SOLVABILITY OF BOUNDARY VALUE PROBLEMS WITH HOMOGENEOUS ORDINARY DIFFERENTIAL OPERATOR (*)

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SOMMARIO. - *Si studia la risolubilità del problema di Dirichlet non lineare*

$$-(|u'|^{p-2}u')' = f(t,u) + g \text{ in } (0,\pi),$$
$$u(0) = u(\pi) = 0,$$

dove f è assoggettata a vari tipi di accrescimento legato agli autovalori dell'operatore differenziale nel membro sinistro. I risultati ottenuti vengono poi generalizzati agli operatori differenziali ordinari quasi-omogenei. Alcuni problemi aperti vengono indicati alla fine.

SUMMARY. - *We study solvability of nonlinear Dirichlet boundary value problem*

$$-(|u'|^{p-2}u')' = f(t,u) + g \text{ in } (0,\pi),$$
$$u(0) = u(\pi) = 0,$$

where the Carathéodory's function f satisfies various types of growth conditions in the second variable. The results are generalized for quasihomogeneous ordinary differential operators of second order.

1. - Introduction

We discuss solvability of strongly nonlinear Dirichlet boundary value problem

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$$\text{(BVP)} \quad \begin{cases} -(|u'|^{p-2}u')' = f(t,u) + g \ \text{ in } \ (0,\pi), \\ u(0) = u(\pi) = 0, \end{cases}$$

where $f(t,s) : [0,\pi] \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory's function satisfying various types of growth conditions depending on the eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ of the eigenvalue problem

$$\text{(EVP)} \quad \begin{cases} -(|u'|^{p-2}u')' = \lambda |u|^{p-2}u \ \text{ in } \ (0,\pi), \\ u(0) = u(\pi) = 0. \end{cases}$$

We shall prove that under certain growth assumptions on the function $f(t,s)$, (BVP) has at least one solution for each $g \in L_1(0,\pi)$.

The method of the proof of this results is based on a shooting argument and the Leray-Schauder degree theory. In the proofs we refer to a recent results concerning $(p\text{-}1)$-homogeneous ordinary differential operator of second order which are formulated in [2] and [4].

In Section 2 we give some preliminary definitions, assumptions and notations. There are also formulated some basic properties of the differential operator in question, which are proved in already mentioned works [2] and [4]. The main existence results are proved in Sections 3 and 4. These results are generalized in Section 5 to the quasihomogeneous ordinary differential operators. In Section 6 we give some suggestions for further research in that direction.

2. - Preliminaries

Let us suppose that $p \geq 2$ is a real number, $g \in L_1(\mathbf{R})$ and $f: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory's function, i.e. measurable in t for all $s \in \mathbf{R}$ and continuous in s for a.a. $t \in \mathbf{R}$. We are concerned with the initial value problem

$$\text{(IVP)} \quad \begin{cases} -(|u'|^{p-2}u')' = f(t,u) + g, \\ u(t_0) = \alpha_1, u'(t_0) = \alpha_2, t \in \mathbf{R}. \end{cases}$$

DEFINITION 2.1 - Let u be a real function of the real variable, suppose u' to be continuous and $|u'|^{p-2}u'$ absolutely continuous on each compact interval in $\mathbf{R}$. If the function u fulfils the initial conditions in (IVP) (resp. the boundary conditions in (BVP)) and the equation holds almost everywhere in $\mathbf{R}$ then u is called a *solution of the initial value problem (IVP)* (resp. *solution of the boundary value problem (BVP)*).

REMARK 2.1 - If we denote $q(\tau) = |\tau|^{p-2}\tau$, $\tau \in \mathbf{R}$, (IVP) may be rewritten into an equivalent vector form

$$\text{(IVP')} \quad \begin{cases} (u'(t), v'(t)) = (q^{-1}(v(t)), f(t,u(t)) + g(t)), t \in \mathbf{R}, \\ (u(t_0), v(t_0)) = (\alpha_1, |\alpha_2|^{p-2}\alpha_2). \end{cases}$$

Let $|f(t,s)| \leq m(t)$, $(t,s) \in \mathfrak{J} \times \mathbf{R}$, where $m(t)$ is Lebesgue-integrable function on the interval $\mathfrak{J} \subset \mathbf{R}$, $t_0 \in \mathfrak{J}$.

It is possible to show that the vector function

$$(q^{-1}(z_2), f(t, z_1) + g(t))$$

satisfies the assumptions stated in [1, p. 43] and we obtain the existence of the solution of (IVP) which is defined for a.a. $t \in \mathfrak{J}$.

REMARK 2.2 - Using a standard regularity argument for ordinary differential equations it is possible to prove that if $g \in C(\mathfrak{J})$, $f \in C(\mathfrak{J} \times \mathbf{R})$, for some interval $\mathfrak{J} \subset \mathbf{R}$ then $|u'|^{p-2}u' \in C^1(\mathfrak{J})$ and the equation (IVP) holds for each $t \in \mathfrak{J}$ (for the precise proof see [2, Th. 3.3, Rem. 4.2]).

REMARK 2.3 - Let us suppose that $\chi(t) \in L_\infty(\mathbf{R})$, $\chi(t) \geqslant \tau > 0$. Then using the shooting argument (e.g. [5]) it is possible to prove that the solution of the initial value problem

$$\text{(2.1)} \qquad \begin{cases} -(|u'|^{p-2}u')' = \chi(t)|u|^{p-2}u, \\ u(t_0) = 0, u'(t_0) = \alpha, \end{cases}$$

is determined uniquely.

REMARK 2.4 - It is proved in [2, Th. 4.4] the following assertion: «The eigenvalues of (EVP) form a sequence

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$$

with the property $\lim_{n \to \infty} \lambda_n = \infty$. To the least eigenvalue λ_1 there corresponds one and only one eigenfunction u (we suppose that $v'(0) = 1$ for each eigenfunction v). Moreover, $u(t) > 0$ for all $t \in (0, \pi)$. If $\lambda_n (n \geqslant 2)$ is an eigenvalue of (EVP) and v_n is the corresponding eigenfunction then there exist exactly $(n-1)$ equidistant zero points of v_n in $(0, \pi)$. To each λ_n there corresponds one and only one eigenfunction v_n and the following relation holds: $\lambda_n = n^p \lambda_1$».

REMARK 2.5 - The proof of the previous assertion is based on the properties of the solution of (IVP) formulated in Remarks 2.1, 2.2 and 2.3 (see [2, p. 176]). Let us remark that from the proof of this assertion it follows that the eigenfunction $v_n (n \geqslant 2)$ may be obtained on the basis of the first eigenfunction u by the following way:

$$v_n : t \longmapsto \begin{cases} -\dfrac{1}{n} u(nt), t \in [2l\dfrac{\pi}{n}, (2l+1)\dfrac{\pi}{n}), \\ +\dfrac{1}{n} u(nt), t \in [0, \pi] \setminus [2l\dfrac{\pi}{n}, (2l+1)\dfrac{\pi}{n}), \end{cases}$$

where $l = 1, 2, \ldots, \dfrac{n}{2}$ if n is even, $l = 1, 2, \ldots, [\dfrac{n}{2}] + 1$ if n is an odd number (see [2, p. 177]).

