetilde{u}(r,t), \widetilde{v}(r,t))$ and $(U(y), V(y))$ exist.

Proof. In this proof, we always assume that $(r,t) \in \Sigma_1$. First, it follows from (48) that for $t - r = 1$

$$R_p(U)(r,t) = \frac{U(1)}{(t+r)(ct-r)^{p-2}} = \frac{C_{11}\varepsilon^p}{(t+r)(ct-r)^{p-2}}.$$

Thus, from (44) and Lemma 2.9,

$$\widetilde{u}(r,t) \geq \frac{2C_{11}\varepsilon^p}{(t+r)(ct-r)^{p-2}} > R_p(U)(r,t)$$

holds for $t-r=1$. Therefore, by the continuity of $\widetilde{u}$ and $R_p(U)$, there exists $N>1$ such that $\widetilde{u}(r,t)>R_p(U)(r,t)$ holds for $(r,t)\in\Omega_1(N)$, where we have set

$$\Omega_1(N)=\{(r,t)\in\Sigma_1:\ ct-r\le cN\}.$$

Suppose that

$$N_1:=\sup\{N>1:\ \widetilde{u}(r,t)>R_p(U)(r,t)\ \text{holds for}\ (r,t)\in\Omega_1(N)\}.$$

is a finite number. Then, we have

$$\min\{\widetilde{u}(r,t)-R_p(U)(r,t):\ (r,t)\in\Omega_1(N_1)\}=0. \tag{50}$$

From (44) and Lemma 2.11, we have for $(r,t)\in\Omega_1(N_1)$

$$\begin{aligned}\widetilde{v}(r,t) &\ge cL_{(3)}(|R_p(U)|^q)(r,t)\\ &\ge \frac{C_{13}}{(t+r)(t-r)^{q-2}}\int_1^{t-r}\frac{t-r-\eta}{(t-r)\eta^{q(p-2)}}H_1\left(\frac{\eta}{t-r}\right)|U(\eta)|^q d\eta\\ &= \frac{V(t-r)}{(t+r)(t-r)^{q-2}}=R_q(V)(r,t).\end{aligned}$$

Thus, from (44) and Lemmas 2.9 and 2.10, we have for $(r,t)\in\Omega_1(N_1)$

$$\begin{aligned}\widetilde{u}(r,t) &\ge L_{(1)}(|\widetilde{v}|^p)(r,t)+L_{(2)}(|R_q(V)|^p)(r,t)\\ &\ge \frac{2C_{11}\varepsilon^p}{(t+r)(ct-r)^{p-2}}+\\ &\quad +\frac{C_{12}}{(t+r)(ct-r)^{p-2}}\int_1^{t-r}\frac{t-r-\eta}{(t-r)\eta^{p(q-2)}}|V(\eta)|^p d\eta\\ &= \frac{C_{11}\varepsilon^p}{(t+r)(ct-r)^{p-2}}+\frac{U(t-r)}{(t+r)(ct-r)^{p-2}}>R_p(U)(r,t),\end{aligned}$$

which contradicts to (50). Therefore, we conclude that $N_1=+\infty$. Hence, the proof is completed. □

Proof of (7) and (9) when $c\ge 1$. Since $H(y)=H_1(y)$ satisfies the assumption (35) in Corollary 2.8, (7) and (9) for the case $c\ge 1$ follows from Corollary 2.8 and Proposition 2.12 □

3. Proof of Theorem 1.3

To prove Theorem 1.3, we shall adapt a blowup criterion on the Cauchy problem to (3) established in Lindblad [18] among other things, which provide us a simple way to get the lower bounds of the life span. The blowup criterion asserts that the solution of the above problem does not blow up as long as $E*|u|^p$ is bounded, where E denotes a fundamental solution of the wave equation. (See also Alinhac [4], Chapter III, Section 3). Subsection 3.2 is devoted to derive basic estimates, from which we establish a priori estimates in Subsections 3.3.

3.1. Blowup criterion

In this subsection, we shall extend the blowup criterion established in [18] in such a way to enable us to apply it for the system of semilinear wave equations (1) when the propagation speeds may be different.

