THE BEHAVIOUR OF BRANCHES OF SOLUTIONS OF NON-LINEAR EIGENVALUE PROBLEMS (*)

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SOMMARIO. - *Si considera il comportamento globale di rami di soluzioni di un problema di autovalori nonlineare. Questi rami si biforcano dagli autovalori sottostanti allo spettro essenziale della linearizzazione. Si danno esempi che mostrano come essi si comportano al tendere degli autovalori allo spettro essenziale.*

SUMMARY. - *We consider the global behaviour of branches of solutions of a nonlinear eigenvalue problem. These branches bifurcate from eigenvalues lying below the essential spectrum of the linearisation. Examples are given showing how they can behave as they approach the essential spectrum.*

1. - Introduction.

We consider equations of the form:

$$F(u) = \lambda u \text{ for } (\lambda, u) \in \mathbf{R} \times H \tag{1.1}$$

(*) Pervenuto in Redazione il 28 marzo 1987.
This paper was presented at the «Colloquium on Topological Methods in BVP's for ODE's», held at the International School for Advanced Studies (S.I.S.S.A.), Trieste, May 1984.

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where H is a real Hilbert space and F is a densely defined mapping with $F(0)=0$. In a certain sense, $F'(0)\equiv S$ is a self-adjoint operator acting in H and Q denotes the infimum of the essential spectrum of S. We consider the global behaviour of branches of non-trivial $(u\neq 0)$ solutions of (1.1) which bifurcate from eigenvalues of S lying below Q.

In section 2, we recall a basic result concerning the global behaviour of such branches. By introducing some additional structure we are led to hypotheses which yield much more precise information about these branches. In section 3, we suppose that

$$F(u)=[S+N(u)]\,u$$

where $N(u)$ is a self-adjoint operator in H for each fixed element u in the domain of F. Thus (1.1) becomes

$$[S+N(u)]\,(u)=u\,, \tag{1.2}$$

and it seems natural to introduce hypotheses about $N(u)$ similar to those commonly used in perturbation theory. In this context we give hypotheses which ensure that branches of solutions extend at least to $\lambda=Q$. It is often the case that (1.1) does not have any non-trivial solutions in the region $\lambda>Q$ and so we seek to determine the behaviour of branches of solutions as λ approaches Q from below. In section 4, we present three examples which represent typical situations of this kind. Finally an appendix clarifies some technical points arising from the assumptions of sections 2 and 3.

2. - A basic result.

Let H be a real Hilbert space with norm, $\|\cdot\|$, and scalar product $\langle .,.\rangle$.

A1) Let $S:\mathfrak{D}(S)\subset H\to H$ be a self-adjoint operator which is bounded below.

The spectrum of S is denoted by $\sigma(S)$ and $\Lambda\equiv\inf\sigma(S)$. The essential spectrum of S is denoted by $\sigma_e(S)$ and $Q=\inf\sigma_e(S)$. If $\sigma_e(S)=\phi$, we set $Q=+\infty$.

A2) Suppose that $\Lambda<Q$.

The set $\mathfrak{D}(S)$ equipped with the graph norm of S:

$$\|u\|_2\equiv\{\|u\|^2+\|Su\|^2\}^{1/2},$$

is a real Hilbert space which we denote by H_2. Since

$$\|u\|\leqslant\|u\|_2\ \forall u\in H_2,$$

H_2 is continuously and densely embedded in H (denoted by $H_2\hookrightarrow H$)

but the injection is compact if and only if $Q = +\infty$. Furthermore, S maps H_2 continuously into H.

A3) Let $M : H_2 \to H$ be a continuous compact mapping such that

$$\lim_{\|u\|_2 \to 0} \|M(u)\| / \|u\|_2 = 0 .$$

We consider the equation,

$$Su + M(u) = \lambda u \text{ for } (\lambda, u) \in \mathbf{R} \times H_2 . \tag{2.1}$$

Clearly $(\lambda, 0)$ is a solution for all $\lambda \in \mathbf{R}$, so we set

$$\mathcal{S} = \{ (\lambda, u) \in \mathbf{R} \times H_2 : \lambda < Q ,\ u \neq 0 \text{ and } Su + M(u) = \lambda u \}.$$

For $A \subset \mathcal{S}$, we set $PA = \{\lambda \in \mathbf{R} : \exists u \in H_2 \text{ with } (\lambda, u) \in A\}$. For $\mu \in \sigma(S) \cap (-\infty, Q)$, let $\mathcal{C}_\mu$ denote the component of $\mathcal{S} \cup \{(\mu, 0)\}$ containing $(\mu, 0)$ and let $\overline{\mathcal{C}}_\mu$ denote its closure $\mathbf{R} \times H_2$. These statements refer to the topology on $\mathcal{S} \cup \{(\mu, 0)\}$ induced from $\mathbf{R} \times H_2$.

THEOREM 2.1 - *Let $\mu < Q$ with $\dim ker(S - \mu I)$ being odd. Then $\mathcal{C}_\mu$ satisfies at least one of the following conditions.*

i) *$\mathcal{C}_\mu$ is unbounded in $\mathbf{R} \times H_2$.*

ii) $\sup P\mathcal{C}_\mu = Q$.

iii) *$\exists \nu \in \sigma(S) \cap (-\infty, Q) \setminus \{\mu\}$ such that $(\nu, 0) \in \overline{\mathcal{C}}_\mu$.*

Proof. This result is a generalisation of a famous theorem due to Rabinowitz [1]. In the above setting it is established in [2].