Let us suppose that $\chi(t) \in L_\infty(\mathfrak{I})$ for sufficiently large interval $\mathfrak{I} \subset \mathbf{R}$, $\lambda > 0$ is a real number. We shall consider the following two initial value problems

$$(2.2) \qquad \begin{cases} -(|w'|^{p-2}w')' = \chi(t)|w|^{p-2}w, \\ w(t_0) = 0, w'(t_0) = \alpha; \end{cases}$$

$$(2.3) \qquad \begin{cases} -(|u'|^{p-2}u')' = \lambda|u|^{p-2}u, \\ u(t_0) = 0, u'(t_0) = \alpha, \end{cases}$$

$t_0 \in \mathfrak{I}$, $\alpha > 0$. Let us denote

$$t_\chi = \inf\{t > t_0; w(t) = 0\},$$
$$t_\lambda = \inf\{t > t_0; u(t) = 0\},$$

and suppose that $t_\lambda \in \mathfrak{I}$. The following assertion will be essential for proving main existence results.

SHOOTING LEMMA. Let us suppose that

$$(2.4) \qquad \chi(t) \geq \lambda, \text{ for a.a. } t \in \mathfrak{I}.$$

Then $t_\chi \leq t_\lambda$.

Proof. There are $t_1 \in (t_0, t_\chi)$, $t_2 \in (t_0, t_\lambda)$ such that $w'(t_1) = u'(t_2) = 0$. Since w and u are concave functions on (t_0, t_χ), resp. (t_0, t_λ), these t_1, t_2 are determined uniquely and $w'(t) < 0$, $u'(t) < 0$, for $t \in (t_1, t_\chi)$, resp. for $t \in (t_2, t_\lambda)$.

We shall define the function $v(t) = u(t - t_1 + t_2)$ (the shift of u). Then this function is the solution of initial value problem

$$(2.5) \qquad \begin{cases} -(|v'|^{p-2}v')' = \lambda|v|^{p-2}v, \\ v(t_0 + t_1 - t_2) = 0, \quad v'(t_0 + t_1 - t_2) = \alpha, \end{cases}$$

and $t_\lambda + t_1 - t_2$ is the first zero point of v in the interval $(t_0 + t_1 - t_2, +\infty)$. Moreover $t_1 \in (t_0 + t_1 - t_2, t_\lambda + t_1 - t_2)$ is the unique point from this interval such that $v'(t_1) = 0$. Let us denote $t_3 = t_\lambda + t_1 - t_2$. We shall prove that $t_3 \geqslant t_\chi$.

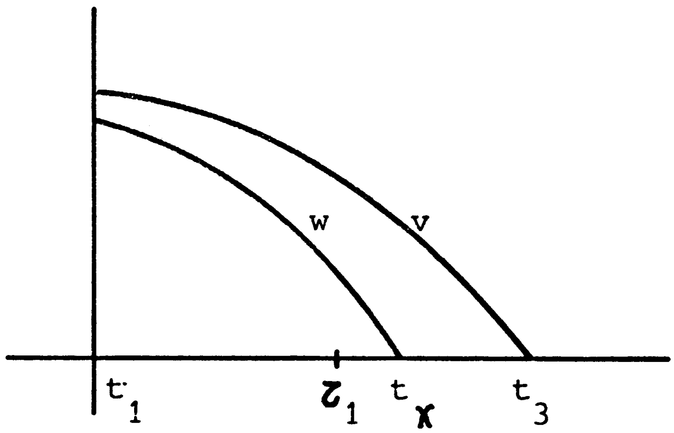

Fig. 1

Suppose to the contrary that $t_\chi > t_3$. Then there exists $\tau_1 \in (t_1, t_3)$ that

$$(2.6) \qquad \left(\frac{v}{w}\right)'(\tau_1) < 0 .$$

Really, in the opposite case

$$\frac{v(\tau)}{w(\tau)} \geqslant \frac{v(t_1)}{w(t_1)} = \text{const.} > 0$$

for all $\tau \in (t_1, t_3)$ and it is impossible because $v(t_3) = 0$. From (2.6) we obtain

$$(2.7) \qquad (v'w - vw')(\tau_1) < 0 .$$

The function $z \mapsto |z|^{p-2} z$ is increasing on **R**. Hence (2.7) is the same as $F(\tau_1) < 0$, where

$$F : t \mapsto (|v'|^{p-2} v'(w)^{p-1} - (v)^{p-1} |w'|^{p-2} w')(t).$$

Since $F(t_1) = 0$ there exists the set $\mathfrak{A} \subset (t_1, \tau_1)$ of positive measure that

$$(2.8) \qquad F(t) < 0, F'(t) < 0, w'(t) < 0, v'(t) < 0$$

hold simultaneously for all $t \in \mathfrak{A}$. Since F is absolutely continuous in $\mathfrak{J}$, the derivative $F'(t)$ exists a.e. in $\mathfrak{J}$. By an elementary calculation we obtain

$$F'(t) = F_1(t) + F_2(t),$$

where

$$F_1(t) = ((|v'|^{p-2} v')'(w)^{p-1} - (v)^{p-1}(|w'|^{p-2} w')')(t),$$
$$F_2(t) = (p-1) v' w'(|v'|^{p-2}(w)^{p-2} - |w'|^{p-2}(v)^{p-2})(t).$$

According (2.8) it is also

$$(v' w - v w')(t) < 0$$

for all $t \in \mathfrak{A}$. Hence

$$(|v'|^{p-2}(w)^{p-2} - (v)^{p-2} |w'|^{p-2})(t) > 0$$

for all $t \in \mathfrak{A}$ and that is why

$$(2.9) \qquad F_2(t) > 0, \ t \in \mathfrak{A} .$$

Putting together (2.8) and (2.9) we conclude

$$(2.10) \qquad F_1(t) < 0, \ t \in \mathfrak{A} .$$

On the other hand from the equations (2.2), (2.5) and from the assumption (2.4) we obtain

$$F_1(t) = \chi(t)(w(t))^{p-1}(v(t))^{p-1} - \lambda(v(t))^{p-1}(w(t))^{p-1} \geqslant 0 ,$$

for a.a. $t \in \mathfrak{A}$. But this is a contradiction with (2.10). Analogously if we put $t_4 = t_0 + t_1 - t_2$, we prove using the same technique that $t_4 \leq t_0$. Hence comming back to u (the solution of (2.3)) we obtain that $t_\chi \leq t_\lambda$. Q.E.D.

REMARK 2.6 - Let us suppose that $\chi(t) \leq \lambda$, for a.a. $t \in \mathfrak{I}$. Then either t_χ does not exist or $t_\chi \geq t_\lambda$.

Let us suppose that $t_\chi \in \mathfrak{I}$ exists. Then we obtain directly from (2.2) (multiplying by w and integrating by parts in (t_0, t_χ)) that

$$(2.11) \qquad 1 = \frac{\int_{t_0}^{t_\chi} |w'(t)|^p \, dt}{\int_{t_0}^{t_\chi} \chi(t) |w(t)|^p \, dt} .$$

If $t_\lambda > t_\chi$, then the function $\tilde{u}(t) = u(\frac{t_\lambda - t_0}{t_\chi - t_0}(t - t_0) + t_0)$ is the solution of

$$\begin{cases} -(|u'|^{p-2} u')' = \tilde{\lambda} |u|^{p-2} u, \\ u(t_0) = u(t_\chi) = 0, \end{cases}$$

and $u > 0$ in (t_0, t_χ), where $\tilde{\lambda} = \lambda (\frac{t_\lambda - t_0}{t_\chi - t_0})^p > \lambda$.