We denote by $\tilde{c}$ the faster propagation speeds in the system, namely,

$$\tilde{c} = \max\{c, 1\},$$

where c is the number in (1), and for $(x_0, t_0) \in \mathbf{R}^3 \times \mathbf{R}$ we denote by $C(x_0, t_0)$ a backward characteristic cone connected with $\tilde{c}$, that is,

$$C(x_0, t_0) = \{(x, t) \in \mathbf{R}^3 \times \mathbf{R} : \ \tilde{c}(t - t_0) < -|x - x_0|\}. \tag{51}$$

We call an open set $\Omega \subset \mathbf{R}^3 \times \mathbf{R}$ an "*influence domain*" if $(x, t) \in \Omega$ implies $\overline{C(x, t)} \subset \Omega$.

Next we introduce a definition of "*weak solution*" for the Cauchy problem of (1).

Definition 3.1. *We say that (u, v) is a weak solution for (1) in an influence domain Ω, if they belongs to*

$$\mathcal{W}(\Omega) = \{(u, v) \in L^q_{loc}(\Omega) \times L^p_{loc}(\Omega) : u(x, t) = v(x, t) = 0 \quad \text{for } t < 0 \text{ and } E_c * |v|^p,\ E_1 * |u|^q \in L^\infty_{loc}(\Omega)\}, \tag{52}$$

where p, q are the number in (1) and satisfy

$$u = \widetilde{u_0} + E_c * |v|^p, \quad v = \widetilde{v_0} + E_1 * |u|^q \quad \text{in } \Omega, \tag{53}$$

where the tilde denotes extension by 0 *for* $\mathbf{R}^n \times (-\infty, 0)$*, the star the convolution, and* E_a *the fundamental solution of the wave equation defined by*

$$E_a(x,t) = (2\pi a)^{-1}\delta((at)^2 - |x|^2)H(t). \tag{54}$$

Here $\delta(s)$ *is the delta function and* $H(s)$ *is the Heaviside function. Besides,*

$$u_0 = \varepsilon\partial_t E_c * f_1 + \varepsilon E_c * g_1, \quad v_0 = \varepsilon\partial_t E_1 * f_2 + \varepsilon E_1 * g_2,$$

where f_j *and* g_j *are the initial data in (1).*

We summarize the properties of u_0 and v_0 in the following. For the proof, see [17].

LEMMA 3.2. *Assume that* f_j *and* g_j *satisfy* (10). *Then* u_0, $v_0 \in C^2(\mathbf{R}^3 \times [0,\infty))$ *and we have*

$$|u_0(x,t)| \le A\varepsilon(1+t+r)^{-1}(1+|ct-r|)^{-1}, \tag{55}$$

$$|v_0(x,t)| \le A\varepsilon(1+t+r)^{-1}(1+|t-r|)^{-1}, \tag{56}$$

where $r = |x|$ *and* A *is a positive constant depending on* p*,* q*,* c *and* R*. Moreover, if* $2 < p \le q$ *holds, then we have for* $(x,t) \in \mathbf{R}^3 \times \mathbf{R}$

$$|E_c * |\widetilde{v_0}|^p(x,t)| \le M_0\varepsilon^p(1+t+r)^{-1}(1+|ct-r|)^{-p^*}, \tag{57}$$

$$|E_1 * |\widetilde{u_0}|^q(x,t)| \le M_0\varepsilon^q(1+t+r)^{-1}(1+|t-r|)^{-q^*}. \tag{58}$$

Here M_0 *is a positive constant depending on* p*,* q*,* c *and* A*.*

The following local existence and uniqueness theorem holds.