3. - Additional structure and sharper conclusions.

From now on we shall assume that the function M in (2.1) has the special form:

$$M(u) = N(u)\, u \text{ for } u \in H_2 , \tag{3.1}$$

where, for each fixed element $u \in H_2$, $N(u)$ is a symmetric linear operator from H_2 into H. Thus the equation (2.1) takes the form

$$[S + N(u)]\, u = \lambda u \text{ for } (\lambda, u) \in \mathbf{R} \times H_2 , \tag{3.2}$$

and we regard $N(u)$ as a perturbation of S.

As in perturbation theory we need the so called form domain associated with S. Let

$$q(u, v) \equiv \langle u, v \rangle + \langle (S - \Lambda I)\, u, v \rangle \text{ for } u, v \in \mathfrak{D}(S)$$

and let H_1 denote the completion of $\mathfrak{D}(S)$ with respect to the norm

$\|u\|_1 \equiv q(u,u)^{1/2}$. Then H_1 is a real Hilbert space which can be characterised as follows:

$$H_1 = \mathfrak{D}((S - \Lambda I)^{1/2}) \text{ and}$$
$$\|u\|_1 = \{\|u\|^2 + \|(S - \Lambda I)^{1/2} u\|^2\}^{1/2} \text{ for } u \in H_1,$$

where $(S - \Lambda I)^{1/2}$ is the (unique) positive self-adjoint square root of $S - \Lambda I$ and $\|\cdot\|_1$ is its graph norm. (See [3,4]). Clearly $H_2 \hookrightarrow H_1 \hookrightarrow H$ and the injections are compact if and only if $Q = +\infty$. For $i = 1,2$, let $B(H_i,H)$ be the space of all bounded linear operators from H_i into H and set

$$|||T|||_i = \sup\{\frac{\|Tu\|}{\|u\|_i} : u \in H_i\backslash\{0\}\} \text{ for } T \in B(H_i,H).$$

Clearly $B(H_1,H) \subset B(H_2,H)$ and $|||T|||_2 \leqslant |||T|||_1 \ \forall T \in B(H_1,H)$.

Let $K(H_2,H) \equiv \{T \in B(H_2,H) : T$ is compact, symmetric and positive$\}$, where $T \in B(H_2,H)$ is said to be:

compact $\Leftrightarrow \overline{T(W)}$ is compact in H whenever W is bounded in H_2,
symmetric $\Leftrightarrow \langle Tu,v\rangle = \langle Tv,u\rangle \ \forall u,v \in H_2$,
positive $\Leftrightarrow \langle Tu,u\rangle \geqslant 0 \ \forall u \in H_2$.

We note that $K(H_2,H)$ is a closed subset of $B(H_2,H)$ and that for $T \in K(H_2,H)$:

i) $\forall \varepsilon > 0 \ \exists c(\varepsilon) > 0$ such that

$$\|Tw\| \leqslant \varepsilon\|Sw\| + c(\varepsilon)\|w\| \quad \forall w \in H_2 \tag{3.3}$$

ii) $S + T$ is a self-adjoint operator in H with domain equal to H_2.

iii) $\sigma(S+T) \subset [\Lambda, \infty)$.

iv) $\sigma_e(S+T) = \sigma_e(S)$.

The property (i) means that T is S-bounded with S-bound zero and follows from the fact $T : H_2 \to H$ is compact ([5] Theorem 9.7). Proerties (ii) to (iv) then follow from [5] Theorem 9.9 since T is symmetric and positive on $\mathfrak{D}(S)$.

We now come to the additional assumptions concerning the operator N in (3.1) that will enable us to sharpen the conclusion of Theorem 2.1.

A4) Let $N : H_1 \to B(H_1,H)$ be a bounded mapping such that

$$N(0) = 0,$$
$$N(u) \in K(H_2,H) \text{ for all } u \in H_2$$
and $$N : H_2 \to B(H_2,H) \text{ is continuous and compact.}$$

A5) Let $f(u) \equiv \langle N(u)u,u\rangle$ for $u \in H_1$. Suppose that

a) $f: H_1 \to \mathbf{R}$ is weakly lower semi-continuous and

b) $\exists\, q > 2$ such that $f(tu) \geqslant t^q f(u) > 0$ for $u \in H_1 \backslash \{0\}$ and $t \geqslant 1$.

The implications of these assumptions for the solutions of (3.2) are established in the following lemmas which lead up to Corollary 3.4.

LEMMA 3.1 - *Let S satisfy* (A1) *and N satisfy* (A4).

i) *The mapping M defined by* (3.1) *satisfies* (A3) *and* $P\mathcal{S} \subset [\Lambda, \infty)$.

ii) *Let $W \subset \mathcal{S}$. Then W is bounded in $\mathbf{R} \times H$ if and only if W is bounded in $\mathbf{R} \times H_2$.*

Proof. i) For $u, v \in H_2$, we have that

$$\| M(u) - M(v) \| \leqslant \| [N(u) - N(v)]\, u \| + \| N(v) [u - v] \|$$
$$\leqslant ||| N(u) - N(v) |||_2 \, \| u \|_2 + ||| N(v) |||_2 \, \| u - v \|_2 .$$

Thus $M: H_2 \to H$ is continuous and, putting $v = 0$, we have

$$\lim \frac{\| M(u) \|}{\| u \|_2} \leqslant \lim ||| N(u) |||_2 = 0 \quad \text{as} \quad \| u \|_2 \to 0 .$$

For the compactness of M, we suppose that $\{u_n\}$ is a bounded sequence in H_2. We must show that there is a subsequence $\{u_{n_k}\}$ such that $\{M(u_{n_k})\}$ converges in H.

