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On the quadratic equations for odeco tensors

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To Giorgio Ottaviani, on the occasion of his 60th birthday.

Abstract. *Elina Robeva discovered quadratic equations satisfied by orthogonally decomposable ("odeco") tensors. Boralevi-Draisma-Horobeţ-Robeva then proved that, over the real numbers, these equations* characterise *odeco tensors. This raises the question to what extent they also characterise the Zariski-closure of the set of odeco tensors over the complex numbers. In the current paper we restrict ourselves to* symmetric tensors of order three*, i.e., of format $n \times n \times n$. By providing an explicit counterexample to one of Robeva's conjectures, we show that for $n \geq 12$, these equations do not suffice. Furthermore, in the open subset where the linear span of the slices of the tensor contains an invertible matrix, we show that Robeva's equations cut out the limits of odeco tensors for dimension $n \leq 13$, and not for $n \geq 14$. To this end, we show that Robeva's equations essentially capture the Gorenstein locus in the Hilbert scheme of n points and we use work by Casnati-Jelisiejew-Notari on the (ir)reducibility of this locus.*

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MS Classification 2020: 15A69 Multilinear algebra, tensor calculus.

1. Introduction

In [11], Robeva discovered quadratic equations satisfied by orthogonally decomposable (odeco) tensors. In [1], it is proved that over the real numbers, these quadratic equations in fact *characterise* odeco tensors.

This raises the question whether Robeva's equations also define (the Zariski closure of) the set of complex odeco tensors. Indeed, Robeva conjectured that they might even generate the prime ideal of this Zariski closure, at least in the case of symmetric tensors [11, Conjecture 3.2]. She proved this stronger statement when the ambient space has dimension at most 3 [11, Figure 2]. In general, however, the answer to the (weaker) question is no, as already pointed out by Koiran in [7]. In this short paper, based on the first author's Master's thesis, we give an explicit symmetric tensor in $(\mathbb{C}^{12})^{\otimes 3}$ that satisfies Robeva's

equations but is not approximable by complex odeco tensors. We do not know whether 12 is the minimal dimension for which this happens, but we show that if we impose a natural, additional *open* condition on the tensor, then Robeva's equations characterise the Zariski closure of the odeco tensors precisely up to dimension 13.

A key idea in [1] is to associate an algebra A to a symmetric three-tensor T and to realise that Robeva's equations express the associativity of that algebra. Furthermore, the symmetry of the tensor implies that A is commutative and that the bilinear form is an invariant form on A; see Section 2.2 for definitions. If, in addition, A contains a unit element – this turns out to be an open condition on T – then A is a *Gorenstein algebra*. Consequently, we can use the results of [2] on (ir)reducibility of the Gorenstein locus in the Hilbert scheme of points in affine space to study the variety defined by Robeva's equations.

In the opposite direction, we use this relation between algebras and tensors to give an elementary proof that the Gorenstein locus in the Hilbert scheme of n points in $\mathbb{A}^n$ has a dimension that grows as $\Theta(n^3)$. This seems surprising at first, since the component containing the schemes consisting of n distinct reduced points has dimension only n^2; on the other hand, it is well-known that the dimension of the Hilbert scheme itself does grow as a cubic function of n.

This relation between algebras and tensors is, of course, not new: a bilinear multiplication on V can be thought of as an element of $V^* \otimes V^* \otimes V$, and in the presence of a bilinear form on V, the copies of V^* may be identified with V. Further properties of the algebra, such as associativity, cut out subvarieties of the corresponding tensor space. A classical reference for varieties of algebras is [4], where the term *algebraic geography* is coined. Another, more closely related paper is [10], whose Remark 4.5 is closely related to Lemma 9.3, and together with [10, Theorem 9.2] gives the cubic lower bound on the dimension of the Hilbert scheme mentioned above. Finally, we note that the Zariski closure of the odeco tensors consists of tensors of minimal border rank; equations for these are studied in the recent paper [6]. In particular, [6, Proposition 1.4], which states that, in the 1-generic locus, the A-Strassen equations are sufficient to characterise tensors of minimal border rank is closely related to our Theorem 2.13.

1.1. Organisation of this paper

In Section 2, we introduce the fundamental notions of this paper, including Robeva's equations, of which we show that they are the *only* quadrics that vanish on odeco tensors. We also state our main results (Theorems 2.10 and 2.13). In Section 3 we extend the well-known decomposition of finite-dimensional unital algebras into products of local algebras to the non-unital case. In Section 4 we recall that the Zariski closure of the odeco tensors is a component of the variety cut out by Robeva's equations; this was already established in [11,

Lemma 3.7]. In Section 5 we show that there are many weakly odeco tensors; combined with later results, this gives a lower bound on the dimension of the Gorenstein locus in the Hilbert scheme of n points in $\mathbb{A}^n$. In Section 6 we show how to unitalise algebras along with an invariant bilinear form to turn them into a local Gorenstein algebra, and vice versa. In Section 7 we use this construction to motivate the search for nilpotent counterexamples to Robeva's conjecture. Then, in Section 8 we find such a counterexample for $n = 12$. In Section 9 we make the connection with the Gorenstein locus in the Hilbert scheme of n points and prove our second main result – that a version of Robeva's conjecture holds in the (open) unital locus precisely up to $n = 13$. Finally, in Section 9.5 we show that the dimension $d(n)$ of that Gorenstein locus is lower-bounded bounded by a cubic polynomial in n. Since it is also trivially upper-bounded by such a polynomial, we have that $d(n) = \Theta(n^3)$.

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2. Set-up

2.1. Weakly and strongly odeco tensors

Let $V_{\mathbb{R}}$ be a finite-dimensional real vector space equipped with a positive-definite inner product $(.|.)$.

Definition 2.1. *A symmetric tensor $T \in S^3V_{\mathbb{R}} \subseteq V_{\mathbb{R}} \otimes V_{\mathbb{R}} \otimes V_{\mathbb{R}}$ is called* orthogonally decomposable *(odeco, for short) if, for some integer $k \geq 0$, T can be written as*

$$T = \sum_{i=1}^{k} v_i^{\otimes 3}$$

where $v_1, \ldots, v_k \in V_{\mathbb{R}}$ are nonzero, pairwise orthogonal vectors. We write $Y(V_{\mathbb{R}}) \subseteq S^3V_{\mathbb{R}}$ for the set of odeco tensors.

Positive-definitiveness of the form implies that $(v_i|v_i) > 0$ for each i, so that $v_1, \ldots, v_k$ are linearly independent. Hence $k \leq n := \dim(V_{\mathbb{R}})$, and $Y(V_{\mathbb{R}})$ is a semi-algebraic set of dimension at most $\binom{n}{2}+n$: the dimension of the orthogonal group plus n degrees of freedom for scaling. It turns out that $Y(V_{\mathbb{R}})$ is in fact the set of real points of an algebraic variety defined by quadratic equations that we will discuss below. Furthermore, the k and the v_i in the decomposition of an odeco tensor T are unique [1, Proposition 7], which implies that the dimension of $Y(V_{\mathbb{R}})$ is precisely $\binom{n}{2} + n$.