THEOREM 3.3. *Suppose that the assumptions of Theorem 1.1 are fulfilled. Assume that* $T > 0$ *satisfy*

$$2^{pq-1}M_0^{p-1}\varepsilon^{p(p-1)}T^2 \le 1, \quad 2^{pq-1}M_0^{q-1}\varepsilon^{p(p-1)}T^2 \le 1, \tag{59}$$

where M_0 *is the constant in Lemma 3.2. Then we have a unique weak solution* (u,v) *for* (1) *in* $\mathbf{R}^3 \times (-\infty, T)$ *verifying*

$$E_c * |v|^p \le 2^p M_0\varepsilon^p, \quad E_1 * |u|^q \le 2^q M_0 \varepsilon^p. \tag{60}$$

and

$$u(x,t) = v(x,t) = 0 \quad \textit{for } |x| \ge \tilde{c}t + R, \ 0 \le t < T, \tag{61}$$

where R *is the number in* (10).

For the proof, see Section 2 of [17]. Theorem 3.3 leads us to the following definition of "*maximal influence domain*".

DEFINITION 3.4. *Let Ω_{max} be the union of all influence domains Ω containing $\mathbf{R}^3 \times (-\infty, 0]$ such that there is a unique weak solution (u, v) for (1) in Ω. Then Ω_{max} is the unique maximal domain with this property.*

The following theorem gives us "*blowup criterion*" for the system (1) and "*additional smoothness*" for the solution. Since the proof of the theorem is analogous to that of Theorems 3.6 and 3.7 in [18], we omit it.

THEOREM 3.5. *Suppose that the assumptions of Theorem 1.3 are fulfilled. If $(x_0, t_0) \in \partial\Omega_{max}$ and*

$$\overline{C(x_0, t_0)} \subset \{(x_0, t_0)\} \cup \Omega_{max}, \tag{62}$$

then either $E_c * |v|^p$ or $E_1 * |u|^q$ is unbounded in $\Omega_{max} \cap B(x_0, t_0 : \rho)$ for any $\rho > 0$. Here $B(x, t : \rho)$ stands for the open ball with radius ρ centered at (x, t).

Moreover, if (u, v) is a weak solution for (1) in Ω_{max}, then

$$u, v \in C^2(\Omega_{max} \cap (\mathbf{R}^3 \times [0, \infty))). \tag{63}$$

Now we are in a position to mention our strategy to prove Theorem 1.1. We denote by $T_0(\varepsilon)$ the supremum of all $T > 0$ such that there is a unique weak solution (u, v) for(1) in $\mathbf{R}^3 \times (-\infty, T)$ satisfying for $(x, t) \in \mathbf{R}^3 \times [0, T)$

$$E_c * |v|^p(x, t) \le 2^{p+1} M \varepsilon^p (1 + r + t)^{-1} (1 + |ct - r|)^{-p^*}, \tag{64}$$

where $M = (1 + R + 2\tilde{c})^{p-1} M_0$ with M_0 the constant in Lemma 3.2 and $r = |x|$. Then we have

$$0 < T_0(\varepsilon) \le T^*(\varepsilon). \tag{65}$$

Indeed, by virtue of (60) and (61), we see that $T_0(\varepsilon) > 0$. While, it is clear that $T_0(\varepsilon) \le T^*(\varepsilon)$, due to (63). Therefore, our task becomes to show (11) through (14) with $T^*(\varepsilon)$ replaced by $T_0(\varepsilon)$. Thanks to the blowup criterion in Theorem 3.5, those estimates follow from suitable a priori estimates. (For the details, see Subsection 3.3 below). Besides, when we prove such a priori estimates, we may assume that the solution u, v of (53) are of class C^2, due to (63).

3.2. Basic estimates

In this subsection, we prepare basic estimates that will be needed to establish a priori estimates in Subsection 3.3. First of all, we introduce the following integral operator

$$L_a(F)(x,t) = \frac{1}{4\pi}\int_0^t (t-s)ds \int_{|\omega|=1} F(x+a(t-s)\omega, s)dS_\omega, \quad (66)$$

where $a > 0$, $(x,t) \in \mathbf{R}^3 \times [0,T)$ and $F \in C(\mathbf{R}^3 \times [0,T))$. Notice that

$$E_a * \widetilde{F}(x,t) = L_a(F)(x,t) \quad \text{for } (x,t) \in \mathbf{R}^3 \times [0,T), \quad (67)$$