According to [4] we have

$$(2.12) \qquad \inf_{\substack{u \in W^{1,p}(0,\pi) \\ u \neq 0}} \frac{\int_{t_0}^{t_\chi} |u'(t)|^p \, dt}{\int_{t_0}^{t_\chi} \tilde{\lambda} |u(t)|^p \, dt} = \frac{\int_{t_0}^{t_\chi} |\tilde{u}'(t)|^p \, dt}{\int_{t_0}^{t_\chi} \tilde{\lambda} |\tilde{u}(t)|^p \, dt} .$$

Obviously (2.11) and (2.12) cannot hold simultaneously. Hence $t_\chi \geq t_\lambda$.

REMARK 2.7 - Analogous results we obtain also for $\alpha < 0$ in (2.2), resp. in (2.3).

3. - First existence result

Suppose that $f : [0, \pi] \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory's function such that

$$(3.1) \qquad |f(t,s)| \leq m(t) + c |s|^{p-1}$$

for all $s \in \mathbf{R}$ and a.a. $t \in [0, \pi]$, where $m \in L_q(0, \pi)$, $1/p + 1/q = 1$ and $c > 0$ is a constant. We shall suppose that there exist limits

$$(3.2) \qquad \lim_{s \to +\infty} \frac{f(t,s)}{|s|^{p-2} s} = \chi_+(t), \quad \lim_{s \to -\infty} \frac{f(t,s)}{|s|^{p-2} s} = \chi_-(t),$$

for a.a. $t \in [0, \pi]$.

THEOREM 3.1 - *Let us suppose that*

(3.3) $\chi_+(t),\ \chi_-(t) \leqslant \mu_0 < \lambda_1$ or

(3.4) $\lambda_i < \mu_i \leqslant \chi_\pm(t) \leqslant \mu_{i+1} < \lambda_{i+1},\ i \in \mathbf{N},$

for a.a. $t \in [0,\pi]$. *Then* (BVP) *has at least one solution for each* $g \in L_1(0,\pi)$.

Proof. Let us denote by $\overset{\circ}{W}{}^{1,p}(0,\pi)$ the usual Sobolev space. It is $\overset{\circ}{W}{}^{1,p}(0,\pi) \hookrightarrow\hookrightarrow L_p(0,\pi)$ and $\overset{\circ}{W}{}^{1,p}(0,\pi) \hookrightarrow\hookrightarrow C([0,\pi])$ (the symbol $\hookrightarrow\hookrightarrow$ denotes the compact imbedding). We shall define operators $J: \overset{\circ}{W}{}^{1,p}(0,\pi) \to W^{-1,q}(0,\pi)$ (dual space), $F: \overset{\circ}{W}{}^{1,p}(0,\pi) \to W^{-1,q}(0,\pi)$ and an element $G \in W^{-1,q}(0,\pi)$ by the following way:

$$(J(u),v) = \int_0^\pi |u'(t)|^{p-2}u'(t)v'(t)\,dt,$$

$$(F(u),v) = \int_0^\pi f(t,u(t))v(t)\,dt,$$

$$(G,v) = \int_0^\pi g(t)v(t)\,dt,$$

$u,v \in \overset{\circ}{W}{}^{1,p}(0,\pi)$, where $(.,.)$ is a pairing between $W^{-1,q}(0,\pi)$ and $\overset{\circ}{W}{}^{1,p}(0,\pi)$. Then J is odd, positively $(p-1)$-homogeneous, one-to-one homeomorphism $\overset{\circ}{W}{}^{1,p}(0,\pi)$ onto its dual and F is completely continuous (see [2]).

In the first step we prove that there exists at least one $u_0 \in \overset{\circ}{W}{}^{1,p}(0,\pi)$,

(3.5) $J(u_0) = F(u_0) + G.$

We use the Leray-Schauder degree theory. Choose $\lambda < \mu_0$, resp. $\lambda \in (\mu_i, \mu_{i+1})$, if (3.3), resp. (3.4), is satisfied. Define an odd, completely continuous, $(p-1)$-homogeneous operator $S: \overset{\circ}{W}{}^{1,p}(0,\pi) \to W^{-1,q}(0,\pi)$ by the following way

$$(S(u),v) = \int_0^\pi |u(t)|^{p-2}u(t)v(t)\,dt,$$

$u,v \in \overset{\circ}{W}{}^{1,p}(0,\pi)$. We shall prove the existence of the ball $B_r(0)$ centred at the origin and with sufficiently large radius $r > 0$ such that

(3.6) $H(u,\tau) \neq 0,$

for all $u \in \partial B_r(0)$ (the boundary of $B_r(0)$), $\tau \in [0,1]$, where

(3.7) $H(u,\tau) = J(u) - \tau F(u) - \tau G - (1-\tau)\lambda S(u),$
$u \in \overset{\circ}{W}{}^{1,p}(0,\pi),\ \tau \in [0,1].$

If (3.6) is not true, there exist sequences $\{u_n\}_{n=1}^\infty, \{\tau_n\}_{n=1}^\infty$ such that $\tau_n \in [0,1]$, $\| u_n \|_{1,p} \to +\infty$ and

$$H(u_n, \tau_n) = 0 .$$

Multiplying the last equality by $1/\| u_n \|_{1,p}^{p-1}$, we obtain with respect to (3.7)

$$J(v_n) - \tau_n \frac{F(u_n)}{\| u_n \|_{1,p}^{p-1}} - \tau_n \frac{G}{\| u_n \|_{1,p}^{p-1}} - (1-\tau_n) \lambda S(v_n) = 0 , \tag{3.8}$$

where $v_n = \dfrac{u_n}{\| u_n \|_{1,p}}$. Hence $\| v_n \|_{1,p} = 1$, for all $n \in \mathbf{N}$, and we may suppose (after passing to suitable subsequence) that $v_n \to v$ in $\overset{0}{W}{}^{1,p}(0,\pi)$, $v_n \to v$ in $C([0,\pi])$ and $\tau_n \to \tau \in [0,1]$. Let us denote by $M = \{t \in [0,\pi];\ v(t) \neq 0\}$. Then it is $|u_n(t)| \to \infty$ for $t \in M$. According to (3.2) it is

$$\frac{f(t,u_n)}{\| u_n \|_{1,p}^{p-1}} \longrightarrow \chi_+(t) | v^+ |^{p-2} v - \chi_-(t) | v^- |^{p-2} v$$

a.e. in M. On the other hand we have (due to (3.1))

$$\left| \frac{f(t,u_n)}{\| u_n \|_{1,p}^{p-1}} \right| \leqslant \frac{m(t)}{\| u_n \|_{1,p}^{p-1}} + c \frac{| u_n |^{p-1}}{\| u_n \|_{1,p}^{p-1}} = \frac{m(t)}{\| u_n \|_{1,p}^{p-1}} + c | v_n |^{p-1} .$$

Hence

$$\frac{f(t,u_n)}{\| u_n \|_{1,p}^{p-1}} \longrightarrow 0 \text{ a.e. in } [0,\pi] \setminus M$$

and

$$\left\| \frac{f(t,u_n)}{\| u_n \|_{1,p}^{p-1}} \right\|_{L_q} \leqslant \text{const.}$$

(with the constant independent of n). Using the Lebesque dominated convergence theorem we obtain

$$\int_0^\pi [\frac{f(t,u_n)}{\| u_n \|_{1,p}^{p-1}} - \chi_+(t) | v^+ |^{p-2} v - \chi_-(t) | v^- |^{p-2} v]^q \, dt \to 0 .$$