We now set $V := \mathbb{C} \otimes V_{\mathbb{R}}$ and extend $(.|.)$ to a complex symmetric bilinear form (*not* a Hermitian form; that setting is studied in [1] under the name *udeco*). We note that the extended bilinear form is nondegenerate, as the inner product $(.|.)$ is positive-definite on the real vector space $V_{\mathbb{R}}$.

DEFINITION 2.2. *A symmetric tensor $T \in S^3V \subseteq V \otimes V \otimes V$ (where the tensor product is over $\mathbb{C}$) is* weakly odeco *if T can be written as*

$$T = \sum_{i=1}^{k} v_i^{\otimes 3}$$

where the v_i are nonzero pairwise orthogonal vectors. It is called strongly odeco *if T admits such a decomposition where, in addition, $(v_i|v_i) \neq 0$. We write $Y(V) \subseteq S^3V$ for the Zariski closure of the set of strongly odeco tensors.*

REMARK 2.3. The set called $\mathrm{SODECO}_n(\mathbb{C})$ in [7] consists of the strongly odeco tensors in $S^3\mathbb{C}^n$. Koiran proves that these are precisely the set of symmetric tensors whose $n \times n$ slices are diagonalisable and commute.

As pointed out above, every element of $Y(V_{\mathbb{R}})$, regarded as an element of S^3V, is strongly odeco. Since the real orthogonal group $O(V_{\mathbb{R}})$ is Zariski dense in the complex orthogonal group $O(V)$, $Y(V)$ is the Zariski closure of $Y(V_{\mathbb{R}})$. On the other hand, due to the presence of isotropic vectors and higher-dimensional spaces in V, the set of weakly odeco tensors strictly contains the set of strongly odeco tensors; we will return to this theme shortly. First we give the easiest example that shows the need for a Zariski closure in the definition of $Y(V)$.

EXAMPLE 2.4. Consider the vector space $V = \mathbb{C}^2$ equipped with the symmetric bilinear form for which $(e_1|e_2) = 1$ and all other products are zero. (Of course, since all nondegenerate symmetric bilinear forms on a finite-dimensional complex vector space are equivalent, we could have changed coordinates such that the bilinear form is the standard form.) Then

$$\begin{aligned} S &= e_1 \otimes e_2 \otimes e_2 + e_2 \otimes e_1 \otimes e_2 + e_2 \otimes e_2 \otimes e_1 \\ &= \frac{1}{2} \lim_{t \to 0} [(t^2 e_1 + t^{-1} e_2)^{\otimes 3} + (t^2 e_1 - t^{-1} e_2)^{\otimes 3}] \end{aligned}$$

shows that S is a limit of strongly odeco tensors and so $S \in Y(\mathbb{C})$. To see that the tensor S is not strongly odeco, it is enough to show that its tensor rank is 3. For order three tensors, the rank of the tensor equals the number of rank one matrices needed to span a space containing the space generated by the slices of the tensor [8, Theorem 3.1.1.1]. The tensor S has slices $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which are not contained in a subspace spanned by two rank 1 matrices, so S has rank 2.

2.2. Commutative algebras from symmetric tensors

We now associate an algebra structure on V to a tensor.

DEFINITION 2.5. *We identify V with V^* via the map $v \mapsto (v|.)$. Then any element $T = \sum_{i=1}^k u_i \otimes v_i \otimes w_i \in V^{\otimes 3}$ can also be regarded as an element of $V^* \otimes V^* \otimes V$, hence defines a bilinear map*

$$\mu_T : V \times V \to V;\ (x, y) \mapsto \mu_T(x, y) = \sum_{i=1}^{k} (u_i|x)(v_i|y)w_i.$$

We call V with μ_T the algebra associated to T.

If T is symmetric, then, first, μ_T is commutative, and second, μ_T satisfies

$$(\mu_T(x, y)|z) = (x|\mu_T(y, z)),$$

i.e., the bilinear form $(.|.)$ is *invariant for the multiplication* μ_T. When T is fixed in the context, then we will often just write xy instead of $\mu_T(x, y)$.

2.3. Odeco implies associative

PROPOSITION 2.6. *If $T \in S^3V$ is weakly odeco, then μ_T is associative.*

This was observed in [1] in the real case, but the argument easily generalises, as follows.

Proof. Write

$$T = \sum_{i=1}^{k} v_i^{\otimes 3}$$

where the v_i are pairwise orthogonal. Let $x, y, z \in V$. Then

$$\begin{aligned}\mu_T(x, \mu_T(y, z)) =& \mu_T(x, \sum_{i=1}^{k} (y|v_i)(z|v_i)v_i) = \sum_{j=1}^{k}\sum_{i=1}^{k} (y|v_i)(z|v_i)(x|v_j)(v_i|v_j)v_j \\ =& \sum_{i=1}^{k} (y|v_i)(z|v_i)(x|v_i)(v_i|v_i)v_i\end{aligned}$$

where the last equality uses that $(v_i|v_j) = 0$ whenever $i \neq j$. A similar computation for $\mu_T(\mu_T(x, y), z)$ yields the exact same result. □

2.4. Robeva's equations

For any fixed x, y, z (e.g. chosen from a basis of V), the condition that $\mu_T(x, \mu_T(y, z))$ equals $\mu_T(\mu_T(x, y), z)$ translates into n quadratic equations for T. All these equations together, with varying x, y, z, are called *Robeva's equations*.

DEFINITION 2.7. *We denote by $X(V) \subseteq S^3V$ the affine variety defined by Robeva's equations, i.e.,*

$$X(V) := \{T \in S^3V \mid \mu_T \text{ is associative}\}.$$

To prove that Robeva's equations are *all* quadratic equations satisfied by strongly odeco tensors, we reinterpret them as follows. The condition that $\mu_T(x, \mu_T(y, z))$ equals $\mu_T(\mu_T(x, y), z)$ means that for every $w \in V$, we have

$$(\mu_T(x, \mu_T(y, z))|w) = (\mu_T(\mu_T(x, y), z)|w),$$

which can be rewritten as

$$(\mu_T(y, z)|\mu_T(x, w)) = (\mu_T(x, y)|\mu_T(z, w)). \tag{1}$$

In other words: Robeva's equations precisely express that the 4-linear map

$$\begin{aligned} T \bullet T : V^4 &\to \mathbb{C} \\ (x, y, z, w) &\mapsto (\mu_T(x, y)|\mu_T(z, w)) \end{aligned}$$

is invariant under arbitrary permutations of (x, y, z, w). This was, in fact, Robeva's original description of these quadratic equations in [11].

PROPOSITION 2.8. *The only quadratic equations vanishing on $Y(V)$ are Robeva's equations.*

Proof. If we consider the natural map

$$\begin{gathered} S^2(S^3V) \to S^2(S^2V) \\ (v_1 \otimes v_2 \otimes v_3) \otimes (w_1 \otimes w_2 \otimes w_3) \mapsto (v_1|w_1)(v_2 \otimes v_3) \otimes (w_2 \otimes w_3), \end{gathered}$$

then for any $T \in S^3V$, we can identify $T \bullet T$ with the image of T under the composition

$$S^3V \xrightarrow{\nu_2} S^2S^3V \to S^2S^2V, \tag{2}$$

where ν_2 is the second Veronese embedding. Then T satisfies Robeva's equations if and only if $T \bullet T \in S^4V \subseteq S^2S^2V$.