when $F \in C^2(\mathbf{R}^3 \times [0,T))$ and $F(x,t) = 0$ for $|x| \geq \tilde{c}t + R$. In addition, we shall use the following notations:

$$\begin{aligned}
\Phi_\kappa(r,t) &= (1+t+r)^{-1-\kappa} \ (\kappa < 0), \\
\Phi_\kappa(r,t) &= (1+t+r)^{-1}(1+|t-r|)^{-\kappa} \ (\kappa > 0), \\
\Phi_0(r,t) &= (1+t+r)^{-1}\left(1+\log(\frac{1+t+r}{1+|t-r|})\right)
\end{aligned} \quad (68)$$

and

$$\Psi_\mu(t) = 1 \ (\mu > 0), \quad \Psi_\mu(t) = (1+t)^{-\mu} \ (\mu < 0), \quad (69)$$
$$\Psi_0(t) = 1 + \log(1+t).$$

Proposition 3.6. *Let $c > 0$, $c' > 0$, $T > 0$ and $F \in C(\mathbf{R}^3 \times [0,T))$.*
(i) *If $\kappa \in \mathbf{R}$ and $\mu \in \mathbf{R}$, we have for $(x,t) \in \mathbf{R}^3 \times [0,T)$*

$$\begin{aligned}
&|L_c(F)(x,t)|\{\Phi_\kappa(r,ct)\}^{-1} \quad (70)\\
&\leq C\Psi_\mu(t+r)\,\|\,|y|(1+|y|+s)^{1+\kappa}(1+|c's-|y||)^{1+\mu}|F(y,s)|\,\|_{L^\infty}
\end{aligned}$$

(ii) *If $\kappa \in \mathbf{R}$ and $\mu > 0$, we have for $(x,t) \in \mathbf{R}^3 \times [0,T)$*

$$\begin{aligned}
&|L_c(F)(x,t)|\{\Phi_\kappa(r,ct)\}^{-1} \quad (71)\\
&\leq C\,\|\,|y|(1+|y|+s)^{1+\kappa+\mu}(1+|c's-|y||)^{1-\mu}|F(y,s)|\,\|_{L^\infty}
\end{aligned}$$

(iii) *If $\kappa \in \mathbf{R}$, $\mu \geq 0$ and $\delta \leq 0$, we have for $(x,t) \in \mathbf{R}^3 \times [0,T)$*

$$\begin{aligned}
&|L_c(F)(x,t)|\{\Phi_\kappa(r,ct)\}^{-1}(1+r+t)^\delta \quad (72)\\
&\leq C\,\|\,|y|(1+|y|+s)^{2+\kappa+\delta}\left(1+\log(\frac{1+c's+|y|}{1+|c's-|y||})\right)^{-\mu}|F(y,s)|\,\|_{L^\infty}
\end{aligned}$$

Here $\|\cdot\|_{L^\infty}=\|\cdot\|_{L^\infty(\mathbf{R}^3\times[0,T))}$, $r=|x|$ *and* C *is a positive constant depending only on* μ, κ, δ, c *and* c'.

For the proof, see Section 3 of [17].

Corollary 3.7. *Let* $A>0$, $c>0$, $c'>0$, $2<p\le q$ *and* $w\in C(\mathbf{R}^3\times[0,T))$.

(i) *If* $w(x,t)$ *satisfies*

$$|w(x,t)|\le A(1+t+r)^{-1}(1+|ct-r|)^{-p^*} \tag{73}$$

for $(x,t)\in\mathbf{R}^3\times[0,T)$, *and if* κ *satisfies either*

$$\kappa\le q^* \quad \textit{and} \quad \kappa<q^*+\beta \tag{74}$$

or

$$\kappa<q^* \quad \textit{and} \quad \kappa\le q^*+\beta \tag{75}$$

with $\beta=qp^*-1$, *then we have*

$$|L_{c'}(|w|^q)(x,t)|\le C_1A^q\Phi_\kappa(r,c't) \quad \textit{for } (x,t)\in\mathbf{R}^3\times[0,T). \tag{76}$$

where $\Phi_\kappa(r,t)$ *was defined in* (68).

