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The Cartesian Closed Topological Hull of the Category of (Quasi-)Uniform Spaces (Revisited)

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Summary. - *This paper provides a concrete description of the cartesian closed topological hull of* q**Unif**, *the category of quasi-uniform spaces and uniformly continuous maps, inside* q**(S)ULim**, *the category of quasi-(semi-)uniform limit spaces and uniformly continuous maps, which also allows to derive a similar and new description of the CCT hull of* **Unif** *inside* **(S)ULim**. *In both cases, the objects of the CCT hull are (quasi-)(semi-)uniform limit spaces whose collection of filters satisfies some natural closure condition, related to the* (q)**Unif**-*bireflection of the space in question.*

1. Introduction

Although being topological is a nice property for a (concrete) category, it may be desirable and useful to have more properties, such as being cartesian closed topological (CCT). However, many categories are not cartesian closed, which has inspired a theory of CCT extensions of such (failing) categories, where the least such CCT extension of a given concrete category, the CCT hull of a category, is especially interesting.

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For instance, in [2], Adámek and Reiterman constructed the CCT hull of **Unif**, the category of uniform spaces (and uniformly continuous maps) and called it **BUnif**, the category of bornological uniform spaces, the objects of which are uniform spaces endowed with a bornology, naturally related to the uniformity.

In [3] (Alderton and Schwarz) and (independently) in [4] (Behling), a concrete isomorphism was used to "transfer" the description mentioned previously into **ULim**, the category of uniform limit spaces (and uniformly continuous maps), thereby providing a description of the CCT hull of **Unif** as a subcategory of **ULim**.

The category **Unif** can be generalized in a number of ways. For instance, one can remove symmetry to obtain q**Unif**, the category of quasi-uniform spaces (and uniformly continuous maps), or one can introduce "approach" aspects to obtain (q)**AUnif**, the category of (quasi-)approach uniform spaces (and uniform contractions) (see [20] or [13]), in the "same" way R. Lowen obtained approach spaces from topological spaces (see e.g. Lowen [12]). However, as **Unif** fails to be cartesian closed, all these resulting categories (being nice extensions of **Unif**) fail to be cartesian closed as well, which raises questions regarding their CCT hulls.

In particular, in [14], the author describes the CCT hulls of **AUnif** and q**AUnif**, which is interesting to be noted here, as it is a "restriction" of those results to a "classical = non-quantified" setting that inspired what is presented here (but to be self-contained in the sequel, we shall not make explicit use of these approach aspects).

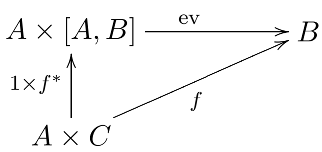
It is the purpose of this paper to consider and describe the CCT hull of q**Unif** (as well as the CCT hull of **Unif**) as a subcategory of q**SULim**, the category of quasi-semi-uniform limit spaces (and uniformly continuous maps), which, in [4], was shown to be a topological universe extending q**Unif** (and **Unif**). Several characterizations of these hulls will be provided which, in case of the CCT hull of **Unif**, add to those mentioned earlier. In case of the CCT hull of q**Unif**, some of the characterizations are quasi-analogues of the descriptions of Adámek and Reiterman or of Alderton and Schwarz or Behling, one of them being considered earlier by Behling in [4] (as noted in hindsight by the author), where the category determined by these properties is shown here to be the CCT hull of q**Unif**, the "symmetric" version of which is the CCT hull of **Unif** (as could be expected), which makes these CCT hulls nicely related.

2. Preliminaries

A *topological* construct will stand for a concrete category over **Set** which is a *well-fibred topological c-construct* in the sense of [1], i.e. each structured source has an initial lift, every set carries only a set of structures and each constant map (or empty map) between two objects is a morphism. Also recall that a construct **A** is *CCT (cartesian closed topological)* if **A** is a topological construct which has *canonical function spaces*, i.e. for every pair (A, B) of **A**-objects the set $\hom(A, B)$ can be supplied with the structure of an **A**-object, denoted by $[A, B]$, such that

(a) the evaluation map $\mathrm{ev} : A \times [A, B] \longrightarrow B$ is an **A**-morphism,

(b) for each **A**-object C and **A**-morphism $f : A \times C \longrightarrow B$, the map $f^* : C \longrightarrow [A, B]$ defined by $f^*(c)(a) = f(a, c)$ is an **A**-morphism (f^* is called the *transpose* of f). Note that given $f : A \times C \longrightarrow B$, the transpose $f^* : C \longrightarrow [A, B]$ is the map which makes the following diagram commute:



In general, categorical concepts and terminology used in this paper (and possibly not recalled here), in particular regarding categorical topology, can be found in [1] and [16]. Furthermore, a functor shall always be assumed to be concrete (unless this is clearly not the case from its definition) and subcategories to be full and isomorphism-closed.

The *CCT hull* of a construct **A** (shortly denoted by CCTH(**A**)) (if it exists) is defined as the smallest CCT construct **B** in which **A** is finally dense (see [9]), where **A** is finally dense in **B** if each **B**-object is a final lift of some structured sink in **A**. Also, from [9] recall that given a CCT construct $\mathbf{C}$ in which $\mathbf{A}$ is finally dense, the CCT hull of $\mathbf{A}$ is the full subconstruct of $\mathbf{C}$ determined by

$$\begin{aligned}\mathrm{CCTH}(\mathbf{A}) :=\{C \in \mathbf{C} \mid\ & \text{there exists an initial source} \\ & (f_i : C \longrightarrow [A_i, B_i])_{i\in I} \text{ where } \forall i \in I : A_i, B_i \in \mathbf{A}\}.\end{aligned}$$

In short, the CCT hull of $\mathbf{A}$ is the initial hull in $\mathbf{C}$ of the power-objects of $\mathbf{A}$-objects.

A more recent survey of such properties and hull concepts can be found in [8] and [18].

First, some necessities regarding uniform spaces and generalizations thereof need to be recalled, where only what is required in the sequel shall be recalled here. For more (background) information the reader is referred to (depending on the topic) e.g. Császár [6], Čech [19], Cook and Fisher [5], Wyler [21], Fletcher and Lindgren [7], Künzi [11], [10], Behling [4] and Preuß [17].

Given a set X, $\mathbf{F}(X)$ stands for the set of all filters on X; if $\mathcal{F} \in \mathbf{F}(X)$, then $\mathbf{U}(\mathcal{F})$ stands for the set of all ultrafilters on X finer than $\mathcal{F}$. In particular, $\mathbf{U}(X) := \mathbf{U}(\{X\})$ stands for the set of all ultrafilters on X. Given $A \subset X$, we recall that $\operatorname{stack} A := \{B \subset X \mid A \subset B\}$, and if A consists of a single point a, we also denote $\dot{a} := \operatorname{stack} a := \operatorname{stack} A$.

If $\mathcal{F} \in \mathbf{F}(X^2)$, then $\mathcal{F}^{-1}$ denotes the filter generated by $\{F^{-1} \mid F \in \mathcal{F}\}$, where, given $F \subset X^2$, we put $F^{-1} := \{(y, x) \mid (x, y) \in F\}$. If $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X^2)$, then $\mathcal{F} \circ \mathcal{G}$ (the *composite* of $\mathcal{F}$ and $\mathcal{G}$) is defined to be the filter on X^2 generated by the filterbasis $\{F \circ G \mid F \in \mathcal{F}, G \in \mathcal{G}\}$, where $F \circ G := \{(x, z) \in X^2 \mid \exists y \in X : (x, y) \in G \text{ and } (y, z) \in F\}$. Besides the "normal" (cartesian) product of sets, maps, filters, ..., we also define the following special product of filters. If $\mathcal{F} \in \mathbf{F}(X^2)$ and $\mathcal{G} \in \mathbf{F}(Y^2)$, then $\mathcal{F} \otimes \mathcal{G}$ denotes the filter generated by $\{F \otimes G \mid F \in \mathcal{F}, G \in \mathcal{G}\}$, where, given $F \subset X^2$ and $G \subset Y^2$, the set $F \otimes G$ is given by $F \otimes G := \{((x, y), (x', y')) \mid (x, x') \in F, (y, y') \in G\}$. Also, given a set X, Δ_X denotes the diagonal of X^2, that is, the set $\{(x, x) \mid x \in X\}$.

Given $F \subset X$, we let

$$\mathbf{S}_q(X,F) := \{\mathcal{F} \in \mathbf{F}(X^2) \mid \mathcal{F} \subset \text{stack}\,\Delta_F \text{ and } F \times F \in \mathcal{F}\}$$
$$\text{and } \mathbf{S}(X,F) := \{\mathcal{F} \in \mathbf{S}_q(X,F) \mid \mathcal{F}^{-1} = \mathcal{F}\},$$

elements of which are called *quasi-semi-uniformities (on F)* and *semi-uniformities (on F)* respectively.

Also let $\mathbf{S}_q(X) := \cup_{F \subset X} \mathbf{S}_q(X,F)$ and $\mathbf{S}(X) := \cup_{F \subset X} \mathbf{S}(X,F)$ denote the collection of quasi-semi-uniformities (in X) and semi-uniformities (in X) respectively, and observe that the set $F \subset X$ such that $\mathcal{F} \in \mathbf{S}_q(X,F)$ is uniquely determined by $\mathcal{F} \in \mathbf{S}_q(X)$, i.e. $\mathbf{S}_q(X,F) \cap \mathbf{S}_q(X,G) = \emptyset$ whenever $F \neq G$. Indeed, if $\mathcal{F} \in \mathbf{S}_q(X,F)$, $\mathcal{G} \in \mathbf{S}_q(X,G)$ and $\mathcal{F} \subset \mathcal{G}$, then it follows that $\Delta_G \subset F \times F$, hence $G \subset F$. Consequently, $\mathbf{S}_q(X,F) \cap \mathbf{S}_q(X,G) \neq \emptyset$ implies that $F = G$.

A *semi-uniform limit space* is a pair $(X, \mathbb{L})$, where X is a set and $\mathbb{L}$ is a set of filters on $X \times X$ such that the following conditions hold:

(SUC_1) $\forall x \in X : \dot{x} \times \dot{x} \in \mathbb{L}$.

(SUC_2) $\forall \mathcal{F} \in \mathbb{L}, \forall \mathcal{G} \in \mathbf{F}(X^2) : \mathcal{F} \subset \mathcal{G} \Rightarrow \mathcal{G} \in \mathbb{L}$.

(SUC_3) $\forall \mathcal{F} \in \mathbf{F}(X^2) : \mathcal{F} \in \mathbb{L} \Rightarrow \mathcal{F}^{-1} \in \mathbb{L}$.

(SUL) $\forall \mathcal{F}, \mathcal{G} \in \mathbb{L} : \mathcal{F} \cap \mathcal{G} \in \mathbb{L}$.