Using (A4), we can find a subsequence $\{u_{n_k}\}$ such that:

$$u_{n_k} \rightharpoonup u \quad \text{weakly in} \quad H_2$$

and
$$N(u_{n_k}) \to T \quad \text{strongly in} \quad B(H_2, H).$$

Furthermore $T \in B(H_2, H)$ is compact and so

$$T u_{n_k} \to T u \quad \text{strongly in} \quad H .$$

Hence $\| M(u_{n_k}) - Tu \| \leqslant \| [N(u_{n_k}) - T]\, u_{n_k} \| + \| T u_{n_k} - Tu \|$
$$\leqslant ||| N(u_{n_k}) - T |||_2 \, \| u_{n_k} \|_2 + \| T u_{n_k} - Tu \| ,$$

proving that $M(u_{n_k})$ converges strongly to Tu in H.

Finally if $(\lambda, u) \in \mathcal{S}$, then

$$\langle Su, u \rangle + \langle N(u)\, u, u \rangle = \lambda \| u \|^2 . \tag{3.4}$$

Thus $\lambda \geqslant \Lambda$ since $N(u)$ is positive.

ii) Suppose that $W \subset \mathcal{S}$ is bounded in $\mathbf{R} \times H$. From (3.4), it follows that

$$\Lambda \| u \|^2 \leqslant \langle Su, u \rangle \leqslant \lambda \| u \|^2 \quad \text{for} \quad (\lambda, u) \in \mathcal{S}$$

and hence we have that W is bounded in $\mathbf{R} \times H_1$.
But for $(\lambda, u) \in \mathcal{S}$,

$$\begin{aligned} \| Su \| &\leqslant |\lambda| \; \| u \| + \| N(u)\, u \| \\ &\leqslant |\lambda| \; \| u \| + ||| N(u) |||_1 \, \| u \|_1 . \end{aligned}$$

From the boundedness of the mapping $N : H_1 \to B(H_1, H)$ it now follows that W is bounded in $\mathbf{R} \times H_2$.

LEMMA 3.2 - *Let S satisfy* (A1). *Suppose that $\{w_n\} \subset \mathfrak{D}(S)$ is such that $w_n \rightharpoonup 0$ weakly in H as $n \to \infty$ and $\| w_n \| = 1$ for all $n \in \mathbf{N}$. Then*

$$\liminf_{n \to \infty} \langle Sw_n, w_n \rangle \geqslant Q \textit{ where } Q = \inf \sigma_e(S).$$

Proof. Let $\{E(\lambda) : \in \mathbf{R}\}$ be the resolution of the identity corresponding to S. Thus

$$S = \int_{\mathbf{R}} \lambda dE(\lambda) = \int_{[\Lambda, \infty)} \lambda dE(\lambda).$$

For $a < Q$, let P denote the orthogonal projection defined by

$$P = \int_{[\Lambda, a]} dE(\lambda).$$

Then $\dim \operatorname{Im} P < \infty$ and so $\lim_{n \to \infty} \| Pw_n \| = 0$.

Furthermore
$$Pu \in \mathfrak{D}(S) \text{ for all } u \in H,$$
$$SPu = PSu \text{ for all } u \in \mathfrak{D}(S),$$

and

$$\langle S[I-P]\,u, (I-P)\,u \rangle \geqslant a \, \| (I-P)\, u \|^2 \quad \forall u \in \mathfrak{D}(S).$$

Hence
$$\begin{aligned} \langle Sw_n, w_n \rangle &= \langle SPw_n, Pw_n \rangle + \langle S(I-P)\, w_n, (I-P)\, w_n \rangle \\ &\geqslant - \| SPw_n \| \; \| Pw_n \| + a \, \| (I-P)\, w_n \|^2 \\ &\geqslant - \max\{ |\Lambda|, a \} \| Pw_n \|^2 + a \{ \| w_n \|^2 - \| Pw_n \|^2 \} \end{aligned}$$

and so
$$\liminf_{n \to \infty} \langle Sw_n, w_n \rangle \geqslant a .$$

LEMMA 3.3 - *Let the assumptions* (A1) *to* (A5) *hold. Then, for each $a < Q$, the set $\mathcal{S} \cap \{ (-\infty, a] \times H_2 \}$ is bounded in $\mathbf{R} \times H_2$.*

Proof. Let us suppose that the set $\mathcal{S} \cap \{ (-\infty, a] \times H_2 \}$ is unbounded. Since $P\mathcal{S} \subset [\Lambda, \infty)$, it follows from Lemma 3.1 (ii) that $\exists$ a sequence

$$\{ (\lambda_n, u_n) \} \in \mathcal{S} \text{ such that } \Lambda \leqslant \lambda_n \leqslant a \text{ and } \| u_n \| \to \infty \text{ as } n \to \infty .$$

Setting
$$w_n = \frac{u_n}{\| u_n \|} \text{ we have that } \| w_n \| = 1$$

and $$\langle Sw_n, w_n \rangle + \frac{f(\| u_n \| w_n)}{\| u_n \|^2} = \lambda_n \text{ for all } u \in \mathbf{N}.$$