Passing to the limit for $n \to \infty$ in (3.8) we obtain that $v_n \to v$ in $\overset{0}{W}{}^{1,p}(0,\pi)$ and

$$\int_0^\pi | v' |^{p-2} v' w' \, dt - \int_0^\pi [\tau \chi_+(t) + (1-\tau)\lambda] | v^+ |^{p-2} v w \, dt + \int_0^\pi [\tau \chi_-(t) + (1-\tau)\lambda] | v^- |^{p-2} v w \, dt = 0 \tag{3.9}$$

holds for all $w \in \overset{0}{W}{}^{1,p}(0,\pi)$. Denote

$$\tilde{\chi}_+(t) = \tau\chi_+(t) + (1-\tau)\lambda,$$

$$\tilde{\chi}_-(t) = \tau\chi_-(t) + (1-\tau)\lambda.$$

It is clear that $\tilde{\chi}_+(t)$ and $\tilde{\chi}_-(t)$ satisfy the inequalities (3.3), (3.4). Using a standard regularity procedure for ordinary differential equations it is possible to prove that if v satisfies (3.9) then $v' \in C([0,\pi])$, $|v'|^{p-2}v'$ is absolutely continuous, the equation

$$-(|v'|^{p-2}v')' - \tilde{\chi}_+(t)|v^+|^{p-2}v + \tilde{\chi}_-(t)|v^-|^{p-2}v = 0$$

holds a.e. in $[0,\pi]$ and $v(0) = v(\pi) = 0$.

Let us consider, now, the initial value problem

$$(3.10)\qquad \begin{cases} -(|v'|^{p-2}v')' - \tilde{\chi}_+(t)|v^+|^{p-2}v + \tilde{\chi}_-(t)|v^-|^{p-2}v = 0, \\ v(0) = 0, v'(0) = \alpha. \end{cases}$$

If (3.3) is satisfied then the same inequality is fulfiled also by $\tilde{\chi}_+$, resp $\tilde{\chi}_-$. Shooting lemma and Remark 2.6 imply that nontrivial solution of (3.10) has no zero point in $(0,\pi]$. Hence there is no such a $v \neq 0$ satisfying (3.9) and this is the contradiction. If (3.4) is satisfied (i.e. it is satisfied also with $\tilde{\chi}_+$ and $\tilde{\chi}_-$) then let us denote by v_i, resp. v_{i+1}, the eigenfunction corresponding to the eigenvalue λ_i, resp. λ_{i+1}. Remark 2.3 (unicity of the solution of (IVP)) implies that for any zero point t_0 of v (the solution of (3.10)) in $[0,\pi]$ it is $v'(t_0) \neq 0$. Then Shooting lemma and Remark 2.6 imply that the zero points of the solution of (3.10) lie strictly between the zero points of v_i and v_{i+1}.

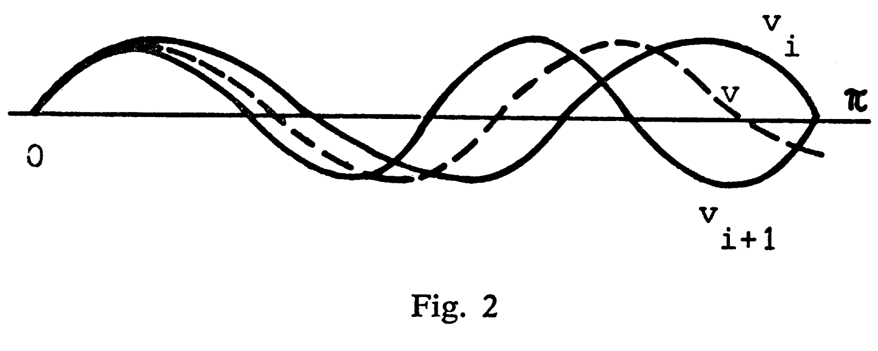

Fig. 2

Hence it is $v(\pi) \neq 0$ and there is no v satisfying (3.9). We have just proved that (3.6) is true. Let us recall that J is one-to-one homeomorphism $W_0^{1,p}(0,\pi)$ onto its dual and it satisfies

$$c_1\|u\|_{1,p}^{p-1} \leq \|J(u)\|_{1,-q} \leq c_2\|u\|_{1,p}^{p-1}$$

for all $u \in W_0^{1,p}(0,\pi)$ with some constants c_1 and c_2 (see [2, Lemma

3.2]). Hence there exists sufficiently large ball $B_\rho(0)$ in $W^{-1,q}(0,\pi)$ that

(3.6') $\quad H(J^{-1}(w),\tau)\neq 0,$

for all $w\in\partial B_\rho(0)$ and $\tau\in[0,1]$.

Now, we can use homotopy invariance property of the Leray-Schauder degree, the fact that S is odd and the Borsuk theorem (see e.g. [4]). On the basis of (3.6') we can write $\deg[w-F(J^{-1}(w))+G; B_\rho(0),0]=\deg[w-\lambda S(J^{-1}(w)); B_\rho(0),0]=$ an odd number.

This implies the existence of at least one $u_0\in W_0^{1,p}(0,\pi)$ such that

(3.5) $\quad J(u_0)=F(u_0)+G$, i.e.

$$\int_0^\pi |u'_0|^{p-2}u'_0 w'\,dt=\int_0^\pi f(t,u_0)w\,dt+\int_0^\pi g w\,dt,$$

for all $w\in W_0^{1,p}(0,\pi)$.

The second step of the proof consists of verifying the fact that u_0 is also the solution of (BVP) in the sense of Definition 2.1. But it may be shown using standard regularity argument for ordinary differential equations (see e.g. [2, Th. 3.3] for the proof of quite analogous assertion).

Q.E.D.

REMARK 3.1 - The assertion of previous theorem can be generalized in various directions. The case of nonexistence of the limits (3.2) will be the subject of the next section. In this section we shall study some other types of sufficient conditions than (3.3), resp. (3.4). We use some results concerning solvability of boundary value problem

(3.11) $$\begin{cases} -(|u'|^{p-2}u')'=\mu|u^+|^{p-2}u-\nu|u^-|^{p-2}u+g,\\ u(0)=u(\pi)=0,\end{cases}$$

which were proved in [2].

REMARK 3.2 - There was proved in [2, Th. 4.5] the following assertion:

«Boundary value problem (3.11) with $g=0$ has a nontrivial solution if and only if $(\mu,\nu)\in A_{-1}$, i.e. one of the following conditions holds:

(i) $\mu=\lambda_1$, ν arbitrary;

(ii) μ arbitrary, $\nu=\lambda_1$;

(iii) $\mu>\lambda_1$, $\nu>\lambda_1$ and

$$\frac{(\mu)^{1/p}(\nu)^{1/p}}{((\mu)^{1/p}+(\nu)^{1/p})(\lambda_1)^{1/p}}\in\mathbf{N},$$

$$\frac{((\mu)^{1/p} - (\lambda_1)^{1/p})(\nu)^{1/p}}{((\mu)^{1/p} + (\nu)^{1/p})(\lambda_1)^{1/p}} \in \mathbf{N},$$

$$\frac{((\nu)^{1/p} - (\lambda_1)^{1/p})(\mu)^{1/p}}{((\mu)^{1/p} + (\nu)^{1/p})(\lambda_1)^{1/p}} \in \mathbf{N},$$

where **N** denotes the set of all positive integers».

The following lines in Fig. 3 give the geometric illustration of the previous classification of the couples (μ, ν).