The semi-uniform limit space $(X, \mathbb{L})$ is called a *principal semi-uniform limit space* if it additionally satisfies

(PrSUL) For any family $(\mathcal{F}_j)_{j \in J} \in \prod_{j \in J} \mathbb{L} : \bigcap_{j \in J} \mathcal{F}_j \in \mathbb{L}$

and it is called a *uniform limit space* if it additionally satisfies

(UL) $\forall \mathcal{F}, \mathcal{G} \in \mathbb{L} : \mathcal{F} \circ \mathcal{G} \in \mathbb{L}$.

Using the prefix *quasi* in the sequel will indicate that condition (SUC_3) need not be satisfied, which leads to various variations of the foregoing, such as for instance a *quasi-semi-uniform limit space*, a *principal quasi-uniform limit space*, ...

Also observe that a (quasi-)semi-uniform limit space $(X, \mathbb{L})$ is principal if and only if there exists a (quasi-)semi-uniformity $\mathcal{U}$ on X

such that $\mathbb{L} = \{\mathcal{F} \in \mathbf{F}(X^2) \mid \mathcal{U} \subset \mathcal{F}\}$ and $(X,\mathcal{U}) = (X,\mathbb{L})$ satisfies (UL) if and only if $\mathcal{U}$ is even a (quasi-)uniformity, meaning that $\mathcal{U}$ additionally satisfies the property: $\forall U \in \mathcal{U}, \exists V \in \mathcal{U} : V \circ V \subset U$.

Given quasi-semi-uniform limit spaces $(X,\mathbb{L}_X)$ and $(Y,\mathbb{L}_Y)$, a map $f : (X,\mathbb{L}_X) \longrightarrow (Y,\mathbb{L}_Y)$ is said to be *uniformly continuous* provided that

$$\forall \mathcal{F} \in \mathbb{L}_X : (f \times f)(\mathcal{F}) \in \mathbb{L}_Y.$$

Quasi-semi-uniform limit spaces and uniformly continuous maps form the objects and morphisms of a topological construct which is denoted by q**SULim**. The other types of spaces which have been considered so far give rise to topological subconstructs of q**SULim**, denoted by **SULim**, (q)**ULim** and $(q)(s)$**Unif** (the previous observation illustrates that the subconstruct consisting of principal (quasi-)(semi-)uniform limit spaces is concretely isomorphic to the the construct of (quasi-)(semi-)uniform spaces.

Next, a number of results are recalled from [4] (which are also obtained in [15]-[14] by considering the particular non-quantified case), where notations and terminology are along the lines of [3] and [15]-[14].

Definition 2.1. *Let* **(q)sug-**(q)**SULim** *be the full subconstruct of* (q)**SULim** *consisting of (quasi-)semi-uniformly generated spaces, i.e. (quasi-)semi-uniform limit spaces* $(X,\mathbb{L})$ *satisfying*

((q)sug) $\quad \forall \mathcal{F} \in \mathbb{L}, \exists \mathcal{H} \in \mathbb{L} \cap \mathbf{S}_{(q)}(X) : \mathcal{H} \subset \mathcal{F}.$

Proposition 2.2. **(q)sug-**(q)**SULim** *is the final hull of* (q)**Unif** *in* (q)**SULim** *and the following relations hold between some of the various constructs considered so far (where* r (c) $: \mathbf{A} \longrightarrow \mathbf{B}$ *means that* $\mathbf{A}$ *is a bi(co)reflective subconstruct of* $\mathbf{B}$*):*

$$\begin{array}{ccccccc}
q\mathbf{Unif} & \xrightarrow{r} & qs\mathbf{Unif} & \xrightarrow{r} & \mathbf{qsug}\text{-}q\mathbf{SULim} & \xrightarrow{c} & q\mathbf{SULim} \\
r\uparrow c & & r\uparrow c & & r\uparrow c & & r\uparrow c \\
\mathbf{Unif} & \xrightarrow{r} & s\mathbf{Unif} & \xrightarrow{r} & \mathbf{sug}\text{-}\mathbf{SULim} & \xrightarrow{c} & \mathbf{SULim},
\end{array}$$

where each construct in the bottom row is obtained by restricting the corresponding top row construct to **SULim** *and each bi(co)reflector* *in the bottom row is also a restriction of the corresponding top one, such as*

$$R : (q)s\mathbf{Unif} \longrightarrow (q)\mathbf{Unif} :$$

$$(X, \mathbb{L}) = (X, \mathcal{U}) \mapsto (X, (q)\mathbf{Unif}(\mathbb{L})) = (X, (q)\mathbf{Unif}(\mathcal{U})).$$

Also, each vertical bicoreflector is a restriction of the bicoreflector $C_s : q\mathbf{SULim} \longrightarrow \mathbf{SULim} : (X, \mathbb{L}) \mapsto (X, \mathbb{L}')$, *where* $\mathbb{L}' := \{\mathcal{F} \in \mathbf{F}(X^2) \mid \mathcal{F} \in \mathbb{L}$ *and* $\mathcal{F}^{-1} \in \mathbb{L}\}$.

PROPOSITION 2.3. **(qsug-)**q**SULim** *is a cartesian closed topological construct. Moreover, initial lifts and function spaces are formed in* **qsug-**q**SULim** *by first forming them in* q**SULim** *and then applying the* **qsug-**q**SULim**-*bicoreflector* C. *Specifically, given a source*

$$(f_i : X \longrightarrow (X_i, \mathbb{L}_i))_{i \in I} (in\ \mathbf{qsug\text{-}}q\mathbf{SULim}),$$

one obtains the initial lift (in **qsug-**q**SULim***)* $\mathbb{L}_X$ *by*

$$\begin{aligned} \mathbb{L}_X := \quad & \{\mathcal{F} \in \mathbf{F}(X^2) \mid \exists \mathcal{H} \in \mathbf{S}_q(X) : \mathcal{H} \subset \mathcal{F} \text{ and} \\ & \forall i \in I : (f_i \times f_i)(\mathcal{H}) \in \mathbb{L}_i\}. \end{aligned}$$

Also, given $(X, \mathbb{L}_X), (Y, \mathbb{L}_Y) \in$ **qsug-**q**SULim**, *the function space* $[(X, \mathbb{L}_X), (Y, \mathbb{L}_Y)]$ *(in* **qsug-**q**SULim***) is given by*

$$(\hom((X, \mathbb{L}_X), (Y, \mathbb{L}_Y)), \mathbb{L}),$$

where

$$\begin{aligned} \mathbb{L} := \quad & \{\Psi \in \mathbf{F}(\hom((X, \mathbb{L}_X), (Y, \mathbb{L}_Y))^2) \mid \exists \Phi \in \mathbf{S}_q(\hom((X, \mathbb{L}_X), \\ & (Y, \mathbb{L}_Y))) : \Phi \subset \Psi \text{ and } \forall \mathcal{F} \in \mathbb{L}_X \cap \mathbf{S}_q(X) : \Phi(\mathcal{F}) \in \mathbb{L}_Y\}. \end{aligned}$$

3. The CCT hull of qUnif

DEFINITION 3.1. *Let* X *be a set and* $E \subset X^2$ *such that* $\Delta_X \subset E$. *Define a Čech closure operator* $E(-)$, *called* E-*enlargement in* X^2, *by*

$$E(-) : \mathcal{P}(X^2) \longrightarrow \mathcal{P}(X^2) : A \mapsto E(A) := E \circ A \circ E.$$

Since it holds for any $A \subset B \subset X^2$ *that* $E(A) \subset E(B)$, *it follows that* $\{E(F) \mid F \in \mathcal{F}\}$ *is a filterbasis whenever* $\mathcal{F} \in \mathbf{F}(X^2)$. *The filter generated by it will be denoted* $E(\mathcal{F})$ *and is called* E-*closure of* $\mathcal{F}$.

Definition 3.2. *Let* $(X,\mathbb{L}) \in q\mathbf{SULim}$*,* $E \in q\mathbf{Unif}(\mathbb{L})$*,* $\mathcal{H} \in \mathbf{S}_q(X)$ *and* $\mathcal{W} \in \mathbf{U}(\mathcal{H})$ *and define* $V_{E,\mathcal{W}}(\mathcal{H}) := \{\mathcal{G} \in \mathbf{S}_q(X) \mid E(\mathcal{G}) \subset \mathcal{W}\}$*.*

Also define $\mathrm{cl}^u_X : \mathcal{P}(\mathbf{S}_q(X)) \longrightarrow \mathcal{P}(\mathbf{S}_q(X)) : \Theta \mapsto \mathrm{cl}^u_X(\Theta)$ *by*

$$\begin{aligned} \mathrm{cl}^u_X(\Theta) := \quad & \{\mathcal{H} \in \mathbf{S}_q(X) \mid \forall\, E \in q\mathbf{Unif}(\mathbb{L}), \\ & \forall\, \mathcal{W} \in \mathbf{U}(\mathcal{H}) : V_{E,\mathcal{W}}(\mathcal{H}) \cap \Theta \neq \emptyset\} \end{aligned}$$

(and it is shown in [14] that $(\mathbf{S}_q(X), \mathrm{cl}^u_X)$ *is a closure space).*

Definition 3.3. *Let* q**EpiUnif** *be the full subconstruct of* **qsug-**q**SULim** *consisting of epi-quasi-uniform spaces (also called closed spaces), that is, objects* $(X,\mathbb{L})$ *satisfying*

$$\mathrm{cl}^u_X(\mathbb{L} \cap \mathbf{S}_q(X)) = \mathbb{L} \cap \mathbf{S}_q(X)$$
$$(\textit{or equivalently } \mathrm{cl}^u_X(\mathbb{L} \cap \mathbf{S}_q(X)) \subset \mathbb{L} \cap \mathbf{S}_q(X)).$$

We are now in a position to state the main result.

Theorem 3.4. q**EpiUnif** *is the cartesian closed topological hull of* q**Unif***.*

This shall be proven in several steps.

STEP 1: First, it needs to be shown that q**Unif** $\subset$ q**EpiUnif**.

Proposition 3.5. q**Unif** *is contained in* q**EpiUnif***.*

Proof. Let $(X,\mathbb{L}) = (X,\mathcal{U}) \in q\mathbf{Unif}$ (where $\mathcal{U}$ is a quasi-uniformity) and let $\mathcal{H} \in \mathbf{S}_q(X)$ such that $\mathcal{H} \in \mathrm{cl}^u_X(\mathbb{L} \cap \mathbf{S}_q(X))$. To show that $\mathcal{H} \in \mathbb{L}$, i.e. $\mathcal{U} \subset \mathcal{H}$, let $E \in \mathcal{U}$ and $\mathcal{W} \in \mathbf{U}(\mathcal{H})$. As $\mathcal{U}$ is a quasi-uniformity, there exist $E_0, E' \in \mathcal{U}$ such that $E_0 \circ E' \circ E_0 \subset E$. Using $\mathcal{H} \in \mathrm{cl}^u_X(\mathbb{L} \cap \mathbf{S}_q(X))$, one finds $\mathcal{G} \in \mathbb{L} \cap \mathbf{S}_q(X)$ such that $E_0(\mathcal{G}) \subset \mathcal{W}$. Since $\mathcal{G} \in \mathbb{L}$, it follows that $\mathcal{U} \subset \mathcal{G}$, hence $E' \in \mathcal{G}$, consequently, $E_0(E') \subset E \in \mathcal{W}$. As this is the case for any $\mathcal{W} \in \mathbf{U}(\mathcal{H})$, it follows that $E \in \mathcal{H}$, which shows that $\mathcal{U} \subset \mathcal{H}$. □

STEP 2: Next we must show that q**EpiUnif** is a cartesian closed topological construct.