Since quadratic equations on $Y(V)$ correspond to linear equations on $\nu_2(Y(V))$, we want to show that the image of (2), when restricted to $Y(V)$, linearly spans S^4V. But (2) maps an odeco tensor $\sum_{i=1}^k v_i^{\otimes 3}$ to the odeco tensor $\sum_{i=1}^k (v_i|v_i)v_i^{\otimes 4}$, and these tensors clearly span S^4V. □

2.5. The main question

By Proposition 2.6, $X(V)$ contains weakly odeco tensors, hence in particular the variety $Y(V)$. On the other hand, the results of [1] imply that the set of real points of $X(V)$ equals $Y(V_{\mathbb{R}})$. This raises the following question.

QUESTION 2.9. *For which dimensions $n = \dim(V)$ is $X(V)$* equal *to $Y(V)$?*

Our partial answer to this question is as follows.

THEOREM 2.10. *For $n = \dim(V) \leq 3$, $X(V)$ equals $Y(V)$. For $n \geq 12$, we have $Y(V) \subsetneq X(V)$.*

Proof. See Section 8. □

In fact, the result for $n \leq 3$ is due to Robeva [11, Figure 2]. Our contribution is a counterexample to [11, Conjecture 3.2] for $n = 12$.

2.6. The existence of a unit

The answer in Theorem 2.10 is unsatisfactory because of the large interval of dimensions $n = \dim(V)$ for which we do not know whether $X(V)$ is strictly larger than $Y(V)$. However, in a certain open subset of S^3V, we *do* know precisely where the two stop being equal.

LEMMA 2.11. *The condition on $T \in X(V)$ that μ_T has a multiplicative unit element is equivalent to the condition that there exists a $x \in V$ such that the multiplication map $L_x : v \mapsto \mu_T(x, v)$ is invertible. This is an open condition on T.*

In the case of ordinary tensors, the analoguous condition is called 1-genericity; see, e.g., [6].

Proof. We write xv instead of $\mu_T(x, v)$. For the implication $\Rightarrow$ take x to be the unit element. For the implication $\Leftarrow$, assume that L_x is invertible. Then in particular there exists an $e \in V$ such that $ex = xe = x$. We then find, for any $y \in V$, that

$$ey = exL_x^{-1}(y) = xL_x^{-1}(y) = y$$

so that e is a unit element.

It follows that (V, μ_T) is not unital if and only if $\det(L_x) = 0$ for all $x \in V$; this is a system of degree n polynomial equations on $T \in S^3(V)$ defining the complement of the unital locus. □

We will refer to the variety in $S^3(V)$ defined by the degree n equations in the proof above as the *non-invertibility locus*. If $T \in S^3(V)$ is not in the non-invertibility locus, then μ_T needs not have a unit element; but it does if

furthermore T lies in $X(V)$ – we have used associativity of μ_T in the proof above.

We define $X^0(V)$ as

$$X^0(V) := \{T \in X(V) \mid \mu_T \text{ is unital}\},$$

and similarly for $Y^0(V)$. Note that $\overline{Y^0(V)} = Y(V)$. This leads to the following weakening of Question 2.9.

QUESTION 2.12. *For which dimensions $n = \dim(V)$ is $X^0(V)$ equal to $Y^0(V)$? In other words, for which dimensions do Robeva's quadrics characterise the set of limits of strongly odeco tensors in the complement of the non-invertibility locus?*

To answer this question, we will prove the following theorem.

THEOREM 2.13. *The number of irreducible components of $X^0(\mathbb{C}^n)$ equals that of the Gorenstein locus in the Hilbert scheme of n points in $\mathbb{A}^n_{\mathbb{C}}$.*

Proof. See Section 9. □

We can now make use of the following result on the irreducibility of Gorenstein loci of Hilbert schemes.

THEOREM 2.14 ([2]). *The Gorenstein locus of n points in $\mathbb{A}^d_{\mathbb{C}}$*

- *is irreducible if $n \leq 13$, or if $n = 14$ and $d \leq 5$.*
- *has 2 irreducible components if $n = 14$ and $d \geq 6$.*

Note that the second item implies reducibility of the Gorenstein locus of n points in $\mathbb{A}^d_{\mathbb{C}}$ for all $n \geq 14$ if $d \geq 6$. Indeed, reducibility of the Gorenstein locus means existence of a non-smoothable point, and if one finds such a point for given pair (n, d), then one can always add to this point more disjoint points (increase d) or embed it in higher dimensions (increase n).

Combining the two previous theorems together with the fact that $Y(\mathbb{C}^n)$ is an irreducible component of $X(\mathbb{C}^n)$, which is proven in Section 4, gives a complete answer to Question 2.12.

COROLLARY 2.15. *The locus $X^0(\mathbb{C}^n)$ is irreducible and equal to $Y^0(\mathbb{C}^n)$ for $n \leq 13$, and is not irreducible and not equal to $Y^0(\mathbb{C}^n)$ for $n \geq 14$.*

3. Decomposing (the algebra of) a tensor in $X(V)$

3.1. Motivation

Recall that if B is a unital finite-dimensional algebra over $\mathbb{C}$, then B is isomorphic to a product $B_1 \times \cdots \times B_k$ of local algebras; here $k = |\operatorname{Spec}(B)|$. In this section we want to establish a similar decomposition for not necessarily unital algebras.

3.2. The unital/nilpotent decomposition

PROPOSITION 3.1. *Let $T \in X(V)$ and equip V with the corresponding commutative, associative multiplication μ_T. Then V has a unique decomposition as a direct sum*

$$V_1 \oplus \cdots \oplus V_k \oplus N\,,$$

where the V_i are nonzero and N is potentially zero, such that N and each V_i is an ideal in (V, μ_T), $(V_i, \mu_T|_{V_i \times V_i})$ is a local unital algebra for each i, and N is a nilpotent algebra. Furthermore, this unique direct sum decomposition is orthogonal.

Accordingly, T decomposes as $T_1 + \cdots T_k + T_N$ with $T_i \in X(V_i)$ and $T_N \in X(N)$; and we have $T \in Y(V)$ if and only if $T_i \in Y(V_i)$ for all i and $T_N \in Y(N)$.

Proof. Note that if $a, b \in V$ belong to different factors in any decomposition of V as a direct sum of ideals, and if the first factor has unit element e, then

$$(a|b) = (ae|b) = (e|ab) = (e|0) = 0;$$

this proves the orthogonality of the decomposition.

That a *unital* finite-dimensional commutative, associative algebra V has a unique product decomposition into local algebras is well-known – it is found by taking a decomposition of 1 into minimal idempotents e_i satisfying $e_i e_j = \delta_{ij} e_i$ and taking $V_i := e_i V$.

To reduce to the unital case, we proceed as follows. If V is nilpotent, then we set $k := 0$ and $N := V$. Otherwise, there exists an element $x \in V$ that is not nilpotent. Let $L_x : V \to V$ be multiplication with x. Then there exists an m such that $\operatorname{im} L_x^m = \operatorname{im} L_x^{m+1} = \ldots$. Set $y := x^m$ and $W := yV$. Then $(L_y)|_{yV}$ is invertible, so the ideal yV is unital by Lemma 2.11. On the other hand, we have $V = yV \oplus \ker(L_y)$: indeed, the dimensions of $yV = \operatorname{im}(L_y), \ker(L_y)$ add up to n, and if $L_y(yv) = 0$, then $L_y^2 v = 0$, so already $L_y v = 0$, so $yv = 0$.