REMARK 3.3 - The plane (μ, ν) is then devided by A_{-1} into an open components. Let us denote by A_1 the union of such components which contain some (λ, λ), $\lambda \in \mathbf{R}$, as an interior point. Then using the ho-

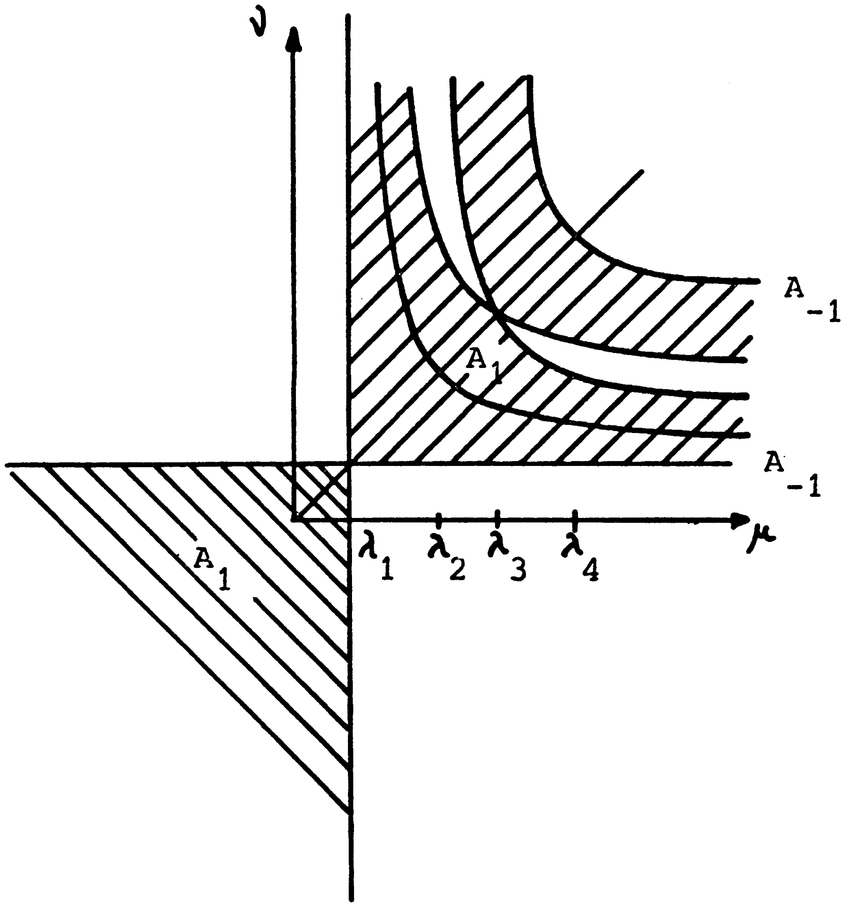

Fig. 3

motopy invariance property of the Leray-Schauder degree it is not hard to see that

$$\deg\,[w - \mu S(J^{-1}(w))^+ + \nu S(J^{-1}(w))^-;\ B_\rho(0), 0] \neq 0, \tag{3.12}$$

for sufficiently large ball $B_\rho(0) \subset W^{-1,q}(0, \pi)$ whenever $(\mu, \nu) \in A_1$ (see [2] for details).

Using this result we can generalize the assertion of Theorem 3.1 in the following sense.

THEOREM 3.2 - *Let K be a rectangle contained in some component of A_1. Let*

$$(3.13)\qquad (\chi_+(x),\ \chi_-(y))\in K,$$

for a.a. $(x,y)\in[0,\pi]\times[0,\pi]$. Then (BVP) *has at least one solution for each $g\in L_1(0,\pi)$.*

Proof. We shall modify the proof of Theorem 3.1. Take some $(\mu,\nu)\in K$. Then we show that

$$(3.14)\qquad H(u,\tau)\neq 0,$$

for all $u\in\partial B_r(0)$, $\tau\in[0,1]$, where $B_r(0)$ is a ball with sufficiently large radius and

$$(3.15)\quad H(u,\tau)=J(u)-\tau F(u)-\tau G-(1-\tau)\mu S(u^+)+(1-\tau)\nu S(u^-),$$
$$u\in \overset{0}{W}{}^{1,p}(0,\pi),\ \tau\in[0,1].$$

If we suppose that (3.14) is not true using similar argument as in the proof of Theorem 3.1 we obtain the existence of $v\in \overset{0}{W}{}^{1,p}(0,\pi)$, $\|v\|_{1,p}=1$ and $\tau\in[0,1]$ such that

$$(3.16)\qquad \int_0^\pi |v'|^{p-2}v'w'\,dt-\int_0^\pi[\tau\chi_+(t)+(1-\tau)\mu]\,|v^+|^{p-2}vw\,dt+$$
$$+\int_0^\pi[\tau\chi_-(t)+(1-\tau)\nu]\,|v^-|^{p-2}vw\,dt=0,$$

for all $w\in \overset{0}{W}{}^{1,p}(0,\pi)$. Denote

$$\tilde\chi_+(t)=\tau\chi_+(t)+(1-\tau)\mu,$$
$$\tilde\chi_-(t)=\tau\chi_-(t)+(1-\tau)\nu.$$

It is clear that $(\tilde\chi_+(x),\ \tilde\chi_-(y))\in K$ for a.a. $(x,y)\in[0,\pi]\times[0,\pi]$ and that v is the solution of the boundary value problem

$$(3.17)\qquad \begin{cases} -(|v'|^{p-2}v')'-\tilde\chi_+(t)|v^+|^{p-2}v+\tilde\chi_-(t)|v^-|^{p-2}v=0,\\ v(0)=v(\pi)=0\end{cases}$$

in the sense of Definition 2.1. Let us suppose at first that K is such that $\mu,\nu>\lambda_1$ for all $(\mu,\nu)\in K$. Since K is compact subset of some component of A_1 there exist $(\mu_1,\nu_1)\in A_{-1}$ and $(\mu_2,\nu_2)\in A_{-1}$ such that there exist nontrivial solutions v_1,v_2 of

$$(3.18)\qquad \begin{cases} -(|v'|^{p-2}v')'-\mu_i|v^+|^{p-2}v+\nu_i|v^-|^{p-2}v=0,\\ v(0)=v(\pi)=0\end{cases}$$

and the number of zero points of v_1, resp. v_2, in $(0,\pi)$ is k, resp. $k+1$. Since μ_i,ν_i are chosen in such a way that $\mu_1\le\mu$, $\nu_1\le\nu$, $\mu_2\ge\mu$, $\nu_2\ge\nu$, for all $(\mu,\nu)\in K$, using Shooting lemma we obtain that the zero points of the nontrivial solutions of (3.17) lie strictly between the zero points of the solutions $v_i, i = 1, 2$, of (3.18) which is a contradiction with the fact that $v(\pi) = 0$.

If K is a subset of a component where $\mu, \nu < \lambda_1$ for all couples (μ, ν), the situation is the same as in Theorem 3.1 (when (3.3) is satisfied).

By the same way as in the proof of Theorem 3.1 we use the homotopy invariance property of the Leray-Schauder degree and the fact that (3.12) is true whenever $(\mu, \nu) \in A_1$. The regularity result then implies the existence of the solution of (BVP) for each $g \in L_1(0, \pi)$ in the sense of Definition 2.1. Q.E.D.