The following general observation about ultrafilters will be useful.

Lemma 3.6. *Let X, Y and Z be sets.*

1. *Let $f : X \longrightarrow Y$ be a map. If $\mathcal{F} \in \mathbf{F}(X)$ and $\mathcal{W} \in \mathbf{U}(f(\mathcal{F}))$, then there exists a $\mathcal{V} \in \mathbf{U}(\mathcal{F})$ such that $f(\mathcal{V}) = \mathcal{W}$.*

2. *Let $g : X \times Y \longrightarrow Z$ be a map. If $\mathcal{F} \in \mathbf{F}(X \times X)$, $\mathcal{G} \in \mathbf{F}(Y \times Y)$ and $\mathcal{W} \in \mathbf{U}((g \times g)(\mathcal{F} \otimes \mathcal{G}))$, then there exists a $\mathcal{Z} \in \mathbf{U}(\mathcal{G})$ such that $(g \times g)(\mathcal{F} \otimes \mathcal{Z}) \subset \mathcal{W}$.*

Proof. (1): Let $F \in \mathcal{F}$ and $W \in \mathcal{W}$. Since $\emptyset \neq (f(F) \cap W) \subset f(F \cap f^{-1}(W))$, it follows that $\emptyset \neq (F \cap f^{-1}(W))$. Hence, there exists $\mathcal{V} \in \mathbf{U}(\mathcal{F}) \cap \mathbf{U}(f^{-1}(\mathcal{W}))$, which also implies $\mathcal{W} \subset f(f^{-1}(\mathcal{W})) \subset f(\mathcal{V})$ and even (since $\mathcal{W}$ is ultra) $f(\mathcal{V}) = \mathcal{W}$.

(2): By (1), we find an ultrafilter $\mathcal{V}$ on $(X \times Y)^2$ such that $\mathcal{F} \otimes \mathcal{G} \subset \mathcal{V}$ and $(g \times g)(\mathcal{V}) = \mathcal{W}$. By $\mathcal{F} \otimes \mathcal{G} \subset \mathcal{V}$, we have $\mathcal{G} \subset \mathcal{Z} := (\mathrm{pr}_Y \times \mathrm{pr}_Y)(\mathcal{V})$. Furthermore, $(g \times g)(\mathcal{F} \otimes \mathcal{Z}) \subset (g \times g)((\mathrm{pr}_X \times \mathrm{pr}_X)(\mathcal{V}) \otimes (\mathrm{pr}_Y \times \mathrm{pr}_Y)(\mathcal{V})) \subset (g \times g)(\mathcal{V}) = \mathcal{W}$. □

Proposition 3.7. *Let $f : (X, \mathbb{L}_X) \longrightarrow (Y, \mathbb{L}_Y)$ be a uniformly continuous map between q**SULim**-objects. Then $\bar{f} : (\mathbf{S}_q(X), \mathrm{cl}^u_X) \longrightarrow (\mathbf{S}_q(Y), \mathrm{cl}^u_Y) : \mathcal{H} \mapsto (f \times f)(\mathcal{H})$ is a continuous map.*

Proof. Let $\Theta \subset \mathbf{S}_q(X)$ and $\mathcal{H} \in \mathrm{cl}^u_X(\Theta)$. To show that $(f \times f)(\mathcal{H}) \in \mathrm{cl}^u_X(\bar{f}(\Theta))$, let $E \in q\mathbf{Unif}(\mathbb{L}_Y)$ and $\mathcal{W} \in \mathbf{U}((f \times f)(\mathcal{H}))$. It follows from the foregoing lemma that there exists $\mathcal{V} \in \mathbf{U}(\mathcal{H})$ such that $(f \times f)(\mathcal{V}) = \mathcal{W}$. Since $(f \times f)^{-1}(E) \in q\mathbf{Unif}(\mathbb{L}_X)$ (by the uniform continuity of $f : (X, q\mathbf{Unif}(\mathbb{L}_X)) \longrightarrow (Y, q\mathbf{Unif}(\mathbb{L}_Y))$), there exists $\mathcal{G} \in \Theta$ such that $(f \times f)^{-1}(E)(\mathcal{G}) \subset \mathcal{V}$. Now observe that $E((f \times f)(\mathcal{G})) \subset (f \times f)\big((f \times f)^{-1}(E)(\mathcal{G})\big)$ (indeed, for any $G \in \mathcal{G}$ we have: $(f \times f)\big((f \times f)^{-1}(E)(G)\big) \subset E((f \times f)(G))$). Consequently, $E((f \times f)(\mathcal{G})) \subset (f \times f)\big((f \times f)^{-1}(E)(\mathcal{G})\big) \subset (f \times f)(\mathcal{V}) = \mathcal{W}$, which shows that $(f \times f)(\mathcal{H}) \in \mathrm{cl}^u_X(\bar{f}(\Psi))$. □

Proposition 3.8. q**EpiUnif** *is bireflective in* **qsug-**q**SULim**, *in particular,* q**EpiUnif** *is a topological construct.*

Proof. It will suffice to show that q**EpiUnif** is initially closed in **qsug-**q**SULim**. To this end, let $(f_i : (X, \mathbb{L}_X) \longrightarrow (X_i, \mathbb{L}_i))_{i \in I}$ be initial in **qsug-**q**SULim** and all $(X_i, \mathbb{L}_i) \in q$**EpiUnif**. To show that $(X, \mathbb{L}_X) \in q$**EpiUnif**, let $\mathcal{H} \in \mathbf{S}_q(X)$ such that $\mathcal{H} \in \mathrm{cl}^u_X(\mathbb{L}_X \cap$

$\mathbf{S}_q(X)$), hence, from the previous proposition and the fact that all $f_i : (X, \mathbb{L}_X) \longrightarrow (X_i, \mathbb{L}_i)$ $(i \in I)$ are uniformly continuous it follows that

$$(f_i \times f_i)(\mathcal{H}) \in \mathrm{cl}^u_{X_i}(\bar{f}_i(\mathbb{L}_X \cap \mathbf{S}_q(X))) \subset \mathrm{cl}^u_{X_i}(\mathbb{L}_i \cap \mathbf{S}_q(X_i)).$$

Furthermore, $(X_i, \mathbb{L}_i) \in q\mathbf{EpiUnif}$ implies that $(f_i \times f_i)(\mathcal{H}) \in \mathbb{L}_i$ $(i \in I)$. Thus, by description of initial lifts given in proposition 2.3, it holds that $\mathcal{H} \in \mathbb{L}_X \cap \mathbf{S}_q(X)$. □

The following result is a quasi (and notational) variation of [3, proposition 3.18].

Proposition 3.9. *Let $(X, \mathbb{L}_X)$ and $(Y, \mathbb{L}_Y)$ be quasi-semi-uniform limit spaces and let*

$$(Z, \mathbb{L}) := [(X, \mathbb{L}_X), (Y, \mathbb{L}_Y)]$$

be the **qsug-qSULim**-*function space. Let $H \subset X$ and $\Delta_Y \subset E \subset Y^2$ and denote $F(H, E) := \{(f, g) \in Z^2 \mid \forall x \in H : (f(x), g(x)) \in E\}$, then $\mathcal{U}_H := \{F(H, E) \mid E \in q\mathbf{Unif}(\mathbb{L}_Y)\}$ is a quasi-uniformity and whenever* $\mathrm{stack}\,\Delta_H \in \mathbb{L}_X$*, we have that $\mathcal{U}_H \subset q\mathbf{Unif}(\mathbb{L})$.*

Proof. It is easily verified that $\mathcal{U}_H$ is a quasi-uniformity (on Z) (observe for instance that $F(H, E) \circ F(H, E') \subset F(H, E \circ E')$).

To prove the latter claim, let $\mathrm{stack}\,\Delta_H \in \mathbb{L}_X$, then it needs to be shown that $1_Z : (Z, q\mathbf{Unif}(\mathbb{L})) \longrightarrow (Z, \mathcal{U}_H)$ is uniformly continuous. Since $(Z, \mathcal{U}_H) \in q\mathbf{Unif}$, it suffices to show that $1_Z : (Z, \mathbb{L}) \longrightarrow (Z, \mathcal{U}_H)$ is uniformly continuous. To this end, let $\Psi \in \mathbb{L} \cap \mathbf{S}_q(X)$. To show that $\mathcal{U}_H \subset \Psi$, let $E \subset q\mathbf{Unif}(\mathbb{L}_Y)$. Since $\mathrm{stack}\,\Delta_H \in \mathbb{L}_X$, it follows from the description of $\mathbb{L}$ in proposition 2.3 that $\Psi(\mathrm{stack}\,\Delta_H) \in \mathbb{L}_Y$, hence $E \in q\mathbf{Unif}(\mathbb{L}_Y) \subset \Psi(\mathrm{stack}\,\Delta_H)$ (as $1_Y : (Y, \mathbb{L}_Y) \longrightarrow (Y, q\mathbf{Unif}(\mathbb{L}_Y))$ is uniformly continuous). Consequently, there exists $\psi \in \Psi$ such that $(\mathrm{ev} \times \mathrm{ev})(\Delta_H \otimes \psi) \subset E$, implying that $\psi \subset F(H, E)$ and therefore $F(H, E) \in \Psi$, hence $\mathcal{U}_H \subset \Psi$. □

Proposition 3.10. *Let $(X, \mathbb{L}_X), (Y, \mathbb{L}_Y) \in$* **qsug-$q$SULim** *and let $(Z, \mathbb{L})$ be the* **qsug-qSULim**-*function space $[(X, \mathbb{L}_X), (Y, \mathbb{L}_Y)]$. If $\mathcal{H} \in \mathbb{L}_X \cap \mathbf{S}_q(X, H)$, then $\bar{\mathcal{H}} : (\mathbf{S}_q(Z), \mathrm{cl}^u_Z) \longrightarrow (\mathbf{S}_q(Y), \mathrm{cl}^u_Y) : \Psi \mapsto \Psi(\mathcal{H})$ is a continuous map.*