Now $\ker L_y$ has strictly lower dimension than V, and by induction we know that $\ker L_y$ is the direct sum of a nilpotent ideal N and an ideal I that is unital as an algebra. Then so is V: it equals the direct sum of the ideals N and $yV \oplus I$, where the latter is unital as an algebra. The decomposition of V into a unital ideal V_0 and a nilpotent ideal V_1 is unique, because V_0 is the space of elements $v \in V$ for which there is an idempotent $e \in V$ with $v \in \operatorname{im} L_e$.

The statements about T are straightforward from the fact that μ_T is the sum of its restrictions to the ideals V_i and N. Here we note that the restrictions of $(.|.)$ to the V_i and to N are non-degenerate, so that the notation $X(V_i), X(N), Y(V_i), Y(N)$ make sense. □

4. $Y(V)$ is a component of $X(V)$

4.1. Motivation

While, as we will see, $Y(V)$ is in general not *equal* to $X(V)$, at least it is an irreducible component. This was already observed in [11]; we paraphrase the argument here.

4.2. A tangent space computation

Here, and later in the paper, we will write $e_1, \ldots, e_n$ for the standard basis of $\mathbb{C}^n$.

PROPOSITION 4.1. *For each V equipped with a nondegenerate symmetric bilinear form, the variety $Y(V)$ of limits of strongly odeco tensors is an irreducible component of the variety $X(V)$ defined by Robeva's quadrics.*

Proof. It suffices to prove that for a suitable tensor $T_0 \in Y(V)$, the tangent space to $X(V)$ at T_0 has dimension equal to $\dim(Y(V)) = \binom{n+1}{2}$. We take $V = \mathbb{C}^n$ and $T_0 = E = \sum_{i=1}^n e_i^{\otimes 3}$.

Let us first write Robeva's equations in coordinates: writing

$$T = \sum_{i,j,k} T_{ijk} e_i \otimes e_j \otimes e_k \,,$$

equation (1) becomes

$$\sum_r T_{jkr} T_{i\ell r} = \sum_r T_{ijr} T_{k\ell r} \,. \tag{3}$$

The equations defining the tangent space at E are given by substituting $T = E + \varepsilon X$ in (3) and taking the coefficients of ε:

$$\delta_{jk} X_{ij\ell} + \delta_{i\ell} X_{ijk} = \delta_{ij} X_{ik\ell} + \delta_{k\ell} X_{ijk}, \tag{4}$$

where δ_{ij} denotes the Kronecker delta.

By taking $i = \ell \neq j \neq k \neq i$ in (4) we find that $X_{ijk} = 0$ for i, j, k pairwise distinct, and by taking $i = \ell \neq j = k$ we find that $X_{iij} = -X_{ijj}$ for all $i \neq j$. But this implies that our tangent space has dimension at most $n + \binom{n}{2} = \binom{n+1}{2}$. □

REMARK 4.2. In a similar manner, one finds that all strongly odeco tensors of tensor rank n are smooth points of $Y(V)$.

5. Many weakly odeco tensors

5.1. Motivation

In this section, we give our first negative answer to Question 2.9 by showing that, for n sufficiently large, there are many more weakly odeco tensors than strongly odeco tensors.

5.2. Weakly odeco tensors from isotropic spaces

Recall that V is a complex vector space of dimension n equipped with a symmetric bilinear form $(.|.)$.

PROPOSITION 5.1. *The variety $X(V)$ contains the union over all (maximal) isotropic subspaces $U \subseteq V$ of S^3U. This union is an affine variety $Z(V) \subseteq S^3V$ of dimension*

$$\binom{\lfloor n/2 \rfloor + 2}{3} + \binom{\lceil n/2 \rceil}{2}.$$

Proof. For the first statement, note that if $u_1, \ldots, u_k$ are elements of an isotropic subspace U of V, then $\sum_i u_i^{\otimes 3}$ is weakly odeco, hence in $X(V)$ by Proposition 2.6.

There is no harm in restricting our attention to maximal isotropic subspaces, i.e., those of dimension $\lfloor n/2 \rfloor$. Hence $Z(V)$ is the projection of the incidence variety

$$\{(U, T) \in \mathrm{Gr}_{\mathrm{iso}}(\lfloor n/2 \rfloor, V) \times S^3V \mid T \in S^3U\}$$

onto the second factor. Since the isotropic Grassmannian is a projective variety, $Z(V)$ is closed. Furthermore, for $U \subseteq V$ isotropic of dimension $\lfloor n/2 \rfloor$ and $T \in S^3U$ *concise*, i.e. such that the associated linear map $S^2U^* \to U$ is surjective, the fibre over T is the single point (U, T), hence $\dim(Z(V))$ equals the dimension of the isotropic Grassmannian, which is the second term above, plus the dimension of S^3U for a fixed isotropic $U \subseteq V$ of dimension $\lfloor n/2 \rfloor$, which is the first term. □

REMARK 5.2. Clearly, $\dim(Z(V))$ grows as a cubic (quasi-)polynomial in n, whereas $\dim(Y(V))$ is a quadratic polynomial in n. Since $X(V) \supseteq Z(V)$, this shows that $X(V) \supsetneq Y(V)$ for all V of sufficiently high dimension. In fact, $\dim(Z(V)) > \dim(X(V))$ for $n \geq 16$. However, we will show with an explicit example that $X(V) \supsetneq Y(V)$ holds already for $n \geq 12$ (and possibly already for smaller n).

REMARK 5.3. The variety $Z(V)$ consists precisely of the tensors $T \in S^3V$ whose algebra is 2-step nilpotent:

$$T \in Z(V) \iff \mu_T(x, \mu_T(y, z)) = 0 \quad \forall x, y, z \in V.$$

One implication is clear: if $T \in Z(V)$, we can write $T = \sum_i u_i^{\otimes 3}$ with the u_i isotropic and pairwise orthogonal, and the computation from the proof of Proposition 2.6 gives that $\mu_T(x, \mu_T(y, z)) = 0$. For the other direction we can work in coordinates: write $V = \mathbb{C}^n$, then the condition

$$\mu_T(e_i, \mu_T(e_j, e_k)) = 0 \quad \forall i, j, k \in \{1, \ldots, n\}$$

is equivalent to

$$\sum_{r=1}^{n} T_{ijr} T_{k\ell r} = 0 \quad \forall i, j, k, \ell \in \{1, \ldots, n\}.$$

But this means that that the space U spanned by the columns of T is isotropic.

6. Unitalisation and de-unitalisation

6.1. Motivation

It is well known that if an associative algebra A, say over $\mathbb{C}$, has no multiplicative unit element, then one can turn A into a unital associative algebra by setting $A' := \mathbb{C}1 \oplus A$ and extending the multiplication on A to A' via $1a' := a'$ for all $a' \in A'$. In this section, we describe a process that also extends an invariant bilinear form.

6.2. Unitalising algebras with invariant forms

Let A be an associative algebra over $\mathbb{C}$ equipped with a bilinear form $(.|.)$ such that $(ab|c) = (a|bc)$ for all $a, b, c \in A$. We do not require A to be commutative or $(.|.)$ to be symmetric.