(ii) *Suppsoe that* $w(x,t)$ *satisfies*

$$|w(x,t)|\le A\Phi_\nu(r,c't) \quad \textit{for } (x,t)\in\mathbf{R}^3\times[0,T). \tag{77}$$

When $\nu>0$, *we have for* $(x,t)\in\mathbf{R}^3\times[0,T)$ *with* $|x|\le mt+R$

$$|L_c(|w|^p)(x,t)|(1+t+r)(1+|ct-r|)^{p^*}\le C_1A^p\Psi_{p\nu-1}(T), \tag{78}$$

where $\Psi_\mu(t)$ *was defined by* (69) *and* m, $R>0$.

While $\nu\le 0$, *then we have for* $(x,t)\in\mathbf{R}^3\times[0,T)$ *with* $|x|\le mt+R$

$$|L_c(|w|^p)(x,t)|(1+t+r)(1+|ct-r|)^{p^*}\le C_1A^p(1+T)^{1-p\nu}. \tag{79}$$

Here $C_1=C_1(c,c',p,q,\kappa,\nu,m,R)$ *is a positive constant independent of* A *and* T.

Proof. First we prove (76). When (74) holds, we can choose μ such that

$$0 < \mu < q^* + \beta - \kappa. \tag{80}$$

Notice that by (73), $\kappa \le q^*$ and (80), we have

$$|y|(1 + |y| + s)^{1+\kappa}(1 + |cs - |y||)^{1+\mu}|w(y, s)|^q \le CA^q.$$

Therefore, (70) with $F = |w|^q$ and $\mu > 0$ yields (76).

On the other hand, when (75) holds, we can choose μ such that

$$0 < \mu < q^* - \kappa. \tag{81}$$

Therefore, (76) follows from (71) with $F = |w|^q$, (73), (81) and $q^* + \beta - \kappa \ge 0$.

Next we prove (78). Applying (70) as $F = |w|^p$, $\kappa = p^* > 0$ and $\mu = p\nu - 1$, and using (77), we get (78), since $r \le mT + R$.

Finally we prove (79). We use (72) as $F = |w|^p$, $\kappa = p^* > 0$ and $\delta = p\nu - 1 \le 0$. Besides, we put $\mu = p$ when $\nu = 0$, $\mu = 0$ when $\nu < 0$. Then (77) and (72) yields (79). The proof is complete. □

3.3. A priori estimates

Let (u, v) be a weak solution for (1) in $\mathbf{R}^3 \times (-\infty, T)$ such that u, $v \in C^2(\mathbf{R}^3 \times [0, T))$. We shall look for upper bounds of

$$w_1 \equiv L_c(|v|^p) = u - u_0, \quad w_2 \equiv L_1(|u|^q) = v - v_0, \tag{82}$$

provided

$$|w_1(x, t)| \le 2^{p+1} M\varepsilon^p (1 + t + r)^{-1}(1 + |ct - r|)^{-p^*} \tag{83}$$

holds for $(x, t) \in \mathbf{R}^3 \times [0, T)$, where M is the number in (64). It is easy to see from (57) and (58) that

$$\begin{aligned} |w_1(x, t)| &\le 2^{p-1}\{M\varepsilon^p(1 + t + r)^{-1}(1 + |ct - r|)^{-p^*} \qquad (84)\\ &\quad + |L_c(|w_2|^p)(x, t)|\}, \\ |w_2(x, t)| &\le 2^{q-1}\{M\varepsilon^q(1 + t + r)^{-1}(1 + |t - r|)^{-q^*} \qquad (85)\\ &\quad + |L_1(|w_1|^q)(x, t)|\}, \end{aligned}$$

since $|a+b|^p \leq 2^{p-1}(|a|^p+|b|^p)$ for a, $b \in \mathbf{R}$ and $p>1$. Moreover (61) implies

$$\text{supp} w_1 \cup \text{supp} w_2 \subset \{|x| \leq \tilde{c}t + R\}, \tag{86}$$

where $\tilde{c} = \max\{c, 1\}$. We shall deal with only the case where $\alpha + p\beta > 0$ and $2 < p \leq q$. For the other cases, see [17].