Proof. Let $\Theta \subset \mathbf{S}_q(Z)$ and $\Psi \in \mathrm{cl}^u_Z(\Theta)$. To show that $\Psi(\mathcal{H}) \in \mathrm{cl}^u_Y(\bar{\mathcal{H}}(\Theta))$, let $E \in q\mathbf{Unif}(\mathbb{L}_Y)$ and $\mathcal{W} \in \mathbf{U}(\Psi(\mathcal{H}))$. It follows from lemma 3.6 that there exists $\mathcal{Z} \in \mathbf{U}(\Psi)$ such that $\mathcal{Z}(\mathcal{H}) = (\mathrm{ev}\times\mathrm{ev})(\mathcal{H}\otimes\mathcal{Z}) \subset \mathcal{W}$. Since $\mathcal{H} \in \mathbb{L}_X \cap \mathbf{S}_q(X,H)$ (and therefore $\mathcal{H} \subset \mathrm{stack}\,\Delta_H$), it also holds that $\mathrm{stack}\,\Delta_H \in \mathbb{L}_X$, hence, by the previous proposition, $F(H,E) \in q\mathbf{Unif}(\mathbb{L})$. As $\Psi \in \mathrm{cl}^u_Z(\Theta)$, there exists $\Phi \in \Theta$ such that $F(H,E)(\Phi) \subset \mathcal{Z}$. Now observe that $E(\Phi(\mathcal{H})) \subset (F(H,E)(\Phi))(\mathcal{H})$ (indeed, for any $\phi \in \Phi$ and $H \times H \supset G \in \mathcal{H}$ we have: $(F(H,E)(\phi))(G) \subset E(\phi(G))$). Consequently, $E(\Phi(\mathcal{H})) \subset (F(H,E)(\Phi))(\mathcal{H}) \subset \mathcal{Z}(\mathcal{H}) \subset \mathcal{W}$, which shows that $\Psi(\mathcal{H}) \in \mathrm{cl}^u_Y(\bar{\mathcal{H}}(\Theta))$. □

PROPOSITION 3.11. *q***EpiUnif** *is closed under formation of function spaces in* **qsug-qSULim**. *Moreover, if $(X,\mathbb{L}_X) \in$* **qsug-qSULim** *and $(Y,\mathbb{L}_Y) \in q$***EpiUnif***, then $[(X,\mathbb{L}_X),(Y,\mathbb{L}_Y)] \in q$***EpiUnif***. In particular, q***EpiUnif** *is a cartesian closed category.*

Proof. Let $(Z,\mathbb{L}) := [(X,\mathbb{L}_X),(Y,\mathbb{L}_Y)]$ (in **qsug-qSULim**). Furthermore, let $\Psi \in \mathbf{S}_q(Z)$ such that $\Psi \in \mathrm{cl}^u_Z(\mathbb{L}\cap\mathbf{S}_q(Z))$. To prove that $\Psi \in \mathbb{L}$, it needs to be shown for any $\mathcal{H} \in \mathbb{L}_X \cap \mathbf{S}_q(X)$ that $\Psi(\mathcal{H}) \in \mathbb{L}_Y$ (by proposition 2.3). To this end, let $\mathcal{H} \in \mathbb{L}_X \cap \mathbf{S}_q(X,H)$, then it follows from the previous proposition and $\mathcal{H} \in \mathbb{L}_X$ that

$$\Psi(\mathcal{H}) \in \mathrm{cl}^u_Y(\bar{\mathcal{H}}(\mathbb{L}\cap\mathbf{S}_q(Z))) \subset \mathrm{cl}^u_Y(\mathbb{L}_Y \cap \mathbf{S}_q(Y)).$$

Since $(Y,\mathbb{L}_Y) \in q$**EpiUnif**, it holds that $\Psi(\mathcal{H}) \in \mathbb{L}_Y$. □

STEP 3: We now turn to showing that the proper density conditions are satisfied.

DEFINITION 3.12. *Let $d_{\mathbb{P}} : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+ : (x,y) \mapsto (x-y)\vee 0$, then it is easily verified that $d_{\mathbb{P}}$ is a pseudo-quasi-metric on $\mathbb{R}^+$, i.e. it satisfies:*

1. *$\forall x \in X : d_{\mathbb{P}}(x,x) = 0$.*
2. *triangle inequality: $\forall x,y,z \in X : d_{\mathbb{P}}(x,z) \le d_{\mathbb{P}}(x,y) + d_{\mathbb{P}}(y,z)$.*

In particular, $\big\{\{(x,y) \in (\mathbb{R}^+)^2 \mid d_{\mathbb{P}}(x,y) < \epsilon\} \mid \epsilon > 0\big\}$ is a filterbasis that generates a quasi-uniformity $\mathcal{U}_{\mathbb{P}}$ and let $\mathbb{R}^+_{\mathbb{P}}$ denote the quasi-uniform space $(\mathbb{R}^+,\mathcal{U}_{\mathbb{P}}) = (\mathbb{R}^+,\mathbb{L}_{\mathbb{P}})$.

Assume without restriction in the following that $X \neq \emptyset$.

Proposition 3.13. *Let* $(X,\mathbb{L}) \in q\mathbf{EpiUnif}$*, then the map* $j : (X,\mathbb{L}) \longrightarrow [[(X,\mathbb{L}),\mathbb{R}^+_{\mathbb{P}}],\mathbb{R}^+_{\mathbb{P}}]$ *defined by* $j(x)(f) = f(x)$ *is a uniformly continuous and initial map (where function spaces and initiality are considered in* **qsug-**q**SULim***).*

Proof. First consider the following diagram and observe that $j := \mathrm{ev}^*_{(X,\mathbb{L}),\mathbb{R}^+_{\mathbb{P}}}$ is the map which makes the following diagram commute:

$$\begin{array}{ccc} [[(X,\mathbb{L}),\mathbb{R}^+_{\mathbb{P}}],\mathbb{R}^+_{\mathbb{P}}] \times [(X,\mathbb{L}),\mathbb{R}^+_{\mathbb{P}}] & \xrightarrow{\mathrm{ev}} & \mathbb{R}^+_{\mathbb{P}} \\ \uparrow{\scriptstyle j\times 1} & \nearrow{\scriptstyle \mathrm{ev}} & \\ (X,\mathbb{L}) \times [(X,\mathbb{L}),\mathbb{R}^+_{\mathbb{P}}] & & \end{array}$$

Hence, by properties of function spaces, j is a uniformly continuous map.

Next, let

$$(\hom((X,\mathbb{L}),\mathbb{R}^+_{\mathbb{P}}),\mathbb{L}_H) := [(X,\mathbb{L}),\mathbb{R}^+_{\mathbb{P}}],$$
$$\text{and } (\hom([(X,\mathbb{L}),\mathbb{R}^+_{\mathbb{P}}],\mathbb{R}^+_{\mathbb{P}}),\mathbb{L}_{HH}) := [[(X,\mathbb{L}),\mathbb{R}^+_{\mathbb{P}}],\mathbb{R}^+_{\mathbb{P}}].$$

To prove that j is initial, it will be shown that for every $\mathcal{H} \in \mathbf{S}_q(X,H)$ $(H \subset X)$ that $\mathcal{H} \notin \mathbb{L}$ implies that $j(\mathcal{H}) \notin \mathbb{L}_{HH}$. It then follows from proposition 2.3 that j is initial.

Let $\mathcal{H} \in \mathbf{S}_q(X,H)$ $(H \subset X)$ be such that $\mathcal{H} \notin \mathbb{L}$. It will be shown that $j(\mathcal{H}) \notin \mathbb{L}_{HH}$ by defining an appropriate $\Psi \in \mathbb{L}_H \cap \mathbf{S}_q(\hom((X,\mathbb{L}),\mathbb{R}^+_{\mathbb{P}}))$ such that $\Psi(\mathcal{H}) = (j(\mathcal{H}))(\Psi) \notin \mathbb{L}_{\mathbb{P}}$, hence, by proposition 2.3, $j(\mathcal{H}) \notin \mathbb{L}_{HH}$.

Defining of $\Psi \in \mathbb{L}_H \cap \mathbf{S}_q(\hom((X,\mathbb{L}),\mathbb{R}^+_{\mathbb{P}}))$:

As $(X,\mathbb{L}) \in q\mathbf{EpiUnif}$ and $\mathcal{H} \notin \mathbb{L}$, it follows that $\mathcal{H} \notin \mathrm{cl}^u_X(\mathbb{L} \cap \mathbf{S}_q(X))$, hence there exist $E \in q\mathbf{Unif}(\mathbb{L})$ and $\mathcal{W} \in \mathbf{U}(\mathcal{H})$ such that:

$$\forall \mathcal{G} \in \mathbf{S}_q(X) : E(\mathcal{G}) \subset \mathcal{W} \Rightarrow \mathcal{G} \notin \mathbb{L}. \qquad (*)$$

We then obtain $d' \in \mathcal{D}(q\mathbf{Unif}(\mathbb{L}))$ and $0 < \delta < \infty$ such that $\{d' < \delta\} \subset E$ (where $\mathcal{D}(\mathcal{U}) := \{d \text{ is pseudo-quasi metric} \mid 1_X : (X,\mathcal{U}) \longrightarrow (X,d) \text{ is uniformly continuous}\}$ is the *uniform quasi-gauge* of a quasi-uniform space $(X,\mathcal{U})$). Letting $d := d' \wedge 2\delta$, it holds that $d \in \mathcal{D}(q\mathbf{Unif}(\mathbb{L}))$ and $\{d < \delta\} \subset E$.

Now let

$$\Phi_1 := \{f{:}X \to \mathbb{R}^+ \mid \exists K_1, K_2 \in \mathbb{R}, \exists x \in X : f = K_1 + (d(-,x) \wedge K_2)\}$$
$$\Phi_2 := \{f{:}X \to \mathbb{R}^+ \mid \exists K_1, K_2 \in \mathbb{R}, \exists x \in X : f = K_1 - (d(x,-) \wedge K_2)\}$$
$$\text{and } \Phi := \Phi_1 \cup \Phi_2.$$

Observe for any $f \in \Phi$ that $f : (X, \mathbb{L}) \longrightarrow \mathbb{R}^+_{\mathbb{P}}$ is uniformly continuous. Indeed, let $f = K_1 + (d(-,x) \wedge K_2) \in \Phi_1$, then

$$\begin{aligned}(f(u) - f(v)) \vee 0 &= (K_1 + (d(u,x) \wedge K_2) - K_1 - (d(v,x) \wedge K_2)) \vee 0 \\ &\leq (d(u,x) - d(v,x)) \vee 0 \\ &\leq (d(u,v) + d(v,x) - d(v,x)) \vee 0 \\ &= d(u,v).\end{aligned}$$

In case $\Phi_2 \ni f = K_1 - (d(x,-) \wedge K_2)$, then

$$\begin{aligned}(f(u) - f(v)) \vee 0 &= (K_1 - (d(x,u) \wedge K_2) - K_1 + (d(x,v) \wedge K_2)) \vee 0 \\ &\leq (d(x,v) - d(x,u)) \vee 0 \\ &\leq (d(x,u) + d(u,v) - d(x,u)) \vee 0 \\ &= d(u,v).\end{aligned}$$

In any case, given $f \in \Phi$, it holds for all $\epsilon > 0$ that $\{d < \epsilon\} \subset (f \times f)^{-1}(\{d_{\mathbb{P}} < \epsilon\})$ $(**)$ (where $\{d < \epsilon\} := \{(u,v) \in X \mid d(u,v) < \epsilon\}$ and $\{d_{\mathbb{P}} < \epsilon\}$ is an analogous abbreviation). This shows that $f : (X, q\mathbf{Unif}(\mathbb{L})) \longrightarrow \mathbb{R}^+_{\mathbb{P}}$ is uniformly continuous and therefore $f : (X, \mathbb{L}) \longrightarrow \mathbb{R}^+_{\mathbb{P}}$ is uniformly continuous (as also $1_X : (X, \mathbb{L}) \longrightarrow (X, q\mathbf{Unif}(\mathbb{L}))$ is uniformly continuous).