We construct a new algebra

$$\widetilde{A} := \mathbb{C}1 \oplus A \oplus \mathbb{C}y$$

with multiplication determined by

$$\begin{aligned}
&1 * x := x, x * 1 := x \text{ for all } x \in \widetilde{A},\\
&a * a' := aa' + (a|a')y \text{ for all } a, a' \in A,\\
&a * y := 0, y * a = 0 \text{ for all } a \in A, \text{ and}\\
&y * y := 0.
\end{aligned}$$

We also extend the form $(.|.)$ to $\widetilde{A}$ by requiring that

$$(1|1) = (a|1) = (1|a) = (a|y) = (y|a) = (y|y) = 0 \text{ for all } a \in A$$

and $(1|y) = (y|1) = 1$.

REMARK 6.1. Let us consider the special case where the multiplication on A is identically zero. If we let $a_1, \ldots, a_n$ be an orthonormal basis of A, then the tensor associated to $\tilde{A}$ is equal to

$$\sum_{i=1}^{n} (1 \otimes a_i \otimes a_i + a_i \otimes 1 \otimes a_i + a_i \otimes a_i \otimes 1) + 1 \otimes 1 \otimes y + 1 \otimes y \otimes 1 + y \otimes 1 \otimes 1.$$

This tensor is known as the Coppersmith-Winograd tensor [3]; it has played a central role in the literature on the complexity of matrix multiplication. We refer the reader to [9] (in particular Chapter 3.4.9) for an overview. In the notation of the latter reference, the above tensor is denoted $T_{n,CW}$.

PROPOSITION 6.2. *The algebra $\widetilde{A}$ is associative, and the form $(.|.)$ on $\widetilde{A}$ is invariant. Furthermore, if $(.|.)$ is nondegenerate or symmetric on A, then its extension to $\widetilde{A}$ has the same property; and if A is commutative and $(.|.)$ is symmetric, then $\widetilde{A}$ is commutative.*

Proof. It suffices to prove the identity $a * (b * c) = (a * b) * c$ for a, b, c ranging over a spanning set of $\widetilde{A}$. If at least one of a, b, c is 1, then the identity is immediate. If none of them is 1 and at least one of them is y, then both sides are zero. So the interesting case is the case where a, b, c are all in A. Then we have

$$a * (b * c) = a * (bc + (b|c)y) = a(bc) + (a|bc)y + 0 = a(bc) + (a|bc)y$$

and

$$(a * b) * c = (ab + (a|b)y) * c = (ab)c + (ab|c)y + 0 = (ab)c + (ab|c)y.$$

These two expressions are equal by associativity of A and invariance of $(.|.)$ on A.

Now we turn to the identity $(a * b|c) = (a|b * c)$ for a, b, c ranging over the same spanning set. If $b = 1$, then the identity is immediate. If $b \in A \oplus \mathbb{C}y$ and $a = 1$, then the identity reads

$$(b|c) = (1|b * c).$$

Now the right-hand side is the coefficient of y in $b * c$. Write $b = b' + \beta y$ and $c = \gamma 1 + c' + \delta y$ with $\beta, \gamma, \delta \in \mathbb{C}$ and $b', c' \in A$. Then the coefficient of y in $b * c$ equals $\beta\gamma + (b'|c')$, and this also equals $(b|c)$. Since a, c play symmetric roles, the identity also holds when $c = 1$. So we are left with the case where $a, b, c \in A \oplus \mathbb{C}y$. But then, since y is perpendicular to A, we have

$$(a * b|c) = (ab|c) = (a|bc) = (a|b * c),$$

as desired.

That the extension of (.|.) inherits the properties of symmetry and non-degeneracy is immediate, and so is the statement about the commutativity of $\widetilde{A}$. □

REMARK 6.3. The space $M := A \oplus \mathbb{C}y$ is a maximal ideal in $\widetilde{A}$, and in particular a non-unital subalgebra of $\widetilde{A}$. This subalgebra has an ideal $\mathbb{C}y$, and the natural map $A \to M/\mathbb{C}y$ is an isomorphism of algebras. We will use this construction below to *de-unitalise* a local Gorenstein algebra in a canonical manner.

LEMMA 6.4. *Suppose that A is commutative and that (.|.) is symmetric. Then A is nilpotent if and only if $\widetilde{A}$ is (unital and) local.*

Proof. If A is nilpotent, then M consists of elements that are nilpotent in $\widetilde{A}$, and hence any element not in M is invertible. Conversely, if $\widetilde{A}$ is local, then M is the unique maximal ideal and its elements are nilpotent. This implies that $A \cong M/\mathbb{C}y$ is nilpotent. □

We now show that each local, unital algebra with an invariant bilinear form arises as $\widetilde{A}$ for some A equipped with a bilinear form.

PROPOSITION 6.5. *Let B be a commutative, local, unital, finite-dimensional algebra with $\dim(B) \geq 2$, equipped with a nondegenerate invariant symmetric bilinear form. Then $B \cong \widetilde{A}$ for some nilpotent algebra A equipped with a nondegenerate invariant symmetric bilinear form.*

Proof. Let M be the maximal ideal of B, and let d be maximal such that M^d is nonzero. Then $d \geq 1$ since $\dim(B) \geq 2$.

We claim that M^d is one-dimensional. Indeed, if it were at least two-dimensional, then $1^\perp \cap M^d$ would contain a nonzero element x. This element would satisfy $(x|1) = 0$ and $(x|z) = (xz|1) = (0|1) = 0$ for all $z \in M$, contradicting the non-degeneracy of (.|.).

Choose a spanning vector $z \in M^d$. Then $(1|z) \neq 0$, and hence we may replace z by a (unique) scalar multiple with $(1|z) = 1$. Furthermore, $z^\perp = M$. We define A as the algebra $M/\mathbb{C}z$ equipped with the induced symmetric bilinear form. We claim that $\widetilde{A} \cong B$ as algebras with bilinear forms. The isomorphism $\phi : \widetilde{A} \to B$ sends $1 \in \widetilde{A}$ to $1 \in B$, $y \in \widetilde{A}$ to $z \in B$ and $m \in A$ to the unique element $m' \in m + \mathbb{C}z \subseteq B$ that satisfies $(1|m') = 0$ in B. All checks are then straightforward. □

Now let V be an n-dimensional complex vector space equipped with a non-degenerate symmetric bilinear form (.|.). Define $\widetilde{V} := \mathbb{C}1 \oplus V \oplus \mathbb{C}y$, equipped with the symmetric bilinear form as above.

Let $T \in X(V)$ and let $V = V_1 \oplus \cdots \oplus V_k \oplus N$ be the decomposition of Proposition 3.1. Let e_i be the unit element in V_i. Now $\widetilde{V} = \mathbb{C}1 \oplus V \oplus \mathbb{C}y$

is also a commutative, associative algebra with invariant symmetric bilinear form (.|.), hence it corresponds to an element $\widetilde{T} \in X(\widetilde{V})$, which in turn gives a decomposition of $\widetilde{V}$ as in Proposition 3.1. The following proposition expresses the latter decomposition into the former.