But $f(\| u_n \| w_n) \geqslant \| u_n \|^q f(w_n) > 0$ since $\| u_n \| \geqslant 1$ (without loss of generality and we have that

$$\Lambda \leqslant \langle Sw_n, w_n \rangle \leqslant \lambda_n \leqslant a \tag{3.5}$$

and $$\| u_n \|^{q-2} f(w_n) \leqslant \lambda_n - \Lambda \leqslant a - \Lambda. \tag{3.6}$$

From (3.5) we see that $\{w_n\}$ is bounded in H_1 and so by passing to a subsequence we can suppose that $w_n \rightharpoonup w$ weakly in H_1. By (A5),

$$f(w) \leqslant \liminf_{n \to \infty} f(w_n).$$

Since $\| u_n \| \to \infty$ and $q > 2$, it follows from (3.6) that $\lim_{n \to \infty} f(w_n) = 0$. Thus $f(w) = 0$ and, by (A5), this implies that $w = 0$. This means that $w_n \to 0$ weakly in H_1 (and hence in H) as $n \to \infty$ and $\| w_n \| = 1$ for all $n \in \mathbf{N}$.

From Lemma 3.2, we conclude that

$$Q \leqslant \liminf_{n \to \infty} \langle Sw_n, w_n \rangle$$

and, returning to (3.5), we see that $a \geqslant Q$. This proves the lemma.

COROLLARY 3.4 - *Suppose that the hypotheses* (A1) *to* (A5) *are satisfied. Let* μ *and* C_μ *be as in Theorem* 2.1. *Then* C_μ *has at least one of the following properties.*

i) $\sup PC_\mu = Q$

ii) $\exists \nu \in \sigma(S) \cap (-\infty, Q) \setminus \{\mu\}$ *such that* $(\nu, 0) \in \overline{C_\mu}$.

Proof. This follows from Theorem 2.1 and Lemmas 3.1 and 3.3. We complete this section by introducing assumptions which eliminate possibility (ii) of Corollary 3.4.

We have already noted that under the summptions (A1) and (A4), for each $u \in H_2$,

$$\sigma_e(S + N(u)) = \sigma_e(S) \subset [Q, \infty).$$

A6) For each $u \in H_2$, the self-adjoint operator $S + N(u)$ has only simple eigenvalues in $(-\infty, Q)$.

REMARKS - 1. In this hypothesis, we admit the possibility the $\sigma(S + N(u)) \cap (-\infty, Q) = \phi$.

2. For $u \in H_2$, let $\lambda_1(u) < \lambda_2(u) < \ldots < \lambda_j(u) < \lambda_{j+1}(u) < \ldots$ denote the set of all eigenvalues of $S + N(u)$ in $(-\infty, Q)$. In particular, $\lambda_j(0)$ are the eigenvalues of S in $(-\infty, Q)$ and so $\lambda_1(0) = \Lambda$.

If $(\lambda, u) \in \mathcal{S}$ then $\lambda \in \sigma(S + N(u)) \cap (-\infty, Q)$ and so there exists $j(u) \in \mathbf{N}$ such that $\lambda = \lambda_{j(u)}(u)$. We claim that the mapping

$$(\lambda, u) \to j(u)$$

is continuous from $\mathcal{S}$ (with the topology induced from $\mathbf{R} \times H_2$) into $\mathbf{N}$ and consequently that it is constant on components of $\mathcal{S}$. The following lemmas establish this fact.

LEMMA 3.5 - *Let* (A1) *and* (A4) *be satisfied. Fix* $u \in H_2$ *and* $\varepsilon > 0$. *There exists* $\delta > 0$ *(depending on* u *and* ε*) such that*

$$\| [N(u) - N(v)] w \| \leqslant \varepsilon \{ \| w \| + \| [S + N(u)] w \| \} \ \forall w \in H_2$$

whenever $\| u - v \|_2 < \delta$.

Proof. Set $R = S + N(u)$.

Then
$$\| Sw \| \leqslant \| Rw \| + \| N(u) w \| \ \ \forall w \in H_2$$
and $\exists c > 0$ such that
$$\| N(u) w \| \leqslant \tfrac{1}{2} \| Sw \| + c \| w \| \ \ \forall w \in H_2.$$

The second inequality follows from (3.3) with $T = N(u)$ and $\varepsilon = \tfrac{1}{2}$. Hence

$$\| Sw \| \leqslant 2 \| Rw \| + 2c \| w \| \ \ \forall w \in H_2.$$

Putting $\alpha = 2(1 + c)$, it follows from (A4) that $\exists \delta > 0$ such that

$$||| N(u) - N(v) |||_2 < \frac{\varepsilon}{\alpha} \text{ whenever } \| u - v \|_2 < \delta.$$

Thus
$$\begin{aligned} \| [N(u) - N(v)] w \| &\leqslant \frac{\varepsilon}{\alpha} \| w \|_2 \leqslant \frac{\varepsilon}{\alpha} \{ \| w \| + \| Sw \| \} \\ &\leqslant \frac{\varepsilon}{\alpha} \{ (2c + 1) \| w \| + 2 \| Rw \| \} \\ &\leqslant \varepsilon \{ \| w \| + \| Rw \| \} \quad \forall w \in H_2. \end{aligned}$$

This proves the lemma.