For any $G \subset X \times X$, let

$$F_0(G) := \{(f,g) \in \Phi \times \Phi \mid \forall (x,y) \in G : d_{\mathbb{P}}(f(x), g(y)) = 0\}$$
$$\text{and } \hat{\Psi} := \{F_0(G) \mid E(G) \notin \mathcal{W}\}.$$

Note that $\hat{\Psi}$ is a filterbasis on $\hom((X, \mathbb{L}), \mathbb{R}^+_{\mathbb{P}})$. Indeed, $\hat{\Psi} \neq \emptyset$, since for any $x \in X$, $E(\dot{x} \times \dot{x}) \not\subset \mathcal{W}$, otherwise, by $(*)$, $\dot{x} \times \dot{x} \notin \mathbb{L}$, a contradiction. Furthermore, such $F_0(G)$ is never a void set, as it always contains pairs of constant (positive) functions. Also, it holds that $F_0(G_1) \cap F_0(G_2) = F_0(G_1 \cup G_2)$, $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$ and $\mathcal{W}$ is an ultrafilter, hence $\hat{\Psi}$ is a filterbasis. Now let Ψ' denote the filter generated by $\hat{\Psi}$ and define $\Psi := \Psi' \cap \operatorname{stack} \Delta_\Phi \in \mathbf{S}_q(\hom((X, \mathbb{L}), \mathbb{R}^+_\mathbb{P}), \Phi)$.

Proving that $\Psi \in \mathbb{L}_H$:

By proposition 2.3, it needs to be shown for any $\mathcal{G} \in \mathbb{L} \cap \mathbf{S}_q(X)$ that $\Psi(\mathcal{G}) \in \mathbb{L}_\mathbb{P}$, or equivalently, $\Psi'(\mathcal{G}) \in \mathbb{L}_\mathbb{P}$ and $(\operatorname{stack} \Delta_\Phi)(\mathcal{G}) \in \mathbb{L}_\mathbb{P}$.

To this end, let $\mathcal{G} \in \mathbb{L} \cap \mathbf{S}_q(X)$ (I) and assume that $\Psi'(\mathcal{G}) \notin \mathbb{L}_\mathbb{P}$, i.e. $\mathcal{U}_\mathbb{P} \not\subset \Psi'(\mathcal{G})$ (II). It follows that $E(\mathcal{G}) \subset \mathcal{W}$. Indeed, if this were not the case, then there would exist $G \in \mathcal{G}$ such that $E(G) \notin \mathcal{W}$, hence $F_0(G) \in \Psi'$, consequently $(\mathrm{ev} \times \mathrm{ev})(G \otimes F_0(G)) \subset \{d_\mathbb{P} = 0\} \in \Psi'(\mathcal{G})$. In particular, $\mathcal{U}_\mathbb{P} \subset \Psi'(\mathcal{G})$, which contradicts (II). Thus, it must indeed be the case that $E(\mathcal{G}) \subset \mathcal{W}$, hence, by $(*)$, $\mathcal{G} \notin \mathbb{L}$, a contradiction with (I). It follows that assumption (II) contradicts with (I), consequently, (I) implies that $\Psi'(\mathcal{G}) \in \mathbb{L}_\mathbb{P}$.

Again, let $\mathcal{G} \in \mathbb{L} \cap \mathbf{S}_q(X)$. It remains to be shown that $(\operatorname{stack} \Delta_\Phi)(\mathcal{G}) \in \mathbb{L}_\mathbb{P}$. As $1_X : (X, \mathbb{L}) \longrightarrow (X, q\mathbf{Unif}(\mathbb{L}))$ is uniformly continuous, it follows that $q\mathbf{Unif}(\mathbb{L}) \subset \mathcal{G}$. In particular, $\{d < \epsilon\} \in \mathcal{G}$ $(\epsilon > 0)$, whereas a reformulation of $(**)$ states that $(\mathrm{ev} \times \mathrm{ev})(\{d < \epsilon\} \otimes \Delta_\Phi) \subset \{d_\mathbb{P} < \epsilon\}$ $(\epsilon > 0)$, hence $\mathcal{U}_\mathbb{P} \subset (\operatorname{stack} \Delta_\Phi)(\mathcal{G})$.

Proving that $\Psi(\mathcal{H}) \notin \mathbb{L}_\mathbb{P}$:

As $\Psi(\mathcal{H}) \subset \Psi'(\mathcal{W})$, it will suffice to show that $\Psi'(\mathcal{W}) \notin \mathbb{L}_\mathbb{P}$. Assume the contrary of the latter, i.e. $\mathcal{U}_\mathbb{P} \subset \Psi'(\mathcal{W})$. In particular, this implies that there exist $G \subset X \times X$ and $U \in \mathcal{W}$ such that

$$E(G) \notin \mathcal{W} \text{ and } (\mathrm{ev} \times \mathrm{ev})(U \otimes F_0(G)) \subset \{d_\mathbb{P} < \delta\}. \qquad \text{(III)}$$

It follows that $U \subset E(G)$. Indeed, let $(x, y) \notin E(G)$ and define $f_1 := \delta - (d(x, -) \wedge \delta)$ and $f_2 := d(-, y) \wedge \delta$, then $f_1, f_2 \in \Phi$ (by definition). Moreover, $(f_1, f_2) \in F_0(G)$. For if this were not the case, then there exists $(x', y') \in G$ such that $(f_1(x') - f_2(y')) \vee 0 = (\delta - (d(x, x') \wedge \delta) - (d(y', y) \wedge \delta)) \vee 0 > 0$, hence $d(x, x') < \delta$ and $d(y', y) < \delta$ (if not, then $(f_1(x') - f_2(y')) \vee 0 = 0$). As $\{d < \delta\} \subset E$, it follows that $(x, x'), (y', y) \in E$, consequently $(x, y) \in E(G)$, which contradicts the fact that $(x, y) \notin E(G)$. Hence, $(f_1, f_2) \in F_0(G)$ and $(f_1(x), f_2(y)) = (\delta, 0) \notin \{d_\mathbb{P} < \delta\}$, consequently, by (III), $(x, y) \notin U$.

Therefore, $U \subset E(G)$, hence $E(G) \in \mathcal{W}$, which contradicts (III). This in turn implies that it must be that $\Psi'(\mathcal{W}) \notin \mathbb{L}_{\mathbb{P}}$. □

STEP 4: Combining all previous steps now leads to the desired conclusion.

THEOREM 3.14. *q**EpiUnif** is the cartesian closed topological hull of q**Unif**.*

Proof. For this to be the case, it is needed (as noted before) that q**EpiUnif** is a cartesian closed topological construct (which has been verified in step 2), that q**Unif** is contained in q**EpiUnif** (which was verified in step 1) and that q**Unif** is finally dense in q**EpiUnif** (indeed, by proposition 2.2, it is even finally dense in **qsug-qSULim**). Also, the class

$$H := \{[(X, \mathbb{L}_X), (Y, \mathbb{L}_Y)] \mid (X, \mathbb{L}_X), (Y, \mathbb{L}_Y) \in q\mathbf{Unif}\}$$

needs to be initially dense in q**EpiUnif**. However, by the previous proposition, for any $(X, \mathbb{L}) \in q$**EpiUnif** there is an initial map $j : (X, \mathbb{L}) \longrightarrow [[(X, \mathbb{L}), \mathbb{R}^+_{\mathbb{P}}], \mathbb{R}^+_{\mathbb{P}}]$ and since the functor $[-, \mathbb{R}^+_{\mathbb{P}}] : q\mathbf{EpiUnif} \longrightarrow q\mathbf{EpiUnif}$ transforms final epi-sinks into initial sources (see [9, lemma 6]) (and by proposition 2.2, $[(X, \mathbb{L}), \mathbb{R}^+_{\mathbb{P}}]$ can be obtained as a final lift of an epi-sink involving q**Unif**-objects), it follows that H is initially dense in q**EpiUnif**. □

4. The CCT hull of (q)Unif (revisited)

As claimed in the introduction, the CCT hulls of q**Unif** and **Unif** can be characterized in several ways which will be considered in this section.

To this end, a closure operator cl that is less complicated than cl^u is introduced first.

DEFINITION 4.1. *Let $(X, \mathbb{L}) \in q\mathbf{SULim}$, $E \in q\mathbf{Unif}(\mathbb{L})$ and $\mathcal{H} \in \mathbf{S}_q(X)$ and define $V_E(\mathcal{H}) := \{\mathcal{G} \in \mathbf{S}_q(X) \mid E(\mathcal{G}) \subset \mathcal{H}\}$.*

Also define $\mathrm{cl}_X : \mathcal{P}(\mathbf{S}_q(X)) \longrightarrow \mathcal{P}(\mathbf{S}_q(X)) : \Theta \mapsto \mathrm{cl}_X(\Theta)$ by

$$\mathrm{cl}_X(\Theta) := \{\mathcal{H} \in \mathbf{S}_q(X) \mid \forall E \in q\mathbf{Unif}(\mathbb{L}) : V_E(\mathcal{H}) \cap \Theta \neq \emptyset\}$$

and observe that cl_X is a topological closure operator.

Also recall the following concept (for instance used in [2]), of which the E-enlargement in X^2, $E(-) : \mathcal{P}(X^2) \longrightarrow \mathcal{P}(X^2) : A \mapsto E(A)$, is a reasonable variation.

Definition 4.2. *Let X be a set and $E \subset X^2$ such that $\Delta_X \subset E$. Define the Čech closure operator $E(-)$, called E-enlargement in X, by*

$$E(-) : \mathcal{P}(X) \to \mathcal{P}(X) : A \mapsto E(A) := \{x \in X \mid \exists a \in A : (x, a) \in E\}.$$

There is the following connection between these E-enlargements.