PROPOSITION 6.6. *We have an orthogonal decomposition*

$$\widetilde{V} = V_1' \oplus \cdots \oplus V_k' \oplus N' \oplus 0$$

into ideals, where $V_i' \subseteq \widetilde{V}$ *is isomorphic to* V_i *via the isomorphism*

$$\phi_i : V_i \to V_i',\ \phi_i(v) := v + (v|e_i)y$$

and where N' *is a local unital algebra spanned by* N, y, *and the unit element* $e_{k+1} := 1 - e_1 - \cdots - e_k$.

Proof. First, ϕ_i is clearly injective. It is also an algebra homomorphism because

$$\begin{aligned}\phi_i(vw) &= vw + (vw|e_i)y = vw + (v|we_i)y \\ &= vw + (v|w)y = (v + (v|e_i)y) * (w + (w|e_i)y) = \phi_i(v) * \phi_i(w).\end{aligned}$$

Now note that if v, w belong to $V_i \neq V_j$, respectively, then

$$\phi_i(v) * \phi_j(w) = (v + (v|e_i)y) * (w + (w|e_j)y) = vw + (v|w)y = 0.$$

This shows that $V_i' * V_j' = \{0\}$. Similarly, we have $N' * V_i = \{0\}$ for all i – e.g., for $v \in V_i$ we have

$$e_{k+1} * \phi_i(v) = (1 - e_1 - \cdots - e_k) * (v + (v|e_i)y) = v + (v|e_i)y - (e_i v + (e_i|v)y) = 0.$$

Finally, e_{k+1} is clearly a unit element in N'. Indeed, we even have an isomorphism $\widetilde{N} \to N'$ of unital algebras with invariant bilinear forms that sends 1 to $1 - e_1 - \cdots - e_k$ and y to y. □

PROPOSITION 6.7. *The map* $T \mapsto \widetilde{T}$ *is a morphism from* $X(V)$ *into* $X(\widetilde{V})$ *that maps* $Y(V)$ *into* $Y(\widetilde{V})$.

We call this morphism the *unitalisation morphism.*

Proof. The first statement is immediate: the algebra structure on $\widetilde{V}$ depends in a polynomial manner on the algebra structure on V. For the last statement, we note that if T is strongly odeco of tensor rank n, then (V, μ_T) is an orthogonal direct sum of n one-dimensional unital ideals. By Proposition 6.6, $(\widetilde{V}, \mu_{\widetilde{T}})$ is then an orthogonal direct sum of n one-dimensional unital ideals and one two-dimensional ideal which, as an algebra with symmetric bilinear form, is

isomorphic to $\mathbb{C}[y]/(y^2)$ with the blinear form determined by $(1|y) = 1$. The latter corresponds to the tensor

$$S := y \otimes y \otimes 1 + y \otimes 1 \otimes y + 1 \otimes y \otimes y$$

from Example 2.4, hence it is a limit of strongly odeco tensors. Consequently, by Proposition 3.1, $\widetilde{T}$ is in $Y(\widetilde{V})$. Since the map $T \mapsto \widetilde{T}$ maps the dense subset of $Y(V)$ of strongly odeco tensors of rank n into $Y(\widetilde{V})$, it maps $Y(V)$ into $Y(\widetilde{V})$. □

Remark 6.8. Unfortunately, we see no reason why, if $T \in X(V)$ satisfies $\widetilde{T} \in Y(\widetilde{V})$, T should be in $Y(V)$. Indeed, the assumption says that $\widetilde{T}$ is a limit of sums with $n + 2$ pairwise orthogonal terms, and we do not see a natural construction that shows that T is a limit of sums with n pairwise orthogonal terms; we do not have a counterexample, though.

7. Nilpotent counterexamples

7.1. Motivation

When one studies the Hilbert scheme of n points in a fixed space for increasing n, and n is taken minimal such that the scheme has more than one irreducible component, then all components other than the main component parameterise subschemes supported in a single point. We will establish a similar result here.

7.2. First counterexamples are nilpotent

Theorem 7.1. *Let $n = \dim(V)$ be minimal such that $X(V) \neq Y(V)$. Then for all $T \in X(V) \setminus Y(V)$ the algebra (V, μ_T) is nilpotent.*

Proof. Let $T \in X(V) \setminus Y(V)$. Decompose $V = V_1 \oplus \cdots \oplus V_k \oplus N$ as in Proposition 3.1, and decompose $T = T_1 + \cdots + T_k + T_N$ accordingly. By Proposition 3.1, either some T_i does not lie in $Y(V_i)$, or T_N does not lie in $Y(N)$. By minimality of n, we find that either $k = 0$ and we are done, or else $k = 1$ and $N = \{0\}$. In the latter case, by Proposition 6.5, the algebra (V, μ_T) equals $\widetilde{A}$ for some nilpotent algebra A of dimension $\dim(V) - 2$ equipped with a nondegenerate symmetric bilinear form. This means that $T = \widetilde{S}$ for some tensor $S \in X(A)$. By minimality of n, S lies in $Y(A)$. But then, by Proposition 6.7, $T = \widetilde{S}$ lies in $Y(V)$, a contradiction. Hence (V, μ_T) is nilpotent, as claimed. □

8. Proof of Theorem 2.10

Proof of Theorem 2.10. Let V be a finite-dimensional complex vector space of dimension $n \geq 12$ and let $(.|.)$ be a nondegenerate symmetric bilinear form

on V. We first show that $X(V)$ is not equal to $Y(V)$ when $n = 12$.

In [5] an explicit 14-dimensional local Gorenstein algebra is constructed which is not smoothable. Call this algebra B, and let $(.|.)$ be a nondegenerate invariant symmetric bilinear form on B. Let M be the maximal ideal of B, and let M^d be its minimal ideal. Then $A := M/M^d$ is a nilpotent algebra and since M^d is the radical of the restriction of $(.|.)$ to M, $(.|.)$ induces a nondegenerate bilinear form on A. Note that $\dim(A) = 12$, so we may assume that V (with its bilinear form) is the underlying vector space of A (with its bilinear form). Let $T \in X(V)$ be the tensor corresponding to the algebra A. We claim that T does not lie in $Y(V)$. Indeed, if it does, then $T = \lim_{i\to\infty} T_i$ for a convergent sequence of strongly odeco tensors T_i. Applying the unitalisation morphism, we obtain $\widetilde{T} = \lim_{i\to\infty} \widetilde{T}_i$. Now $\widetilde{T}$ is the structure tensor of the algebra $\tilde{A}$, which by (the proof of) Proposition 6.5 is isomorphic to B.

However, by (the proof of) Proposition 6.7, each $\widetilde{T}_i$ is the direct product of 12 one-dimensional ideals and one copy of $\mathbb{C}[x]/(x^2)$. In particular, each $\widetilde{T}_i$ corresponds to a smoothable algebra, and B is smoothable, as well. This contradicts the choice of B, finishing the proof in the case $n = 12$. If $n > 12$, one can embed the algebra $A = M/M^d$ from above into the vector space V by $A' := A \oplus \mathbb{C}^{n-12}$ and then proceed as in the proof above. □

9. Proof of Theorem 2.13

We set $V := \mathbb{C}^n$, equipped with the standard symmetric bilinear form $\beta_0(u, v) := \sum_i u_i v_i$. Recall that $X^0(V)$ is the variety of tensors corresponding to *unital* associative algebras on V for which β_0 is invariant. We want to show that $X^0(V)$ has the same number of irreducible components as $\mathcal{H}^{\mathrm{Gor}}$.