LEMMA 3.6 - *Let* (A1) *to* (A6) *be satisfied. Fix* $u \in H_2$ *and choose* $a \in (-\infty, Q) \setminus \sigma(S + N(u))$. *Let* k *be the number of eigenvalues of* $S + N(u)$ *in* $(-\infty, a)$. *Then* $\exists \delta > 0$ *(depending on* u *and* a*) such that* $S + N(v)$ *has exactly* k *eigenvalues in* $(-\infty, a)$ *whenever* $\| u - v \|_2 < \delta$. *Furthermore for* $1 \leqslant j \leqslant k$,

$$| \lambda_j(u) - \lambda_j(v) | \to 0 \ \ as \ \ \| u - v \|_2 \to 0.$$

Proof. Let $\hat{\delta}(S + N(u), S + N(v))$ denote the gap between the operators $S + N(u)$ and $S + N(v)$ as defined in [3] IV § 2.4.

Given $\varepsilon \in (0, 1)$ it follows from Lemma 3.5 that $\exists \delta > 0$ such that

$$\| [N(u) - N(v)] w \| \leqslant \varepsilon \| w \| + \varepsilon \| [S + N(u)] w \| \ \ \forall w \in H_2$$

whenever $\| u - v \|_2 < \delta$. Thus, by Theorem 2.14 of [3] IV § 2.4,

$$\hat{\delta}(S + N(u),\ S + N(v)) \leqslant \frac{\sqrt{2}\varepsilon}{1-\varepsilon}$$

whenever $\| u - v \|_2 < \delta$.

The result now follows from Theorem 3.16 of [3] IV § 3.4. since

$$\sigma(S + N(v)) \subset [\Lambda, \infty) \quad \forall v \in H_2.$$

THEOREM 3.7 - *Let* (A1) *to* (A6) *hold and let* $\mu = \lambda_k(0)$.

Then i) $\lambda = \lambda_k(u)$ *for all* $(\lambda, u) \in C_\mu$,

ii) $PC_\mu = [\mu, Q)$,

iii) $(Q, 0) \notin \bar{C}_\mu$.

Proof. By Lemma 3.6, the mapping $(\lambda, u) \to j(u)$ is continuous from $\mathcal{S}$ to **R**. Since $(\mu, 0) \to k$, it follows that $j(u) = k\ \forall (\lambda, u) \in C_\mu$. This proves (i).

Similarly we see that if $(\nu, 0) \in \bar{C}_\mu$ then $\nu = \mu$. This proves (iii) and from Corollary 3.4 we deduce that $\sup PC_\mu = Q$.

From the positivity of $N(u)$ and the Courant-Weyl min-max characterisation [4, Vol. IV Theorem XIII.1] of the eigenvalues of $S + N(u)$, we have that $\lambda_k(u) \geqslant \lambda_k(0)$ for all $u \in H_2$. For $(\lambda, u) \in C_\mu$ we have just established that $\lambda = \lambda_k(u)$ and so we have that $\lambda \geqslant \lambda_k(0) = \mu$. This completes the proof.

REMARK - The assumption (A6) is quite restrictive. [It can be verified for equations involving second order ordinary differential operators or integral operators with oscillation kernels]. The following weaker assumption is appropriate for second order elliptic operators.

A6)* For each $u \in H_2$, the lowest eigenvalue of $S + N(u)$ in $(-\infty, Q)$ is simple.

REMARK - Again (A6)* admits the possibility that

$$\sigma(S + N(u)) \cap (-\infty, Q) = \phi.$$

Adapting slightly the above proofs we obtain the following result.

THEOREM 3.7* - *Let* (A1) *to* (A5) *and* (A6)* *hold. Then* C_Λ *has the following properties*

i) $\lambda = \lambda_1(u)$ *for all* $(\lambda, u) \in C_\Lambda$

ii) $PC_\Lambda = [\Lambda, Q)$

iii) $(Q, 0) \notin \bar{C}_\Lambda$.

Under the hypotheses of Theorem 3.7 or 3.7*, when $Q < \infty$, we would like to be able to distinguish been three primitive types of behaviour of a component C_μ near $\lambda = Q$.

(P1) $\lim_{\lambda \to Q} \| u \| = +\infty$ for $(\lambda, u) \in C_\mu$.

(P2) C_μ is bounded in $\mathbf{R} \times H$ but C_μ does not contain a sequence $\{(\lambda_n, u_n)\}$ converging in $\mathbf{R} \times H_2$ with $\lim_{n \to \infty} \lambda_n = Q$.

(P3) C_μ is bounded in $\mathbf{R} \times H$ and every sequence $\{(\lambda_n, u_n)\}$ in C_μ such that $\lim_{n \to \infty} \lambda_n = Q$ contains a subsequence converging in $\mathbf{R} \times H_2$.

In case (2), C_μ is bounded but not relatively compact in $\mathbf{R} \times H_2$, whereas in case (3), C_μ is relatively compact in $\mathbf{R} \times H_2$. To discuss these two cases, we consider a sequence $\{(\lambda_n, u_n)\} \subset C_\mu$ such that $\lim_{n \to \infty} \lambda_n = Q$ and $\{\| u_n \|\}$ is bounded. By Lemma 3.1 (ii), $\{\| u_n \|_2\}$ is bounded and hence $\exists$ a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \rightharpoonup u$ weakly in H_2 and $N(u_{n_k}) \to T$ strongly in $B(H_2, H)$. Since $T \in B(H_2, H)$ is compact, we have that $M(u_{n_k}) = N(u_{n_k})\, u_{n_k} \to Tu$ strongly in H and so (Q, u) satisfies $Su + Tu = Qu$.