Proposition 4.3. *Let $\Delta_X \subset E \subset X \times X$ be symmetric (i.e. $E^{-1} = E$), then $\mathcal{H} \in \mathbf{S}_q(X, H)$ implies that $E(\mathcal{H}) \in \mathbf{S}_q(X, E(H))$.*

Proof. First observe that as E is symmetric, it follows that $\Delta_{E(H)} \subset E(\Delta_H)$ (since for any $x \in E(H)$, there exists $h \in H$ such that $(x, h) \in E$, $(h, h) \in \Delta_H$ and $(h, x) \in E$) and $E(H \times H) = E(H) \times E(H)$. Indeed, if $(x, y) \in E(H \times H)$, then there exists $(h, h') \in H \times H$ such that $(x, h) \in E$ and $(h', y) \in E$, hence $(x, y) \in E(H) \times E(H)$. Conversely, if $(x, y) \in E(H) \times E(H)$, then there exists $(h, h') \in H \times H$ such that $(x, h) \in E$ and $(y, h') \in E = E^{-1}$, hence $(x, y) \in E(H \times H)$.

Since $H \times H \in \mathcal{H}$, it already follows that $E(H) \times E(H) \in E(\mathcal{H})$. To show that $E(\mathcal{H}) \subset \operatorname{stack} \Delta_{E(H)}$, let $G \in \mathcal{H}$, then $\Delta_H \subset G$, hence $\Delta_{E(H)} \subset E(\Delta_H) \subset E(G)$. □

The following result is most useful in handling (in the present setting) the *immer* elusive (and/or illusive) ultrafilters.

Lemma 4.4. *If $\mathcal{F} \in \mathbf{F}(X)$ and $\Psi \subset \mathbf{F}(X)$, then the following are equivalent:*

1. *$\forall \mathcal{W} \in \mathbf{U}(\mathcal{F}), \exists \mathcal{G} \in \Psi : \mathcal{G} \subset \mathcal{W}$.*
2. *For any family $(\sigma(\mathcal{G}))_{\mathcal{G} \in \Psi}$ such that $\sigma(\mathcal{G}) \in \mathcal{G}$ $(\mathcal{G} \in \Psi)$, there exists a finite set $\Psi' \subset \Psi$ such that $\bigcup_{\mathcal{G} \in \Psi'} \sigma(\mathcal{G}) \in \mathcal{F}$.*

Proof. $\boxed{1 \Rightarrow 2}$ Let $(\sigma(\mathcal{G}))_{\mathcal{G} \in \Psi}$ be a family such that $\sigma(\mathcal{G}) \in \mathcal{G}$ $(\mathcal{G} \in \Psi)$. Suppose the second claim does not hold, then it follows that the family $\mathcal{F} \cup \{X \setminus \sigma(\mathcal{G}) \mid \mathcal{G} \in \Psi\}$ has the finite intersection property and is therefore contained in some ultrafilter $\mathcal{W} \in \mathbf{U}(\mathcal{F})$. By (1), there exists $\mathcal{G} \in \Psi$ such that $\mathcal{G} \subset \mathcal{W}$. This implies that both $\sigma(\mathcal{G}) \in \mathcal{G} \subset \mathcal{W}$ and $X \setminus \sigma(\mathcal{G}) \in \mathcal{W}$, which is a contradiction.

$\boxed{2 \Rightarrow 1}$ Suppose (1) does not hold, then there exists some $\mathcal{W} \in \mathbf{U}(\mathcal{F})$ such that for all $\mathcal{G} \in \Psi$ there exists $\sigma(\mathcal{G}) \in \mathcal{G}$ such that $\sigma(\mathcal{G}) \notin \mathcal{W}$ $(*)$. Applying (2) on the family $(\sigma(\mathcal{G}))_{\mathcal{G}\in\Psi}$ yields a finite set $\Psi' \subset \Psi$ satisfying $\bigcup_{\mathcal{G}\in\Psi'} \sigma(\mathcal{G}) \in \mathcal{F}$. As $\mathcal{F} \subset \mathcal{W}$ and $\mathcal{W}$ is an ultrafilter, there is some $\mathcal{G} \in \Psi'$ such that $\sigma(\mathcal{G}) \in \mathcal{W}$, which contradicts $(*)$. ☐

Proposition 4.5. *Given* $(X, \mathbb{L}) \in$ **qsug-**q**SULim**, *the following are equivalent:*

1. $\mathrm{cl}^u_X(\mathbb{L} \cap \mathbf{S}_q(X)) = \mathbb{L} \cap \mathbf{S}_q(X)$ *(i.e.* $(X, \mathbb{L}) \in q$**EpiUnif***).*
2. $\mathrm{cl}_X(\mathbb{L} \cap \mathbf{S}_q(X)) = \mathbb{L} \cap \mathbf{S}_q(X)$.
3. $\forall H \subset X : \big(\mathrm{stack}\,\Delta_H \in \mathrm{cl}_X(\mathbb{L} \cap \mathbf{S}_q(X)) \Rightarrow \mathrm{stack}\,\Delta_H \in \mathbb{L}\big)$ *and* $\forall H \subset X : \big(\mathrm{stack}\,\Delta_H \in \mathbb{L} \Rightarrow q\mathbf{Unif}(\mathbb{L})_{|H} \in \mathbb{L}\big)$ *(where* $q\mathbf{Unif}(\mathbb{L})_{|H}$ *is the restriction of* $q\mathbf{Unif}(\mathbb{L})$ *to* $H \times H$*).*
4. *If* $H \subset X$ *is such that* $\forall E \subset q\mathbf{Unif}(\mathbb{L}), \exists G \subset X : \mathrm{stack}\,\Delta_G \in \mathbb{L}$ *and* $H \subset (E \cap E^{-1})(G), (*)$ *then* $\mathrm{stack}\,\Delta_H \in \mathbb{L}$. *Also,* $\forall H \subset X : \big(\mathrm{stack}\,\Delta_H \in \mathbb{L} \Rightarrow q\mathbf{Unif}(\mathbb{L})_{|H} \in \mathbb{L}\big)$.
5. $\forall H \subset X : \big(\mathrm{stack}\,\Delta_H \in \mathrm{cl}^u_X(\mathbb{L} \cap \mathbf{S}_q(X)) \Rightarrow \mathrm{stack}\,\Delta_H \in \mathbb{L}\big)$ *and* $\forall H \subset X : \big(\mathrm{stack}\,\Delta_H \in \mathbb{L} \Rightarrow q\mathbf{Unif}(\mathbb{L})_{|H} \in \mathbb{L}\big)$.

Proof. $\boxed{1 \Rightarrow 2}$ Observe that for any $\Theta \subset \mathbf{S}_q(X)$, it holds that $\mathrm{cl}_X(\Theta) \subset \mathrm{cl}^u_X(\Theta)$.

$\boxed{2 \Rightarrow 3}$ The first claim follows immediately from (2).

To show the latter claim, let $H \subset X$ be such that $\mathrm{stack}\,\Delta_H \in \mathbb{L}$. By (2), it will suffice to show that $q\mathbf{Unif}(\mathbb{L})_{|H} \in \mathrm{cl}_X(\mathbb{L} \cap \mathbf{S}_q(X))$. To this end, let $E \in q\mathbf{Unif}(\mathbb{L})$, then it follows that $E(\mathrm{stack}\,\Delta_H) \subset q\mathbf{Unif}(\mathbb{L})_{|H}$. Indeed, observe that $E \cap (H \times H) \subset E(\Delta_H)$ (since for any $(x, y) \in E \cap (H \times H)$, it holds that $(x, y) \in E$, $(y, y) \in \Delta_H$ and $(y, y) \in E$). As $\mathrm{stack}\,\Delta_H \in \mathbb{L} \cap \mathbf{S}_q(X)$, it follows that $q\mathbf{Unif}(\mathbb{L})_{|H} \in \mathrm{cl}_X(\mathbb{L} \cap \mathbf{S}_q(X)) \subset \mathbb{L}$.

$\boxed{3 \Rightarrow 4}$ The latter claim is clear. As for the first claim, let $H \subset X$ be such that $(*)$ holds. By (3), it will suffice to show that $\mathrm{stack}\,\Delta_H \in$

$\mathrm{cl}_X(\{\mathrm{stack}\,\Delta_G \mid \mathrm{stack}\,\Delta_G \in \mathbb{L}\})$. To this end, let $E \in q\mathbf{Unif}(\mathbb{L})$, then it follows from $(*)$ that there exists $G \subset X$ such that $\mathrm{stack}\,\Delta_G \in \mathbb{L}$ and $H \subset (E \cap E^{-1})(G)$, hence

$$\begin{aligned} E(\mathrm{stack}\,\Delta_G) &\subset (E\cap E^{-1})(\mathrm{stack}\,\Delta_G) \quad (\text{as } E\cap E^{-1} \subset E)\\ &\subset \mathrm{stack}\,\Delta_{(E\cap E^{-1})(G)} \quad (\text{by proposition 4.3})\\ &\subset \mathrm{stack}\,\Delta_H \quad (\text{as } H \subset (E\cap E^{-1})(G)),\end{aligned}$$

which shows that $\mathrm{stack}\,\Delta_H \in \mathrm{cl}_X(\{\mathrm{stack}\,\Delta_G \mid \mathrm{stack}\,\Delta_G \in \mathbb{L}\})$.

$\boxed{4 \Rightarrow 5}$ Again, the latter claim is clear. As for the first claim, let $H \subset X$ be such that $\mathrm{stack}\,\Delta_H \in \mathrm{cl}^u_X(\mathbb{L} \cap \mathbf{S}_q(X))$. By (4), it will suffice to show that $(*)$ holds. To this end, let $E \in q\mathbf{Unif}(\mathbb{L})$. As $q\mathbf{Unif}(\mathbb{L})$ is, by definition, a quasi-uniformity, it follows that there exists $E' \in q\mathbf{Unif}(\mathbb{L})$ such that $E' \circ E' \subset E$. If we let $\mathcal{F} := \mathrm{stack}\,\Delta_H$ and $\Psi := \{E'(\mathcal{G}) \mid \mathcal{G} \in \mathbb{L} \cap \mathbf{S}_q(X)\}$, then $\mathrm{stack}\,\Delta_H \in \mathrm{cl}^u_X(\mathbb{L} \cap \mathbf{S}_q(X))$ implies that (1) in the foregoing lemma is satisfied. Consequently, applying (2) of the foregoing lemma to the family $(E'(E' \cap G^2))_{\mathcal{G} \in \mathbb{L} \cap \mathbf{S}_q(X,G), G \subset X}$ (note that $E' \in q\mathbf{Unif}(\mathbb{L}) \subset \mathcal{G}$, whenever $\mathcal{G} \in \mathbb{L}$) leads to $n \in \mathbb{N}_0$ and

$$\mathcal{G}_1, \ldots, \mathcal{G}_n \in \mathbb{L} \cap \mathbf{S}_q(X) : E'(E' \cap G_1^2) \cup \ldots \cup E'(E' \cap G_n^2) \in \mathrm{stack}\,\Delta_H,$$

hence,

$$\Delta_H \subset E'(E' \cap G_1^2) \cup \ldots \cup E'(E' \cap G_n^2).$$

Letting $G := G_1 \cup \ldots \cup G_n$, this implies that

$$H \subset (E \cap E^{-1})(G_1) \cup \ldots \cup (E \cap E^{-1})(G_n) = (E \cap E^{-1})(G).$$

Indeed, let $h \in H$, then there exists $1 \leq i \leq n$ such that $(h,h) \in E'(E' \cap G_i^2)$, hence there exists $(x,y) \in E' \cap G_i^2$ such that $(h,x) \in E'$ and $(y,h) \in E'$, consequently $(h,x) \in E' \subset E$ and $(x,h) \in E' \circ E' \subset E$. As $x \in G_i$, it follows that $h \in (E \cap E^{-1})(G_i)$. Since it also holds that $\mathcal{G}_1 \cap \ldots \cap \mathcal{G}_n \subset \mathrm{stack}\,\Delta_{G_1} \cap \ldots \cap \mathrm{stack}\,\Delta_{G_n} = \mathrm{stack}\,\Delta_G$ and $\mathcal{G}_1, \ldots, \mathcal{G}_n \in \mathbb{L}$, and therefore $\mathrm{stack}\,\Delta_G \in \mathbb{L}$, it has been shown that $(*)$ holds.