REMARK 3.4 - Let us remark that the assumptions (3.3), (3.4) are the special case of the assumptions (3.13). Really, (3.3) is equivalent to $(\chi_+(x), \chi_-(y)) \in K$ for a.a. $(x, y) \in [0, \pi] \times [0, \pi]$, where

$$K = \{ (\mu, \nu) \in \mathbf{R}^2 ;\ \mu, \nu \leq \mu_0 < \lambda_1 \}.$$

The assumption (3.4) is equivalent to $(\chi_+(x), \chi_-(y)) \in K$, for a.a. $(x, y) \in [0, \pi] \times [0, \pi]$, where

$$K = \{ (\mu, \nu) \in \mathbf{R}^2 ;\ \lambda_i < \mu_i \leqslant \mu \leqslant \mu_{i+1} < \lambda_{i+1},$$
$$\lambda_i < \mu_i \leqslant \nu \leqslant \mu_{i+1} < \lambda_{i+1} \},\ i = 1, 2, \ldots .$$

4. - Second existence result

Let us suppose that a Carathéodory's function $f : [0, \pi] \times \mathbf{R} \to \mathbf{R}$ satisfies (3.1). We shall suppose, now, that there exist finite

$$\limsup_{s \to \pm\infty} \frac{f(t, s)}{|s|^{p-2} s} = \chi^{\pm\infty}(t),$$

$$\liminf_{s \to \pm\infty} \frac{f(t, s)}{|s|^{p-2} s} = \chi_{\pm\infty}(t),$$

for a.a. $t \in [0, \pi]$.

The assertion of Theorem 3.1 can be generalized also in the following way.

THEOREM 4.1 - *Let us suppose that*

(4.1) $\quad \chi_{\pm\infty}(t),\ \chi^{\pm\infty}(t) \leqslant \mu_0 < \lambda_1$ *or*

(4.2) $\quad \lambda_i < \mu_i \leqslant \chi_{\pm\infty}(t) \leqslant \mu_{i+1} < \lambda_{i+1},$

$\quad \lambda_i < \mu_i \leqslant \chi^{\pm\infty}(t) \leqslant \mu_{i+1} < \lambda_{i+1},\ i \in \mathbf{N},$

for a.a. $t \in [0, \pi]$. *Then* (BVP) *has at least one solution for each* $g \in L_1(0, \pi)$.

Proof. We shall prove the assertion under the assumptions (4.2) for some $i \in \mathbf{N}$. Then it will be clear how to proceed if (4.1) is satisfied. We use the same notation as in the proof of Theorem 3.1. Our aim, at first, is to prove the existence of at least one $u_0 \in W_0^{1,p}(0,\pi)$ such that

(3.5) $$J(u_0) = F(u_0) + G$$

is satisfied.

It is sufficient to show that the homotopy

$$H : W_0^{1,p}(0,\pi) \times [0,1] \to W^{-1,q}(0,\pi)$$

defined by (3.7) satisfies (3.6). Let us suppose that this is not true. Then there exist

$\{u_n\}_{n=1}^{\infty} \subset W_0^{1,p}(0,\pi)$, $\{\tau_n\}_{n=1}^{\infty} \subset [0,1]$, $\| u_n \|_{1,p} \to \infty$ and $H(u_n, \tau_n) = 0$. If we denote $v_n = u_n / \| u_n \|_{1,p}$ then after passing to subsequences we can suppose that $v_n \rightharpoonup v$ in $W_0^{1,p}(0,\pi)$, $v_n \to v$ in $C([0,\pi])$, $\tau_n \to \tau \in [0,1]$ and

(4.3) $$\int_0^{\pi} [|v'_n|^{p-2} v'_n - |v'_m|^{p-2} v'_m] w' \, dt =$$
$$= \int_0^{\pi} [\tau_n h_n(t) - \tau_m h_m(t)] w \, dt +$$
$$+ \int_0^{\pi} [\tau_n \frac{g}{\| u_n \|_{1,p}^{p-1}} - \tau_m \frac{g}{\| u_m \|_{1,p}^{p-1}}] w \, dt +$$
$$+ \lambda \int_0^{\pi} [(1-\tau_n) |v_n|^{p-2} v_n - (1-\tau_m) |v_m|^{p-2} v_m] w \, dt ,$$

for all $w \in W_0^{1,p}(0,\pi)$, $n, m \to \infty$, where

$$h_k(t) = \frac{f(t, u_k(t))}{\| u_k \|_{1,p}^{p-1}} .$$

Let us put $w = v_n - v_m$ in (4.3). Since $(v_n - v_m) \to 0$ in $C([0,\pi])$, $n, m \to \infty$ and $\| h_k \|_{L^q} \leqslant$ const. for all $k \in \mathbf{N}$ (f satisfies condition (3.1)), we have

$$\int_0^{\pi} [\tau_n h_n(t) - \tau_m h_m(t)] (v_n - v_m) \, dt \to 0 ,$$

for $n, m \to \infty$. From this and from inequality

$$(J(u) - J(v), u - v \geqslant c \| u - v \|_{1,p}^{p}$$

for all $u, v \in W_0^{1,p}(0,\pi)$ with some constant $c > 0$ (see [2, Lemma 3.2]) we obtain with respect to (4.3) that

$$\| v_n - v_m \|_{1,p}^{p} \to 0 ,$$

for $n, m \to \infty$. Hence $v_n \to v$ in $W_0^{1,p}(0,\pi)$ and $\| v \|_{1,p} = 1$.

We may suppose that (at least for some subsequence)

$$h_k \rightharpoonup h_1 , h_1 \in L^q(0,\pi).$$

Taking into account $L^q(0,\pi) \hookrightarrow\hookrightarrow W^{-1,q}(0,\pi)$ and passing to the limit (for $n \to \infty$) in

$$(4.4) \qquad \int_0^\pi |v'_n|^{p-2} v'_n w' \, dt - \tau_n \int_0^\pi h_n w \, dt - \tau_n \int_0^\pi \frac{g}{\|u_n\|_{1,p}^{p-1}} w \, dt -$$

$$- (1 - \tau_n) \lambda \int_0^\pi |v_n|^{p-2} v_n w \, dt = 0$$

we obtain that

$$(4.5) \qquad \int_0^\pi |v'|^{p-2} v' u' \, dt - \tau \int_0^\pi h_1 u \, dt - (1 - \tau) \lambda \int_0^\pi |v|^{p-2} v u \, dt = 0 ,$$

holds for all $u \in \overset{0}{W}{}^{1,p}(0,\pi)$. We shall show that

$$h_1(t) = h(t) |v(t)|^{p-2} v(t),$$

where $h(t) \in [\mu_i, \mu_{i+1}]$ for a.a. $t \in (0,\pi)$, $v(t) \neq 0$, and $h(t) = 0$ for a.a. $t \in (0,\pi), v(t) = 0$. Let us suppose, at first, that there exists some set $A \subset N_0 = \{\tau \in [0,\pi]; v(\tau) = 0\}$, meas $A > 0$ such that $h_1(t) > 0$, $t \in A$. Then (3.1) implies that

$$h_n(t) = \frac{f(t, u_n(t))}{\|u_n\|_{1,p}^{p-1}} \leqslant \frac{m(t)}{\|u_n\|_{1,p}^{p-1}} + C |v_n(t)|^{p-1}.$$

Since $v_n \to v$ in $C([0,\pi])$ we have $h_n(t) \to 0$ a.e. in A, and $|h_n(t)| \leq \tilde{m}(t)$, for a.a. $t \in (0,\pi), \tilde{m}(t) \in L^1(0,\pi)$.