First we show that there is a number κ verifying

$$0 < \kappa \leq q^*, \quad 1/p < \kappa < q^* + \beta. \tag{87}$$

Indeed, $\alpha + p\beta > 0$ implies $q^* + \beta > 1/p$. Moreover, since $\alpha \geq \beta$ for $p \leq q$, we see that $\alpha > 0$ when $\alpha + p\beta > 0$ and $1 < p \leq q$. Therefore, we have $1/p < q^*$. Hence, we can find κ verifying (87). For such $\kappa > 0$ we have

Proposition 3.8. *Assume that $\alpha + p\beta > 0$, $2 < p \leq q$ and that κ satisfies* (87). *Let w_1 and w_2 be as in* (82) *and suppose that* (83) *holds. Then there is a number $\varepsilon_0 = \varepsilon_0(p, q, c, M)$ such that we have for $0 < \varepsilon \leq \varepsilon_0$ and $(x, t) \in \mathbf{R}^3 \times [0, T)$*

$$|w_2(x,t)| \leq 2^q M \varepsilon^q (1+t+r)^{-1}(1+|t-r|)^{-\kappa}, \tag{88}$$

$$|w_1(x,t)| \leq 2^p M \varepsilon^p (1+t+r)^{-1}(1+|ct-r|)^{-p^*}. \tag{89}$$

Proof. First we take ε_0 $(0 < \varepsilon_0 \leq 1)$ so small that

$$2^{q(p+1)} C_1 M^q \varepsilon_0^{q(p-1)} \leq M, \quad 2^{pq} C_1 M^p \varepsilon_0^{p(q-1)} \leq M, \tag{90}$$

where C_1 is the constant in Corollary 3.7. Since κ satisfies (74) by (87), it follows from (83) and (76) with $c' = 1$ that

$$|L_1(|w_1|^q)(x,t)|(1+r+t)(1+|t-r|)^{\kappa} \leq C_1(2^{p+1} M \varepsilon^p)^q.$$

Using (90), (85) and $\kappa \leq q^*$, we get (88) for $0 < \varepsilon \leq \varepsilon_0$.

Moreover, since $p\kappa - 1 > 0$ by (87), it follows from (88) and (78) that

$$|L_c(|w_2|^p)(x,t)|(1+r+t)(1+|ct-r|)^{p^*} \leq C_1(2^q M \varepsilon^q)^p.$$

By (90) and (84), we obtain (89). This completes the proof. □

Proof of Theoem 1.3 when $\alpha + p\beta > 0$: As we have discussed at the end of Subsection 2.1, it suffices to show that (11) through (14) with $T^*(\varepsilon) = T_0(\varepsilon)$ hold. By the definition of $T_0(\varepsilon)$, (83) holds for $T = T_0(\varepsilon)$. We consider only the case where $\alpha + p\beta > 0$.

Suppose that $T_0(\varepsilon) < +\infty$. We see from (88) that $w_2 = E_1 * |u|^q$ is bounded in $\mathbf{R}^3 \times [0, T_0(\varepsilon))$. Hence, by Theorem 3.5 we see that

$$(\mathbf{R}^3 \times [0, T_0(\varepsilon)]) \cap \partial\Omega_{max} = \emptyset. \tag{91}$$

While from (89), we get for $(x, t) \in \mathbf{R}^3 \times [0, T_0(\varepsilon))$

$$|E_c * |v|^p(x,t)| \leq 2^p M \varepsilon^p (1 + r + t)^{-1}(1 + |ct - r|)^{-p^*}, \tag{92}$$

which is a sharper estimate than (64). We thus have a contradiction to the definition of $T_0(\varepsilon)$, hence $T_0(\varepsilon) = +\infty$. This completes the proof of Theorem 1.1 when $\alpha + p\beta > 0$. □

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