$\boxed{5 \Rightarrow 1}$ Let $\mathcal{H} \in \mathbf{S}_q(X,H)$ $(H \subset X)$ be such that $\mathcal{H} \in \mathrm{cl}^u_X(\mathbb{L} \cap \mathbf{S}_q(X))$. From the definition of cl^u_X and the fact that $\mathcal{H} \subset \mathrm{stack}\,\Delta_H$, it follows easily that $\mathrm{stack}\,\Delta_H \in \mathrm{cl}^u_X(\mathbb{L} \cap \mathbf{S}_q(X))$, hence, by (5), stack $\Delta_H \in \mathbb{L}$, consequently, again by (5), $q\mathbf{Unif}(\mathbb{L})_{|H} \in \mathbb{L}$. It will therefore suffice to show that $q\mathbf{Unif}(\mathbb{L})_{|H} \subset \mathcal{H}$. To this end, let $E \in q\mathbf{Unif}(\mathbb{L})$ and $\mathcal{W} \in \mathbf{U}(\mathcal{H})$. As $q\mathbf{Unif}(\mathbb{L})$ is a quasi-uniformity, there exists $E' \in q\mathbf{Unif}(\mathbb{L})$ such that $E' \circ E' \circ E' \subset E$. It then follows from $\mathcal{H} \in \mathrm{cl}^u_X(\mathbb{L} \cap \mathbf{S}_q(X))$ that there exists $\mathcal{G} \in \mathbb{L}$ such that $E'(\mathcal{G}) \subset \mathcal{W}$. Also, $\mathcal{G} \in \mathbb{L}$ implies that $q\mathbf{Unif}(\mathbb{L}) \subset \mathcal{G}$, in particular, $E' \in \mathcal{G}$ and therefore $E'(E') \subset E \in \mathcal{W}$. Thus, it has been shown for all $\mathcal{W} \in \mathbf{U}(\mathcal{H})$ that $q\mathbf{Unif}(\mathbb{L}) \subset \mathcal{W}$, consequently, $q\mathbf{Unif}(\mathbb{L}) \subset \mathcal{H}$ and also $q\mathbf{Unif}(\mathbb{L})_{|H} \subset \mathcal{H}$ (as $H \times H \in \mathcal{H}$). ☐

Remark 4.6. *(1) The description given in (4) in the previous proposition is a quasi analogue of the one given for* CCTH(**Unif**) *by Alderton and Schwarz in [3] or by Behling in [4] (which is also mentioned lateron, see (4) in proposition 4.10). As such, it is also a property (among additional ones) determining a category* **BQUGPULim** *that Behling considers in [4] to show that*

$$\mathrm{CCTH}(q\mathbf{Unif}) \subset \mathbf{BQUGPULim} \subsetneqq \mathrm{TUH}(q\mathbf{Unif})$$

(the latter refers to the topological universe hull), where it is now shown that $q\mathbf{EpiUnif} = \mathrm{CCTH}(q\mathbf{Unif}) = \mathbf{BQUGPULim}$*, which can also be described by any one of the previously stated equivalent characterizations.*

(2) Verifying whether one of those conditions is satisfied can be assisted by the following observation: given $(X, \mathbb{L}) \in \mathbf{qsug}\text{-}q\mathbf{SULim}$ *and* $H \subset X$*, the following are equivalent:*

- *(a)* $\mathrm{stack}\,\Delta_H \in \mathrm{cl}^u_X(\mathbb{L} \cap \mathbf{S}_q(X))$.
- *(b)* $\mathrm{stack}\,\Delta_H \in \mathrm{cl}^u_X(\{\mathrm{stack}\,\Delta_G \mid \mathrm{stack}\,\Delta_G \in \mathbb{L}\})$.
- *(c)* $\mathrm{stack}\,\Delta_H \in \mathrm{cl}_X(\mathbb{L} \cap \mathbf{S}_q(X))$.
- *(d)* $\mathrm{stack}\,\Delta_H \in \mathrm{cl}_X(\{\mathrm{stack}\,\Delta_G \mid \mathrm{stack}\,\Delta_G \in \mathbb{L}\})$.
- *(e)* $\forall E \subset q\mathbf{Unif}(\mathbb{L}), \exists G \subset X : \mathrm{stack}\,\Delta_G \in \mathbb{L}$ *and* $H \subset (E \cap E^{-1})(G)$.

Indeed, $\boxed{4 \Rightarrow 5}$ *in the foregoing proof actually shows that (a) implies (e), while* $\boxed{3 \Rightarrow 4}$ *in the foregoing proof shows that (e) implies (d). Combining this with the observation that (d) ⇒ (c) ⇒ (a) and (d) ⇒ (b) ⇒ (a) proves the required equivalence.*

By design, CCTH(q**Unif**) $\subset q$**SULim**, but it is possible to show more.

PROPOSITION 4.7. q**EpiUnif** $\subset q$**ULim**.

Proof. Let $(X, \mathbb{L}) \in q$**EpiUnif** and let $\mathcal{F}, \mathcal{G} \in \mathbb{L}$. As $(X, \mathbb{L}) \in$ **qsug-**q**SULim**, there exist $\mathcal{H}_F \in \mathbb{L} \cap \mathbf{S}_q(X, F)$ and $\mathcal{H}_G \in \mathbb{L} \cap \mathbf{S}_q(X, G)$ such that $\mathcal{H}_F \subset \mathcal{F}$ and $\mathcal{H}_G \subset \mathcal{G}$. In particular, stack $\Delta_F \in \mathbb{L}$ and stack $\Delta_G \in \mathbb{L}$, hence stack $\Delta_{F\cup G} \in \mathbb{L}$, consequently, by the previous proposition, $q\mathbf{Unif}(\mathbb{L})_{|F\cup G} \in \mathbb{L}$.

Also,

$$\mathcal{F} \supset \mathcal{H}_F \supset q\mathbf{Unif}(\mathbb{L})_{|F} \supset q\mathbf{Unif}(\mathbb{L})_{|F\cup G}$$

and

$$\mathcal{G} \supset \mathcal{H}_G \supset q\mathbf{Unif}(\mathbb{L})_{|G} \supset q\mathbf{Unif}(\mathbb{L})_{|F\cup G},$$

hence

$$\mathcal{F} \circ \mathcal{G} \supset q\mathbf{Unif}(\mathbb{L})_{|F\cup G} \circ q\mathbf{Unif}(\mathbb{L})_{|F\cup G} = q\mathbf{Unif}(\mathbb{L})_{|F\cup G} \in \mathbb{L}.$$

□

Next, these results are "restricted" down to results regarding the CCT hull of **Unif**.

DEFINITION 4.8. *Define* **EpiUnif** $:= q$**EpiUnif** $\cap$ **SULim**.

LEMMA 4.9. *Let* $(X, \mathbb{L}) \in$ **SULim** *and* $\mathcal{H} \in \mathbf{S}_q(X)$, *then*

1. $\mathcal{H} \in \mathrm{cl}^u_X(\mathbb{L} \cap \mathbf{S}_q(X)) \Longleftrightarrow \mathcal{H} \cap \mathcal{H}^{-1} \in \mathrm{cl}^u_X(\mathbb{L} \cap \mathbf{S}(X))$.
2. $\mathcal{H} \in \mathrm{cl}_X(\mathbb{L} \cap \mathbf{S}_q(X)) \Longleftrightarrow \mathcal{H} \cap \mathcal{H}^{-1} \in \mathrm{cl}_X(\mathbb{L} \cap \mathbf{S}(X))$.

Proof. In both cases, the implication $\boxed{\Leftarrow}$ is easily verified.

$\boxed{1, \Rightarrow}$ Let $E \in q\mathbf{Unif}(\mathbb{L}) = \mathbf{Unif}(\mathbb{L})$ (by proposition 2.2), hence, it can be assumed that E is symmetric. Also let $\mathcal{W} \in \mathbf{U}(\mathcal{H} \cap \mathcal{H}^{-1})$, hence either $\mathcal{W} \in \mathbf{U}(\mathcal{H})$ or $\mathcal{W} \in \mathbf{U}(\mathcal{H}^{-1})$, and in the latter case, $\mathcal{W}^{-1} \in \mathbf{U}(\mathcal{H})$. As $\mathcal{H} \in \mathrm{cl}^u_X(\mathbb{L} \cap \mathbf{S}_q(X))$, there exists $\mathcal{G} \in \mathbb{L} \cap \mathbf{S}_q(X))$ such that $E(\mathcal{G} \cap \mathcal{G}^{-1}) \subset E(\mathcal{G}) \subset \mathcal{W}$ (in the first case) or $E(\mathcal{G} \cap \mathcal{G}^{-1}) \subset E(\mathcal{G}) \subset \mathcal{W}^{-1}$ (in the latter case). In either case, it also holds that $\mathcal{G} \cap \mathcal{G}^{-1} \in \mathbb{L} \cap \mathbf{S}(X)$ (as $(X, \mathbb{L}) \in$ **SULim**) and $E(\mathcal{G} \cap \mathcal{G}^{-1})^{-1} =$

$E(\mathcal{G}\cap\mathcal{G}^{-1})$ (since E and $\mathcal{G}\cap\mathcal{G}^{-1}$ are symmetric). Thus, in either case, $E(\mathcal{G}\cap\mathcal{G}^{-1})\subset\mathcal{W}$, which shows that $\mathcal{H}\cap\mathcal{H}^{-1}\in \mathrm{cl}^u_X(\mathbb{L}\cap\mathbf{S}(X))$.