In general, however, $T \neq N(u)$ and (Q, u) is *not* a solution of (3.2). Clearly it is useful to distinguish cases where (Q, u) does satisfy (3.2). The following hypothesis is sufficient to ensure this.

(A7) $N(u_n) \to N(u)$ strongly in $B(H_2, H)$ whenever $u_n \rightharpoonup u$ weakly in H_2.

In the above discussion of the sequence $\{(\lambda_n, u_n)\}$, the assumption (A7) implies that $T = N(u)$ and hence (Q, u) satisfies (3.2). Furthermore it follows as in Lemma 3.6 that $u \neq 0$ and we note finally that:

$$u_{n_k} \to u \text{ strongly in } H \text{ if and only if } u_{n_k} \to u \text{ strongly in } H_2.$$

4. - Three examples.

Let $H = L^2(0, \infty)$ with $\langle u, v \rangle = \int_0^\infty u(x)\, v(x)\, dx$. A linear operator $S : \mathfrak{D}(S) \subset H \to H$ is defined by:

$$\mathfrak{D}(S) = \{u \in H : u'' \in H \text{ and } u(0) = 0\}$$

$$Su(x) = -u''(x) - \frac{u(x)}{x} \text{ for } u \in \mathfrak{D}(S) \text{ and } x > 0.$$

It is well known [3, 4] that S is self-adjoint in H and that

$$\sigma(S) = \{-\frac{1}{(2k)^2} : k \in \mathbf{N}\} \cup [0, \infty).$$

In fact, $-\frac{1}{(2k)^2}$ is a simple eigenvalue of S whose eigenfunction has exactly k zeros in $[0, \infty)$ and all the zeros are simple. There are no eigenvalues of S in $[0, \infty)$. Thus we see that S satisfies the hypotheses (A1) and (A3) with $\Lambda = -\frac{1}{4}$ and $Q = 0$.

Furthermore the form domain H_1, and the graph space, H_2, associated with S co-incide (up to equivalence of norms) with the Sobolev spaces which are usually denoted by

$$\overset{\circ}{W}{}_2^1(0, \infty) \text{ and } \overset{\circ}{W}{}_2^1(0, \infty) \cap W_2^2(0, \infty) \text{ respectively}.$$

The three examples which we consider all involve the operator S defined above. They differ only in the choice of non-linear term, $M(u) = N(u)\, u$, and in each case $N(u)$ is a multiplication operator in $H = L^2(0, \infty)$ which we write in the form:

$$N(u)\, v = \frac{1}{x} q(u)(x)\, v(x) \quad \text{for } u, v \in H_1 \text{ and } x > 0.$$

For the three examples ($i = 1, 2$ and 3) the function $q(u)$ is defined as follows:

$$q_1(u)(x) = u(x)^2,$$

$$q_2(u)(x) = \int_0^\infty u(y)^2\, dy,$$

$$q_3(u)(x) = \int_0^x u(y)^2\, dy + \int_x^\infty \frac{x}{y} u(y)^2\, dy.$$

Thus the examples are boundary value problems on $(0, \infty)$ which can be written as,

$$Su + N_i(u)\, u = \lambda u \quad \text{for } (\lambda, u) \in \mathbf{R} \times H_2$$

with $$N_i(u)\, v(x) = -q_i(u)(x)\, v(x) \quad \text{for } x > 0.$$

Example 1:

$$-u''(x) - \frac{1}{x}\{1 - u(x)^2\}\, u(x) = \lambda u(x) \quad \text{for } x > 0$$

$$u(0) = 0 \text{ and } u \in L^2(0, \infty).$$

Example 2:

$$- u''(x) - \frac{1}{x}\{1 - \| u \|^2\} \, u(x) = \lambda u(x) \quad \text{for } x > 0$$
$$u(0) = 0 \text{ and } u \in L^2(0, \infty).$$

Example 3:

$$- u''(x) - \frac{1}{x}\{1 - \| u \|^2 + \int_x^\infty (1 - \frac{x}{y}) u(y)^2 \, dy\} \, u(x) = \lambda u(x) \quad \text{for } x > 0$$
$$u(0) = 0 \text{ and } u \in L^2(0, \infty).$$

LEMMA 4.1 - a) In examples 1 and 3, the hypotheses (A1) to (A7) are all satisfied — b) In example 2, the hypotheses (A1) to (A6) are all satisfied, but (A7) does not hold.

REMARKS - 1. The examples 2 and 3 are similar in that, for $i = 2, 3$,

$$\lim_{x \to \infty} q_i(u)(x) = \| u \|^2 \quad \forall u \in H_1.$$

However, examples 1 and 3 satisfy (A7) whereas example 2 does not.