Take $w(t) = \chi_A(t), t \in [0,\pi]$ (where χ_A is the characteristic function of A). We may use the Lebesque convergence theorem to obtain $\lim_{n\to\infty} \int_0^\pi h_n(t) w(t) \, dt = \int_0^\pi \lim_{n\to\infty} h_n(t) \chi_A(t) \, dt = 0$.

On the other hand

$$\int_0^\pi h_1(t) \, w(t) \, dt = \int_A h_1(t) \, dt > 0 ,$$

which contradicts to $h_n \to h_1$ in $L^q(0,\pi)$. Hence $h_1(t) = 0$ for a.a. $t \in N_0$. We show, now, that $h_1(t) \geq \mu_i |v(t)|^{p-1} v(t)$ for a.a. $t \in N_+ = \{\tau \in [0,\pi]; v(\tau) > 0\}$. Let us suppose that this is not true, i.e. there is $A \subset N_+$, meas $A > 0$, such that $h_1(t) < \mu_i |v(t)|^{p-2} v(t)$ for $t \in A$. For each $t \notin N_0$ we have $|u_n(t)| \to +\infty$. Hence laking into account $v_n \to v$ in $C([0,\pi])$ we have

$$\liminf_{n\to\infty} h_n(t) = \liminf_{n\to\infty} \frac{f(t, u_n(t))}{\|u_n\|_{1,p}^{p-1}} =$$

$$= \liminf_{n\to\infty} \frac{f(t, u_n(t))}{|u_n(t)|^{p-2} u_n(t)} |v_n(t)|^{p-2} v_n(t) \geq \mu_i |v(t)|^{p-2} v(t),$$

for a.a. $t \in A$ according to (4.2). Take $w(t) = \chi_A(t)$, $t \in [0,\pi]$. Using Fatou's lemma we obtain

$$\liminf_{n\to\infty} \int_0^\pi h_n(t)\, w(t)\, dt \geqslant \int_0^\pi \liminf_{n\to\infty} h_n(t)\, \chi_A(t)\, dt \geqslant \mu_i \int_A |v(t)|^{p-2} v(t)\, dt.$$

On the other hand

$$\int_0^\pi h_1(t)\, w(t)\, dt = \int_A h_1(t)\, dt < \mu_i \int_A |v(t)|^{p-2} v(t)\, dt.$$

Hence $h_1(t) \geqslant \mu_i |v(t)|^{p-2} v(t)$ for a.a. $t \in N_+$. Analogously we prove that $h_1(t) \leq \mu_{i+1} |v(t)|^{p-2} v(t)$ for a.a. $t \in N_+$ and

$$\mu_{i+1} |v(t)|^{p-2} v(t) \leq h_1(t) \leq \mu_i |v(t)|^{p-2} v(t),$$

for a.a. $t \in N_- = \{\tau \in [0, \pi]; v(\tau) < 0\}$. The function $h(t)$ is, now, of the following form

$$h(t) = 0,\ t \in N_0,\ h(t) = \frac{h_1(t)}{|v(t)|^{p-2} v(t)},\ t \notin N_0.$$

The considerations made above imply that $h(t) \in [\mu_i, \mu_{i+1}]$ for a.a. $t \in (0, \pi)$ and hence $\tau h(t) + (1-\tau)\lambda = \chi(t) \in [\mu_i, \mu_{i+1}]$ for a.a. $t \in (0, \pi)$. Regularity result [2, Th. 3.3] implies that the solution of (4.5) is also the solution of

$$\text{(4.6)} \qquad \begin{cases} -(|v'|^{p-2} v')' = \chi(t) |v|^{p-2} v,\ t \in (0, \pi), \\ v(0) = v(\pi) = 0 \end{cases}$$

in the sense of Definition 2.1. Particularly, it is $v \in C^1([0, \pi])$.

The uniqueness result (Remark 2.3) then implies that the set $N_0 = \{t \in [0, \pi]; v(t) = 0\}$ is finite and that $v'(t) \neq 0$ for all $t \in N$. Let us consider, now, the initial value problem

$$\text{(4.7)} \qquad \begin{cases} -(|v'_\alpha|^{p-2} v'_\alpha)' = \chi(t) |v_\alpha|^{p-2} v_\alpha, \\ v_\alpha(0) = 0,\ v'_\alpha(0) = \alpha \neq 0. \end{cases}$$

Applying Shooting lemma we obtain that the zero points of the solution of (4.7) in $(0, \pi]$ lie strictly between the zero points of v_i and v_{i+1}, where v_i, resp. v_{i+1}, is eigenfunction corresponding to λ_i, resp. λ_{i+1}. Hence it is $v_\alpha(\pi) \neq 0$ for each $\alpha \neq 0$. This implies that (4.6) has only trivial solution which proves that the homotopy H satisfies (3.6). The remainder of the proof is the same as in the case of Theorem 3.1.

Q.E.D.

REMARK 4.1 - Applying the results from [2] to the equation (3.11) it is possible to generalize the assumptions (4.1), (4.2) of Theorem 4.1 in the following sense.

THEOREM 4.2 - *Let K be a rectangle contained in some component of A_1. Let*

$$\text{(4.8)} \qquad (\chi_{\pm\infty}(x),\ \chi^{\pm\infty}(y)) \in K$$

for a.a. $(x, y) \in [0, \pi] \times [0, \pi]$. Then (BVP) *has at least one solution* *for each* $g \in L_1(0,\pi)$.

The proof of this assertion is a modification of the proof of Theorem 4.1. The only difference is that we consider the homotopy defined by the relation (3.15) and then we use (3.12) (see Remark 3.3).

REMARK 4.2 - Similarly as in Remark 3.4 it is easy to see that the assumptions (4.1), (4.2) are the special cases of the assumption (4.8).

5. - Quasihomogeneous operators

Let us keep the notation of operators J, S and $F: \overset{0}{W}{}^{1,p}(0,\pi) \to \to W^{-1,q}(0,\pi)$ from Section 3.

DEFINITION 5.1 - The operator $A: \overset{0}{W}{}^{1,p}(0,\pi) \to W^{-1,q}(0,\pi)$ is said to be *$(p-1)$-quasihomogeneous with respect to J* if $t_n \to 0$, $u_n \to u_0$ in $\overset{0}{W}{}^{1,p}(0,\pi)$ and $t_n^{p-1} A(\frac{u_n}{t_n}) \to G \in W^{-1,q}(0,\pi)$ imply $J(u_0) = G$.

DEFINITION 5.2 - The operator $A: \overset{0}{W}{}^{1,p}(0,\pi) \to W^{-1,q}(0,\pi)$ is said to be a *$(K, L, p-1)$-homeomorphism* of $\overset{0}{W}{}^{1,p}(0,\pi)$ onto $W^{-1,q}(0,\pi)$ if

(i) A is a homeomorphism of $\overset{0}{W}{}^{1,p}(0,\pi)$ onto $W^{-1,q}(0,\pi)$;

(ii) there exist real numbers $K > 0, L > 0$ such that

$$L \| u \|_{1,p}^{p-1} \leqslant \| A(u) \|_{-1,q} \leqslant K \| u \|_{1,p}^{p-1}.$$

REMARK 5.1 - To the operators of the type mentioned above it is devoted Chapter II in [4].

The assertion of Theorem 3.1 may be generalized also in the following way.

THEOREM 5.1 - *Let A be an odd $(K, L, p-1)$-homeomorphism which is $(p-1)$-quasihomogeneous with respect to J. Let $f: [0,\pi] \times \mathbf{R} \to \mathbf{R}$ satisfies* (3.1), (4.1) *and* (4.2).