$\boxed{2,\Rightarrow}$ Let $E\in q\mathbf{Unif}(\mathbb{L})=\mathbf{Unif}(\mathbb{L})$ (by proposition 2.2), hence, again assume E to be symmetric. As $\mathcal{H}\in\mathrm{cl}_X(\mathbb{L}\cap\mathbf{S}_q(X))$, there exists $\mathcal{G}\in\mathbb{L}\cap\mathbf{S}_q(X))$ such that $E(\mathcal{G}\cap\mathcal{G}^{-1})\subset E(\mathcal{G})\subset\mathcal{H}$. Since it also holds that $\mathcal{G}\cap\mathcal{G}^{-1}\in\mathbb{L}\cap\mathbf{S}(X)$ (as $(X,\mathbb{L})\in\mathbf{SULim}$) and $E(\mathcal{G}\cap\mathcal{G}^{-1})^{-1}=E(\mathcal{G}\cap\mathcal{G}^{-1})$ (since E and $\mathcal{G}\cap\mathcal{G}^{-1}$ are symmetric), it follows that $E(\mathcal{G}\cap\mathcal{G}^{-1})\subset\mathcal{H}\cap\mathcal{H}^{-1}$, which shows that $\mathcal{H}\cap\mathcal{H}^{-1}\in \mathrm{cl}_X(\mathbb{L}\cap\mathbf{S}(X))$. □

PROPOSITION 4.10. *Given* $(X,\mathbb{L})\in$ **sug-SULim***, the following are equivalent to the statement that* $(X,\mathbb{L})\in\mathbf{EpiUnif}=q\mathbf{EpiUnif}\cap\mathbf{SULim}$*:*

1. $\mathbf{S}(X)\cap\mathrm{cl}^u_X(\mathbb{L}\cap\mathbf{S}(X))=\mathbb{L}\cap\mathbf{S}(X)$.
2. $\mathbf{S}(X)\cap\mathrm{cl}_X(\mathbb{L}\cap\mathbf{S}(X))=\mathbb{L}\cap\mathbf{S}(X)$.
3. $\forall H\subset X:\big(\text{stack }\Delta_H\in\mathrm{cl}_X(\mathbb{L}\cap\mathbf{S}(X))\Rightarrow\text{stack }\Delta_H\in\mathbb{L}\big)$ *and* $\forall H\subset X:\big(\text{stack }\Delta_H\in\mathbb{L}\Rightarrow\mathbf{Unif}(\mathbb{L})_{|H}\in\mathbb{L}\big)$ *(where* $\mathbf{Unif}(\mathbb{L})_{|H}$ *is the restriction of* $\mathbf{Unif}(\mathbb{L})$ *to* $H\times H$*).*
4. *If* $H\subset X$ *is such that*
$$\forall E\subset\mathbf{Unif}(\mathbb{L}),\exists G\subset X:\text{stack }\Delta_G\in\mathbb{L}\ \textit{and}\ H\subset E(G),$$
then stack $\Delta_H\in\mathbb{L}$.
Also, $\forall H\subset X:\big(\text{stack }\Delta_H\in\mathbb{L}\Rightarrow\mathbf{Unif}(\mathbb{L})_{|H}\in\mathbb{L}\big)$.
5. $\forall H\subset X:\big(\text{stack }\Delta_H\in\mathrm{cl}^u_X(\mathbb{L}\cap\mathbf{S}(X))\Rightarrow\text{stack }\Delta_H\in\mathbb{L}\big)$ *and* $\forall H\subset X:\big(\text{stack }\Delta_H\in\mathbb{L}\Rightarrow\mathbf{Unif}(\mathbb{L})_{|H}\in\mathbb{L}\big)$.

Proof. By definition of $\mathbf{EpiUnif}=q\mathbf{EpiUnif}\cap$ **sug-SULim**, it will suffice to show that given $(X,\mathbb{L})\in$ **sug-SULim**, each of the items mentioned here is equivalent to the respective item of proposition 4.5.

Let us consider (1) for example. Clearly, (1) of proposition 4.5 implies the present (1). Conversely, assume that the present (1) holds and let $\mathcal{H}\in\mathrm{cl}^u_X(\mathbb{L}\cap\mathbf{S}_q(X))$ (where $\mathcal{H}\in\mathbf{S}_q(X)$). It then follows from the foregoing lemma that $\mathcal{H}\cap\mathcal{H}^{-1}\in\mathrm{cl}^u_X(\mathbb{L}\cap\mathbf{S}(X))$, hence, by the present (1), $\mathcal{H}\cap\mathcal{H}^{-1}\subset\mathcal{H}\in\mathbb{L}$.

The remaining cases can be proven analogously (by also observing that $q\mathbf{Unif}(\mathbb{L}) = \mathbf{Unif}(\mathbb{L})$ (by proposition 2.2) and that it is sufficient in (4) to consider $E \in \mathbf{Unif}(\mathbb{L})$ that are symmetric). □

Theorem 4.11 ([2], [3], [4]). **EpiUnif** *is the cartesian closed topological hull of* **Unif**.

Proof. It is shown in [3] and [4] (by using [2]) that the CCT hull of **Unif** is the full subconstruct of $\mathbf{ULim} \cap \mathbf{sug\text{-}SULim}$ whose objects satisfy the property stated in (4) of the previous proposition. Hence, by the previous proposition and the fact that $\mathbf{EpiUnif} = q\mathbf{EpiUnif} \cap \mathbf{SULim} \subset q\mathbf{ULim} \cap \mathbf{SULim} = \mathbf{ULim}$, it follows that **EpiUnif** is the CCT hull of **Unif**. □

The restriction also behaves nicely in the following sense (cfr. [4, proposition 2.4.8]).

Proposition 4.12. **EpiUnif** *is bireflective and bicoreflective in* q**EpiUnif**, *hence*

$$\begin{array}{ccccccc}
q\mathbf{Unif} & \xrightarrow{r} & q\mathbf{EpiUnif} & \xrightarrow{r} & \mathbf{qsug\text{-}}q\mathbf{SULim} & \xrightarrow{c} & q\mathbf{SULim} \\
r \Big\uparrow c & & r \Big\uparrow c & & r \Big\uparrow c & & r \Big\uparrow c \\
\mathbf{Unif} & \xrightarrow{r} & \mathbf{EpiUnif} & \xrightarrow{r} & \mathbf{sug\text{-}SULim} & \xrightarrow{c} & \mathbf{SULim}
\end{array}$$

Proof. The first claim is clear and as for the latter claim, it suffices to show that the bicoreflector $C_s : q\mathbf{SULim} \longrightarrow \mathbf{SULim}$ restricts to a **EpiUnif**-bicoreflector in q**EpiUnif**.

To this end, let $(X, \mathbb{L}) \in q\mathbf{EpiUnif}$, then it follows from proposition 2.2 that $(X, \mathbb{L}') := C_s(X, \mathbb{L}) \in \mathbf{sug\text{-}SULim}$. Next, let $\mathcal{H} \in \mathbf{S}(X) \cap \mathrm{cl}_{(X,\mathbb{L}')}(\mathbb{L}' \cap \mathbf{S}(X))$, then also $\mathcal{H} \in \mathrm{cl}_{(X,\mathbb{L})}(\mathbb{L} \cap \mathbf{S}(X))$ (as $1_X : (X, \mathbb{L}') \longrightarrow (X, \mathbb{L})$ is uniformly continuous and by proposition 3.7), hence $\mathcal{H} \in \mathbb{L}$ and therefore $\mathcal{H} \in \mathbb{L}'$ (since $\mathcal{H}$ is symmetric). □

Remark 4.13. *(1) While the authors of [3] worked under the "roof" of* **ULim**, (q)**SULim** *was chosen as a superconstruct here, since it allows easier working in the larger (and more convenient) superconstruct* (q)**SULim** *(as* q**ULim** *is shown not to be cartesian closed in [4]). However, in both cases, the situation turns out to "collapse" into* (q)**ULim** *(by itself, so to speak).*

(2) As mentioned earlier, the description given in (4) was obtained (by Alderton and Schwarz [3] and Behling [4]) by using a concrete isomorphism to place the CCT hull of **Unif** *into* **ULim** *(or* **SULim** *for that matter). However, in this case one can proceed in reverse order. By considering the correspondence indicated in [4, remark 2.4.7.3] (or noted just before [3, theorem 3.14]) in a quasi setting, one easily finds a concrete isomorphism between q***EpiUnif** *and a quasi analogue of the description of Adámek and Reiterman in [2].*

More precisely, one then obtains $\mathrm{CCTH}(q\mathbf{Unif})$ *as the category of quasi-bornological uniform spaces* $(X,\mathcal{U},\mathcal{A})$*, where* $(X,\mathcal{U})$ *is a quasi-uniform space and* $\mathcal{A}$ *is a bornology on* X*, i.e.* $\mathcal{A} \subset \mathcal{P}(X)$ *(elements of* $\mathcal{A}$ *are called bounded sets) such that (i) each finite subset is in* $\mathcal{A}$*, (ii) if* $P \in \mathcal{A}$ *and* $Q \subset P$*, then* $Q \in \mathcal{A}$ *and (iii) if* $P, Q \in \mathcal{A}$*, then* $P \cup Q \in \mathcal{A}$*, such that*

1. $(1_X : (X,\mathcal{U})_{|B} \longrightarrow (X,\mathcal{U}))_{B\in\mathcal{A}}$ *is final in* $q\mathbf{Unif}$.
2. $\mathcal{A}$ *is* $\mathcal{U}$*-closed, i.e.* $\mathcal{A}$ *contains each set* $M \subset X$ *with the property that*

$$\forall E \subset \mathcal{U}, \exists G \in \mathcal{A} : M \subset (E \cap E^{-1})(G).$$

Let $q\mathbf{BUnif}$ *denote the category of quasi-bornological uniform spaces whose morphisms from* $(X,\mathcal{U}_X,\mathcal{A}_X)$ *to* $(Y,\mathcal{U}_Y,\mathcal{A}_Y)$ *are those uniformly continuous maps* $f : (X,\mathcal{U}_X) \longrightarrow (Y,\mathcal{U}_Y)$ *which preserve bounded sets, i.e.* $P \in \mathcal{A}_X$ *implies* $f(P) \in \mathcal{A}_Y$.

The correspondence is given by

$$q\mathbf{EpiUnif} \longrightarrow q\mathbf{BUnif}:$$

$$(X,\mathbb{L}) \mapsto \big(X, q\mathbf{Unif}(\mathbb{L}), \{G \subset X \mid \mathrm{stack}\,\Delta_G \in \mathbb{L}\}\big),$$

$$q\mathbf{BUnif} \longrightarrow q\mathbf{EpiUnif}:$$

$$(X,\mathcal{U},\mathcal{A}) \mapsto \big(X, \{\mathcal{F} \in \mathbf{F}(X^2) \mid \exists H \in \mathcal{A} : \mathcal{U}_{|H} \subset \mathcal{F}\}\big).$$

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