2. The proof of Lemma 4.1 in easily given once the following facts are recalled.

i) $| u(x) | \leqslant \| u \|_1 \quad \forall u \in W_2^1(0, \infty)$

ii) $\left| \frac{u(x)}{x} \right| \leqslant \| u \|_2 \quad \forall u \in H_2$

iii) $\left\| \frac{u(x)}{x} \right\| \leqslant 2 \| u' \| \leqslant 2 \| u \| \quad \forall u \in H_1$ (Hardy's inequality)

iv) $q_i(u)$ is bounded on $(0, \infty)$ and $\lim_{x \to 0} q_i(u)(x)$ exists

v) From a bounded sequence $\{u_n\}$ in H_2 we can extract a subsequence $\{u_{n_k}\}$ such that

$$u_{n_k} \rightharpoonup u \text{ weakly in } H_2,$$
$$u_{n_k} \to u \text{ strongly in } C^1[0, z] \text{ for each } z > 0,$$
$$\| u_{n_k} \| \to \gamma \text{ as } n_k \to \infty.$$

For $i = 1$ and 3,

$$q_i(u_{n_k}) \to q_i(u) \text{ uniformly on } [0, \infty)$$

whereas $\quad q_2(u_{n_k}) \to \gamma^2$ uniformly on $[0, \infty)$

and $\quad q_2(u) = \| u \|^2 \leqslant \gamma^2$. (In general, $q_2(u) < \gamma^2$).

For $k \in \mathbf{N}$, let C_k^i denote the component of solutions of example i branching from the eigenvalue $-\frac{1}{(2k)^2}$ of S. By Lemma 4.1,

C_k^i satisfies the conclusion of Theorem 3.7 and we are able to distinguish between the phenomena P(1), P(2) and P(3) discussed at the end of section 3.

RESULT - For $i = 1, 2$ and 3 and $k \in \mathbf{N}$, the component C_k^i exhibits the behaviour P(i) near $\lambda = 0$.

Example 1: Suppose that C_k^1 contains a sequence $\{(\lambda_n, u_n)\}$ such that $\lambda_n \to 0$ and $\{\| u_n \|\}$ is bounded. As in the discussion following the statements P(1) to P(3) in section 3, since the hypotheses (A7) is satisfied by example 1, we can suppose that example 1 has a solution $(0, u)$ with $u \not\equiv 0$. Furthemore since u_n has exactly k zeros in $[0, \infty)$, the function u has at most k zeros in $[0, \infty)$. Replacing u by $-u$, if necessary, and recalling that $\lim_{x \to 0} u(x) = 0$ (since $u \in H_2$), we see that there exists $Z > 0$ such that $0 < u(x) < 1$ for all $x \geqslant Z$ and so $-u''(x) = \frac{1}{x}\{1 - u(x)^2\} u(x) > 0$ for all $x \geqslant Z$. This contradicts the fact that $u \in L^2(0, \infty)$ and so C_k^1 has the behaviour P(1).

For related results concerning differential equations of a slightly different type, see [6].

Example 2: This problem can be solved explicitly using scaled eigenfunctions of S. Let φ_k denote the eigenfunction of S associated with the eigenvalue $-(\frac{1}{2k})^2$ and normalised so that

$$\| \varphi_k \| = 1 \text{ and } \varphi'_k(0) > 0.$$

Then one easily finds that

$$C_k^2 = \{ (p_k(\alpha), \pm r_k(\alpha)) \in \mathbf{R} \times H_2 : 0 < \alpha \leqslant 1 \}$$

where $\quad p_k(\alpha) = -\dfrac{\alpha^2}{(2k)^2}$

and $\quad r_k(\alpha)(x) = \sqrt{\alpha(1-\alpha)}\, \varphi_k(\alpha x)$ for $x \geqslant 0$.

As $\quad \alpha \to 1$, $(p_k(\alpha), \pm r_k(\alpha)) \to (-\dfrac{1}{(2k)^2}, 0)$ strongly in $\mathbf{R} \times H_2$.

As $\quad \alpha \to 0$, $p_k(\alpha) \to 0$

$$\| r_k(\alpha) \| = \sqrt{1-\alpha} \to 1$$

and $\quad r_k(\alpha) \rightharpoonup 0$ weakly in H (and H_2).

Furthermore all non-trivial solutions of example 2 belong to

$\bigcup_{k=1}^{\infty} C_k^2$. Thus we see that example 2 exhibits the behaviour P(2).

Example 3: After appropriate scaling of the variables, this problem reduces to the radial version of the Hartee equation for a two electron atom discussed in [7]. It is shown in [7] that C_k^3 exhibits the behaviour P(3) for all $k \in \mathbf{N}$.

REMARK - Adapting slightly the proof of Lemma 1 [8], we see that, for $i = 1, 2$ and 3, the example i has no non-trivial solutions in $(0, \infty) \times H_2$.

Appendix.

In sections 2 and 3, we have introduced the graph space H_2 and then the form domain H_1 associated with the self-adjoint operator S in H. For applications it is important, if not essential, to identify these spaces, up to equivalence of norms with known structures (such as Sobolev spaces). Again perturbation theory is a useful tool. We suppose that $S = A + V$ where A is some self-adjoint operator for which the graph space and form domain have already been identified and we seek conditions on the operator V such that $S = A + V$ has the same graph space and form domain (up to equivalence of norms) as A.

(H1) $A : \mathfrak{D}(A) \subset H \to H$ is a self-adjoint operator which is positive.

Let $T : \mathfrak{D}(T) \subset H \to H$ be a closed densely defined operator such that $A = T^*T$ where $T^* : \mathfrak{D}(T^*) \subset H \to H$ is the adjoint of T in H. [We can take $T = A^{1/2}$ the (unique) positive self-adjoint square root of A, but this is not always the most convenient choice. In any case $\mathfrak{D}(T) = \mathfrak{D}(A^{1/2})$ and $\|Tu\| = \|A^{1/2}u\| \ \forall u \in \mathfrak{D}(T)$].