Then for each $g \in L_1(0,\pi)$ there exists at least one $u \in \overset{0}{W}{}^{1,p}(0,\pi)$ such that

(5.1) $\qquad A(u) = F(u) + G$

(for the definition of G see the proof of Theorem 3.1*)*.

Proof. We shall consider the homotopy
$H(u,\tau) = A(u) - \tau F(u) - \tau G - (1-\tau)\lambda S(u)$, $u \in \overset{0}{W}{}^{1,p}(0,\pi)$, $\tau \in [0,1]$, and we shall prove that

(5.2) $\qquad H(u,\tau) \neq 0$,

for all $u \in \partial B_r(0)$, $\tau \in [0,1]$, where $B_r(0)$ has sufficiently large radius

$r > 0$. Then the assertion will follow from the fact that S is odd, A is odd $(K, L, p-1)$-homeomorphism and from the Leray-Schauder degree theory. Let us verify (5.2). Suppose to the contrary that there exist $\{u_n\}_{n=1}^{\infty}$ $\{\tau_n\}_{n=1}^{\infty}$ such that $\tau_n \in [0,1]$, $\|u_n\|_{1,p} \to \infty$ and $H(u_n, \tau_n) = 0$. Multiplying the last equality by $1/\|u_n\|_{1,p}^{p-1}$ we obtain

$$\frac{1}{\|u_n\|_{1,p}^{p-1}} A(v_n \|u_n\|_{1,p}) - \tau_n \frac{F(u_n)}{\|u_n\|_{1,p}^{p-1}} - \tau_n \frac{G}{\|u_n\|_{1,p}^{p-1}} - $$
$$- (1-\tau_n)\lambda S(v_n) = 0,$$

where $v_n = u_n / \|u_n\|_{1,p}$ (and $v_n \rightharpoonup v$ after possibly passing to subsequence). Analogously as in the proof of Theorem 4.1 we obtain that

$$\tau_n \frac{F(u_n)}{\|u_n\|_{1,p}^{p-1}} + \tau_n \frac{G}{\|u_n\|_{1,p}^{p-1}} + (1-\tau_n)\lambda S(v_n) \to \tilde{G}$$

in $W^{-1,q}(0,\pi)$. Since

$$\frac{A(u_n)}{\|u_n\|_{1,p}^{p-1}} \geqslant L,$$

it is $\tilde{G} \neq 0$. Hence $v \neq 0$ and $J(v) = \tilde{G}$ (according to Def. 5.1), i.e. (4.5) is satisfied. But as it was shown in the proof of Theorem 3.1 on the basis of Shooting lemma the relation (4.5) together with $v \neq 0$ leads to the contradiction. That is why (5.2) is satisfied and the proof is completed. Q.E.D.

REMARK 5.2 - It is easy to see (using the homotopy invariance property of the Leray-Schauder degree) that if A is an odd $(K, L, p-1)$-homeomorphism which is $(p-1)$-quasihomogeneous with respect to J then

(3.12') $$\deg [w - \mu S(A^{-1}(w))^+ + \nu S(A^{-1}(w))^-; B_\rho(0), 0] \neq 0,$$

for sufficiently large ball $B_\rho(0) \subset W^{-1,q}(0,\pi)$, whenever $(\mu, \nu) \in A_1$ (see Remark 3.3). Then using (3.12') instead of (3.12) we can prove the generalization of Theorem 5.1 in the sense that the assumptions (4.1) and (4.2) are replaced by the general assumption (4.8) (see [7]).

REMARK 5.3 - It is possible to verify that the operator A defined by

$$(Au, v) = \int_0^\pi (1 + |u'|^{p-2}) u'v' \, dt,$$

$u, v \in \overset{0}{W}{}^{1,p}(0,\pi)$ is odd, $(K, L, p-1)$-homeomorphism $\overset{0}{W}{}^{1,p}(0,\pi)$ onto $W^{-1,q}(0,\pi)$ which is $(p-1)$-quasihomogeneous with respect to J (to verify it we can proceed by the same way as in [2, Lemma 3.2]). If the function $f: [0,\pi] \times \mathbf{R} \to \mathbf{R}$ is given by the relation

$$f(t,s) = \lambda(t)[c_1 |s^+|^{p-2} s - c_2 |s^-|^{p-2} s + |s|^{p-2}],$$

where $c_1\lambda(t), c_2\lambda(t) \in [\lambda_i + \delta, \lambda_{i+1} - \delta]$ for a.a. $t \in (0, \pi)$ with some small $\delta > 0$.

Then there exists at least one $u \in \overset{0}{W}{}^{1,p}(0, \pi)$ such that

$$\int_0^\pi (1 + |u'|^{p-2}) u' v' \, dt = \int_0^\pi c_1 \lambda(t) |u^+|^{p-2} uv \, dt -$$

$$- \int_0^\pi c_2 \lambda(t) |u^-|^{p-2} uv \, dt + \int_0^\pi \lambda(t) |u|^{p-2} v \, dt + \int_0^\pi gv \, dt$$

holds for each $v \in \overset{0}{W}{}^{1,p}(0, \pi)$.

6. - Final remarks

REMARK 6.1 - It would be probably interesting to check if the similar results it is possible to prove also for strongly nonlinear Dirichlet boundary value problem

$$\text{(BVP')} \quad \begin{cases} (|u''|^{p-2} u'')'' = f(t, u) + g, \\ u(0) = u'(0) = u(\pi) = u'(\pi) = 0. \end{cases}$$

REMARK 6.2 - The studying of the similar nonlinear Dirichlet boundary value problem with partial differential operator of the type

$$\begin{cases} -\nabla(|\nabla u|^{p-2} \nabla u) = f(x, u) + g \text{ in } \Omega \subset \mathbf{R}^N, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

would be probably more difficult (see [7]).

It seems that for $p \geqslant 2$, p is near 2 (i.e. for $p \in [2, 2 + \delta)$ for some small $\delta > 0$) we may prove analogous result as in Theorem 3.1 using the properties of the linearized partial differential operator of second order which we obtain for $p = 2$.

But it is not clear how to proceed in the case of general $p \geqslant 2$.

REMARK 6.3 - It would be interesting to study the case when χ_+ and χ_- crosses an eigenvalue of (EVP), e.g. $\chi_\pm(t) = \lambda_i$ in the set of positive measure in $[0, \pi]$.

REMARK 6.4 - Solvability of (BVP) in more general setting will be studied in prepared paper [7].

REFERENCES

[1] E. A. CODDINGTON, N. LEVINSON, *Theory of ordinary differential equations*, McGraw-Hill, New York - Toronto - London 1955.

[2] P. DRÁBEK, *Ranges of a-homogeneous operators and their perturbations*, Čas. pest. mat. 105 (1980), 167-183.

[3] S. FUČÍK, *Solvability of nonlinear equations and boundary value problems*, D. Riedel Publishing Company, Holland, 1980.

[4] S. FUČÍK, J. NEČAS, J. SOUČEK, V. SOUČEK, *Spectral analysis of nonlinear operators*, JČSMF, Praha, 1973.

[5] P. DRÁBEK, *Nonlinear operator equations*, Graduate Thesis, Charles University, Prague, 1977.

[6] A. KUFNER, O. JOHN, S. FUČÍK, *Function spaces*, Academia, Prague, 1977.

[7] L. BOCCARDO, D. GIACCHETTI, M. KUČERA, P. DRÁBEK, *Generalization of Fredholm alternative for nonlinear differential operators*, Nonlinear Analysis, T.M.A. 10 (1986), 1083-1103.