9.1. Locating the unit element

LEMMA 9.1. *The map $u : X^0(V) \to V$ that assigns to a tensor T the unit element of (V, μ_T) is a morphism of quasi-affine varieties.*

Proof. For given T, the unit element $u = u(T)$ is the solution to the system of linear equations $\mu_T(u, e_i) = e_i$ for $i = 1, \ldots, n$. For each $T \in X^0(V)$, this system has a unique solution. This means that we can cover $X^0(V)$ with open affine subsets in which some subdeterminant of the coefficient matrix has nonzero determinant, and on such an open subset the map u is morphism with a formula in which that determinant appears in the denominator. These morphisms glue to a global morphism u. □

9.2. A map from $X^0(V)$ to the Hilbert scheme

We write $R := \mathbb{C}[x_1, \ldots, x_n]$, denote by $\mathcal{H}$ the Hilbert scheme of n points in $\mathbb{A}^n$, and by $\mathcal{H}^{\mathrm{Gor}}$ the open subscheme of $\mathcal{H}$ parameterising Gorenstein schemes. In fact, since we care only about irreducible components, we may and will replace both of these by the corresponding reduced subvarieties, and we will only speak of $\mathbb{C}$-valued points of these varieties. Points in $\mathcal{H}$ will be regarded as ideals in R of codimension n. To define a morphism from an affine variety B over $\mathbb{C}$ to $\mathcal{H}$, it suffices to indicate a subscheme of $B \times \mathbb{A}^n$ (product over $\mathbb{C}$), flat over B, such that the fibre over each $b \in B$ is defined by such a codimension-n ideal.

Take $B = X^0(V)$. A tensor $T \in X^0(V)$ gives rise to the ideal $I_T := \ker(\phi_T)$, where $\phi_T : \mathbb{C}[x_1, \ldots, x_n] \to (V, \mu_T)$ is the homomorphism of associative algebras that maps x_i to e_i and 1 to the unit element $u(T)$ from Lemma 9.1. The ideals I_T have vector space codimension n in R and together define a subscheme of $X^0(V) \times \mathbb{A}^n$ flat over $X^0(V)$. Hence we have described a morphism $\Phi : X^0(V) \to \mathcal{H}$. Since any algebra corresponding to a tensor in $X^0(V)$ has a nondegenerate invariant bilinear form, $\Phi(T) \cong (V, \mu_T)$ is Gorenstein for each $T \in X^0(V)$, so Φ is a morphism $X^0(V) \to \mathcal{H}^{\mathrm{Gor}}$. We want to use Φ to compare irreducible components of $\mathcal{H}^{\mathrm{Gor}}$ and $X^0(V)$. However, the map Φ is not an isomorphism, so some care is needed for this. We first describe the image of Φ; the following is immediate.

LEMMA 9.2. *The image of Φ consists of all codimension-n ideals $I \in \mathcal{H}^{\mathrm{Gor}}$ such that $x_1, \ldots, x_n \in R$ map to a basis of R/I and moreover the bilinear form on R/I for which this basis is orthonormal is invariant for the multiplication in R/I.* □

The following lemma shows that, as far as irreducible components are concerned, it is no real restriction to consider ideals I modulo which $x_1, \ldots, x_n$ is a basis.

LEMMA 9.3. *The locus $\mathcal{H}_0^{\mathrm{Gor}}$ in $\mathcal{H}^{\mathrm{Gor}}$ of ideals I in R for which $x_1, \ldots, x_n$ maps to a vector space basis of R/I is open and dense in $\mathcal{H}_0^{\mathrm{Gor}}$. Consequently, $\mathcal{H}_0^{\mathrm{Gor}}$ has the same number of irreducible components as $\mathcal{H}^{\mathrm{Gor}}$.*

This is well-known to the experts – see [10, Remark 4.5] – but we include a quick proof.

Proof. The condition on I can be expressed by the non-vanishing of certain determinants; this shows that $\mathcal{H}_0^{\mathrm{Gor}}$ is open. For density, suppose that some component C of $\mathcal{H}^{\mathrm{Gor}}$ does not meet $\mathcal{H}_0^{\mathrm{Gor}}$, and let I_0 be a point in C such that the image of $\langle x_1, \ldots, x_n \rangle_{\mathbb{C}}$ in R/I_0 has maximal dimension, say $m < n$. After a linear change of coordinates (which preserves all components of $\mathcal{H}^{\mathrm{Gor}}$ and hence in particular C), we may assume that $x_{m+1}, \ldots, x_n$ are in I_0, and

there exists a monomial r in $x_1, \ldots, x_m$ of degree $\neq 1$ such that $x_1, \ldots, x_m, r$ are linearly independent in R/I_0.

Now, for $a \in \mathbb{C}$, consider the nonlinear automorphism $\psi_a : R \to R$ that maps all $x_i, i \neq n$ to themselves but x_n to $x_n + a \cdot r$. The map $(a, I) \mapsto \psi_a^{-1}(I)$ defines an action of the additive group $(\mathbb{C}, +)$ on $\mathcal{H}^{\mathrm{Gor}}$, and since the additive group is irreducible, this action preserves all components of $\mathcal{H}^{\mathrm{Gor}}$. Since, for $a \neq 0$, $x_1, \ldots, x_m, x_n + ar$ are linearly independent modulo I_0, their pre-images $x_1, \ldots, x_m, x_n$ under ψ_a are linearly independent modulo $I_a := \psi_a^{-1}(I_0)$. Since I_a is in C, this contradicts the maximality assumption in the choice of I_0.

The last statement is now immediate. □

9.3. Varieties $Z_2 \to Z_1 \to \mathcal{H}_0^{\mathrm{Gor}}$ with the same number of components

In what follows, we will identify V with the space in R spanned by the variables $x_1, \ldots, x_n$, via the identification $e_i \mapsto x_i$. Each point in $\mathcal{H}_0^{\mathrm{Gor}}$ defines a unital, commutative, associative algebra structure on V. The structure constant tensor in $(S^2V^*) \otimes V$ of this algebra does not necessarily lie in $X^0(V)$, though, because the standard form β_0 may not be invariant for it.

LEMMA 9.4. *Let Z_1 be the subvariety*

$$\{(I, [\beta]) \in \mathcal{H}_0^{\mathrm{Gor}} \times \mathbb{P}(S^2V^*) \mid \beta \text{ is invariant for } R/I\} \subseteq \mathcal{H}_0^{\mathrm{Gor}} \times \mathbb{P}(S^2V^*).$$

Then the projection $Z_1 \to \mathcal{H}_0^{\mathrm{Gor}}$ is surjective and induces a bijection on irreducible components.

