A NOTE ON A THEOREM OF KHAN (*)

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SOMMARIO. - *T è un'applicazione di uno spazio metrico completo* (X, d) *in sè, tale che*

$$d(Tx, Ty) \leq K \frac{d(x, Tx)\, d(x, Ty) + d(y, Ty)\, d(y, Tx)}{d(x, Ty) + d(y, Tx)}$$

dove $0 \leq K < 1$, *e* $x, y \in X$. *Noi consideriamo ciò che accade se* $d(x, Ty) + d(y, Tx) = 0$.

SUMMARY. - *T is a mapping of the complete metric space* (X, d) *into itself satisfying*

$$d(Tx, Ty) \leq K \frac{d(x, Tx)\, d(x, Ty) + d(y, Ty)\, d(y, Tx)}{d(x, Ty) + d(y, Tx)}$$

where $0 \leq K < 1$, *and* $x, y \in X$. *We consider what happens if* $d(x, Ty) + d(y, Tx) = 0$.

In a recent paper, see [1], M. S. Khan gives the following theorem:

THEOREM. *Let* (X, d) *be a complete metric space and* $T: X \to X$ *satisfy*

$$d(Tx, Ty) \leq K \frac{d(x, Tx)\, d(x, Ty) + d(y, Ty)\, d(y, Tx)}{d(x, Ty) + d(y, Tx)} \tag{A}$$

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where $0 \leq K < 1$ *and* $x, y \in X$. *Then* T *has a unique fixed point.*

In the proof of his theorem, Khan does not consider the possibility that

$$d(x, Ty) + d(y, Tx) = 0.$$

If x_0 is an arbitrary point in X and $x_n = Tx_{n-1}$ for $n = 1, 2, \ldots$, then it follows that

$$d(x_n, x_{n+1} \leq K \frac{d(x_{n-1}, x_n)\, d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+1})\, d(x_n, x_n)}{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}$$

$$= K \frac{d(x_{n-1}, x_n)\, d(x_{n-1}, x_{n+1})}{d(x_{n-1}, x_{n+1})} = K\, d(x_{n-1}, x_n),$$

only if $d(x_{n-1}, x_{n+1}) \neq 0$. His proof therefore breaks down if $T^2 x_{n-1} = x_{n-1}$ for some n and all we will be able to deduce is that T^2 has a fixed point x_{n-1}. If the sequence $\{x_n\}$ is a sequence of distinct points then it will of course be convergent and its limit point x will be a fixed point of T.

We cannot exclude from the theorem the possibility that

$$d(x, Ty) + d(y, Tx) = 0$$

for some x, y in X since if x is the fixed point of T

$$d(x, Tx) + d(x, Tx) = 2\, d(x, x) = 0$$

and to exclude the possibility that

$$d(x, Ty) + d(y, Tx) = 0$$

for some distinct x, y in X is probably too restrictive. It would probably be best to ammend the theorem so that inequality (A) holds if

$$d(x, Ty) + d(y, Tx) \neq 0$$

and that

$$d(Tx, Ty) = 0$$

if

$$d(x, Ty) + d(y, Tx) = 0.$$

This would then imply that if we did indeed have $d(x_{n-1}, x_{n+1})=0$ for some n, then

$$x_{n-1}=x_n=x_{n+1}$$

and so x_{n-1} would be a fixed point of T.

A trivial example showing that a mapping T can satisfy inequality (A) for all distinct x, y in X with

$$d(x, Ty)+d(y, Tx)\neq 0,$$

is as follows: let $X=\{0, 1\}$ with metric

$$d(x, y)=|x-y|$$

for $x, y=0, 1$. Define a mapping T on X by

$$T(0)=1, \quad T(1)=0.$$

Inequality (A) is satisfied with $K=\frac{1}{2}$ for all cases with

$$d(x, Ty)+d(y, Tx)\neq 0$$

but T has no fixed point. T^2 however has two distinct fixed points.

A less trivial example is as follows: let $X=\{0, 1, 2, 3, \ldots\}$ with metric

$$d(0, 1)=1,$$

$$d(0, x)=d(1, x)=2,$$

for $x=2, 3, \ldots$,

$$d(x, y)=2,$$

for $x, y=2, 3, \ldots$ and $x\neq y$ and

$$d(x, x)=0,$$

for $x=0, 1, 2, \ldots$. Define a mapping T on X by

$$T(2x)=0, \quad T(2x+1)=1,$$

for $x=0, 1, 2, \ldots$. Inequality (A) is again satisfied with $K=\frac{1}{2}$, for all cases with

$$d(x, Ty) + d(y, Tx) \neq 0$$

but T has no fixed points.

The above comments also apply to the other theorems in [1].

REFERENCES

[1] M. S. Khan, « *A fixed point theorem for metric spaces* », Rend. Ist. di Matem. Univ. Trieste, vol. VIII (1976).