The graph space H_A of A is $\mathfrak{D}(A)$ with the norm

$$\|u\|_A = \{\|u\|^2 + \|Au\|^2\}^{1/2} \quad \forall u \in \mathfrak{D}(A)$$

and the form domain H_T of A is $\mathfrak{D}(T)$ with the norm

$$\|u\|_T = \{\|u\|^2 + \|Tu\|^2\}^{1/2} \quad \forall u \in \mathfrak{D}(T).$$

The form domain is independent of T by the remarks concerning $A^{1/2}$.

(H2) $V \in B(H_A, H)$ is symmetric and compact.

Then

i) $\forall \varepsilon > 0 \ \exists c(\varepsilon) > 0$ such that

$$\|Vu\| \leqslant \varepsilon \|Au\| + c(\varepsilon)\|u\| \quad \forall u \in H_A$$

ii) $S = A + V$ is self-adjoint on H_A and is bounded below.

iii) The graph space H_2 and the form domain H_1 of S co-incide (up to equivalence of norms) with H_A and H_T.

iv) $\sigma_e(A+V) = \sigma_e(A)$.

Example: Let $H = L^2(0,\infty)$ and A be defined by:

$$\mathfrak{D}(A) = \{u \in H : u'' \in H \text{ and } u(0) = 0\},$$
$$Au = -u''.$$

Then $A = T^* T$ where

$$\mathfrak{D}(T) = \{u \in H : u' \in H \text{ and } u(0) = 0\}$$
$$Tu = u',$$

$$\mathfrak{D}(T^*) = \{u \in H : u' \in H\}$$
$$Tu = -u'.$$

Thus A is self-adjoint and positive. Furthermore H_A and H_T co-incide (up to equivalence of norms) with Sobolev spaces:

$$\mathfrak{D}(T^*) = W_2^1(0,\infty),\ H_T = \mathfrak{D}(T) = \overset{\circ}{W}{}_2^1(0,\infty)$$

$$H_A = \mathfrak{D}(A) = W_2^2(0,\infty) \cap \overset{\circ}{W}{}_2^1(0,\infty).$$

Let V be the multiplication operator defined by:

$$\mathfrak{D}(V) = \{u \in H : u' \in H \text{ and } u(0) = 0\}$$

$$Vu(x) = -\frac{u(x)}{x} \text{ for } u \in \mathfrak{D}(V) \text{ and } x > 0.$$

By Hardy's inequality, $\| Vu \| \leqslant 2 \| u' \| \quad \forall u \in \mathfrak{D}(V)$. To show that V satisfies (H2) we need only prove that $V \in B(H_A, H)$ is compact. For this we consider a sequence $\{u_n\}$ which is bounded in $H_A = W_2^2(0,\infty) \cap \overset{\circ}{W}{}_2^1(0,\infty)$. We can extract a subsequence $\{u_{n_k}\}$ such that

$$u_{n_k} \rightharpoonup u \text{ weakly in } H_A$$

and $$u_{n_k} \to u \text{ strongly in } C^1[0,z] \text{ for each } z > 0.$$

Then for $z > 0$, we have

$$\| Vu_{n_k} - Vu \|^2 = \int_0^z \frac{1}{x^2} \{u_{n_k}(x) - u(x)\}^2 dx + \int_z^\infty \frac{1}{x^2} \{u_{n_k}(x) - u(x)\}^2 dx$$

$$\leqslant z \| u_{n_k} - u \|^2_{C^1[0,z]} + \frac{1}{z^2} \| u_{n_k} - u \|^2_{L^2(0,\infty)}$$

since $u_{n_k}(0) = u(0) = 0$. Hence we have that

$$\limsup_{k\to\infty} \| Vu_{n_k} - Vu \| \leqslant \frac{1}{z} \limsup_{k\to\infty} \| u_{n_k} - u \|_{L^2(0,\infty)} \quad \forall z > 0 .$$

It follows from this that $\lim_{k\to\infty} \| Vu_{n_k} - Vu \| = 0$ and so V satisfies (H2).

Note that for this example the operator $S = A + V$ is the one used for the three examples in section 4.

REFERENCES

[1] RABINOWITZ, P. H., *Some global results for non-linear eigenvalue problems,* J. Functional Analysis, 7 (1971), 487-513.

[2] STUART, C. A., *Global properties of components of solutions of non-linear second order ordinary differential equations on the half-line,* Ann. Scuola Norm. Sup. Pisa, 29 (1975), 265-286.

[3] KATO, T., *Perturbation Theory for Linear Operators,* Springer, Berlin (1966).

[4] REED, M. and SIMON, B., *Mathematical Physics,* Vol. I: *Functional Analysis;* Vol. II: *Fourier Analysis, Self-Adjointness;* Vol. IV: *Analysis of Operators,* Academic Press, New York (1972), (1975) and (1978).

[5] WEIDMANN, J., *Linear Operators in Hilbert Spaces,* Springer, New York, (1980).

[6] DANCER, E. N., *Boundary-value problems for ordinary differential equations on infinite intervals II,* Quart. J. Math. Oxford, 28 (1977), 101-115.

[7] STUART, C. A., TOLAND, J. F. and WILLIAMS, P. G., *Excited states in the Hartree Approximation for the hydrogen ion H^-,* Proc. R. Soc. Lond., A388 (1983), 229-246.

[8] STUART, C. A., *An example in non-linear functional analysis: the Hartree equation,* J. Math. Anal. Appl., 49 (1975), 725-733.