Proof. Indeed, every (possibly degenerate) invariant bilinear form on R/I is of the form $\beta(r, s) = \ell(rs)$ for a unique linear form $\ell \in (R/I)^*$, namely, the form $\ell(r) := \beta(1, r)$. Moreover, since for $I \in \mathcal{H}_0^{\mathrm{Gor}}$ the space V is a vector space complement of I in R, the natural map $(R/I)^* \to V^*$ is a linear bijection. We conclude that, in fact, Z_1 is isomorphic to $\mathcal{H}_0^{\mathrm{Gor}} \times \mathbb{P}(V^*)$ via the map that sends $(I, [\beta])$ to $(I, [v \mapsto \beta(1, v)])$. So each component of Z_1 is just a component of $\mathcal{H}_0^{\mathrm{Gor}}$ times the projective space $\mathbb{P}(V^*)$. □

LEMMA 9.5. *Let Z_2 be the subvariety*

$$\{((I, [\beta]), g) \in Z_1 \times \mathrm{GL}(V) \mid g[\beta] = [\beta_0]\} \subseteq Z_1 \times \mathrm{GL}(V).$$

Then the projection $Z_2 \to Z_1$ has dense image and induces a bijection between irreducible components.

Proof. For the first statement, if $I \in \mathcal{H}_0^{\mathrm{Gor}}$, then by definition there are *non-degenerate* invariant bilinear forms on R/I. These correspond to a dense open

subset of $\mathbb{P}(V^*)$ via the correspondence in the proof above. This shows that $Z_2 \to Z_1$ has dense image. This image, U, is open in Z_1.

Next we claim that, in the analytic topology, $Z_2 \to U$ is a fibre bundle with fibre the group $\mathbb{C}^* \cdot \mathrm{O}(\beta_0) \subseteq \mathrm{GL}(V)$; here $\mathrm{O}(\beta_0)$ is the orthogonal group of the form β_0. To see this, it consider a point $(I, [\beta_1]) \in U$. By definition of U, there exists a $g_1 \in \mathrm{GL}(V)$ such that $g_1[\beta_1] = [\beta_0]$. Furthermore, there exists a holomorphic map γ defined in an open neighbourhood Ω in $\mathbb{P}S^2V^*$ of $[\beta_0]$ to $\mathrm{GL}(V)$ such that $\gamma([\beta_0]) = \mathrm{id}_V$ and $\gamma([\beta])[\beta] = [\beta_0]$ for all $[\beta] \in \Omega$. Essentially, $\gamma([\beta])$ is found by the Gram-Schmidt algorithm – note that in this algorithm one has to divide by square roots of complex numbers, which, since $[\beta]$ is close to $[\beta_0]$, are close to 1; this can be done holomorphically.

Now the map

$$\begin{aligned}&(U \cap (\mathcal{H}_0^{\mathrm{Gor}} \times g_1^{-1}\Omega)) \times (\mathbb{C}^* \cdot \mathrm{O}(\beta_0)) \to Z_2,\\ &((I, [\beta]), g) \mapsto ((I, [\beta]), g \cdot \gamma(g_1[\beta]) \cdot g_1)\end{aligned}$$

trivialises the map $Z_2 \to Z_1$ over an open neighbourhood of $(I, [\beta_1])$; here we use that $\mathbb{C}^* \cdot \mathrm{O}(\beta_0)$ is the stabiliser of $[\beta_0]$ in $\mathrm{GL}(V)$.

Now since $Z_2 \to U$ is a fibre bundle with *irreducible* fibre $\mathbb{C}^* \cdot \mathrm{O}(V)$ – this is where it is important that we work with the projective space $\mathbb{P}S^2V^*$ rather than S^2V^*; the orthogonal group $\mathrm{O}(V)$ itself has two components! – that map induces a bijection between irreducible components. □

9.4. Completing the proof

Proof of Theorem 2.13. Recall that $X^0(V)$ parameterises the unital associative, commutative algebra structures on V such that β_0 is invariant for the multiplication. Now consider the map

$$\mathrm{GL}(V) \times X^0(V) \to \mathcal{H}_0^{\mathrm{Gor}}, (g, T) \mapsto g \cdot \Phi(T)$$

By Lemma 9.2, this map is surjective. Since $\mathrm{GL}(V)$ is irreducible, the left-hand side has as many irreducible components as $X^0(V)$. This shows that $X^0(V)$ has at least as many irreducible components as $\mathcal{H}_0^{\mathrm{Gor}}$, hence as $\mathcal{H}^{\mathrm{Gor}}$ by Lemma 9.3.

For the converse, by Lemmas 9.4 and 9.5, $\mathcal{H}_0^{\mathrm{Gor}}$ has as many irreducible components as Z_2. Now we claim that the morphism

$$\begin{aligned}Z_2 &\to (S^2V^*) \otimes V\\ ((I, [\beta]), g) &\mapsto \text{ the structure constant tensor of } R/(g \cdot I)\end{aligned}$$

has as image the variety $X^0(V)$. Indeed, if $((I, [\beta]), g)$ lies in Z_2, then β is an invariant symmetric bilinear form for the multiplication on R/I, and $g\beta \in \mathbb{C}^* \cdot \beta_0$ is an invariant symmetric bilinear form for the multiplication on

$R/(g \cdot I)$; this therefore corresponds to an element in $X^0(V)$. We conclude that the number of components of $X^0(V)$ is also at most that of $\mathcal{H}^{\mathrm{Gor}}$. This concludes the proof. □

9.5. Cubic dimension growth for the Gorenstein locus

We conclude this paper with an observation on the dimension of $\dim(\mathcal{H}^{\mathrm{Gor}})$.

PROPOSITION 9.6. *The dimension of $\mathcal{H}^{\mathrm{Gor}}$, and hence that of the Hilbert scheme $\mathcal{H}$ of n points in $\mathbb{A}^n$, is lower-bounded by a cubic polynomial in n for $n \to \infty$.*

Note that this was already known for $\mathcal{H}$ by [10, Theorem 9.2]. The algebras constructed there are of the form $A := \mathbb{C}[x_1, \ldots, x_d]/(V + \mathfrak{m}^3)$ where $\mathfrak{m}$ is the maximal ideal $(x_1, \ldots, x_d)$ and where $V \subseteq \mathfrak{m}^2/\mathfrak{m}^3$ has the correct codimension $r = n - 1 - d$ for this quotient to have dimension n. Since any 1-dimensional subspace of $\mathfrak{m}^2/(V + \mathfrak{m}^3)$ is a minimal ideal in A, A is not Gorenstein unless $\mathfrak{m}^2/(V + \mathfrak{m}^3)$ is one-dimensional, in which case $r = 1$. However, to obtain cubic behaviour in n, one needs r to grow linearly with d. So the cubic-dimensional locus in [10] is a non-Gorenstein part of the Hilbert scheme.

Proof. The unitalisation morphism sends $X(\mathbb{C}^{n-2})$ into $X^0(\mathbb{C}^n)$ by Proposition 6.7, and it does so injectively. This means that the latter variety has dimension at least that of $Z(\mathbb{C}^{n-2})$, which is lower-bounded by a cubic polynomial by Proposition 5.1. Furthermore, the morphism $\Phi : X^0(\mathbb{C}^n) \to \mathcal{H}^{\mathrm{Gor}}$ is also injective. □

REMARK 9.7. The coefficient of n^3 in $\dim(Z(\mathbb{C}^n))$ equals $\frac{1}{48}$, which is considerably smaller than the coefficient $\frac{2}{27}$ in [10] for the lower bound on the dimension of the Hilbert scheme of n points in $\mathbb{A}^n$. We do not know whether the $\frac{1}{48}$ can be improved.

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