SOME FIXED POINT THEOREMS IN METRIC SPACES (*)

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SOMMARIO. - *Si dimostra una generalizzazione di un teorema di punto fisso di Picard-Banach e di uno di Kannan.*

SUMMARY. - *A generalization of the fixed point theorem of Picard-Banach and of Kannan is given.*

1. Let (X, d) be a complete metric space. The fixed point theorem of Picard-Banach ([1], [7]) and of Kannan [4] are well-known. The purpose of the present paper is to give a generalization of these theorems.

THEOREM 1. *Let (X, d) be a complete metric space, and $f: X \rightarrow X$ a mapping for which there exist numbers $\alpha, \beta \in \mathbb{R}_+$, $\alpha + 2\beta < 1$, such that*

$$d(f(x), f(y)) \leq \alpha\, d(x, y) + \beta\,(d(x, f(x)) + d(y, f(y))) \tag{1}$$

for all $x, y \in X$.

Then f has a unique fixed point.

REMARK. For $\beta = 0$, we have the condition of Picard-Banach

$$d(f(x), f(y)) \leq \alpha\, d(x, y), \quad 0 < \alpha < 1, x, y \in X \tag{2}$$

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and for $\alpha = 0$, the condition of Kannan

(3) $d(f(x), f(y)) \leq \beta (d(x, f(x)) + d(y, f(y))),\ 0 < \beta < \frac{1}{2}, x, y \in X,$

R. Kannan [4] proved that the condition (2) and (3) are independent. The following example show that conditions (1), (2) (3), are also independent.

EXAMPLE. Let $X = [0, 1]$, $d(x, y) = |x - y|$, and

$$f(x) = \begin{cases} \frac{7x}{20} & \text{for } 0 \leq x \leq \frac{1}{2} \\ \frac{3x}{10} & \text{for } \frac{1}{2} < x \leq 1 \end{cases}$$

$f: X \to X$ is discontinuous at $x = \frac{1}{2}$. Condition (2) is not satisfied. Neither condition (3) is satisfied if we take $x = \frac{1}{2}, y = 0$. It is easily seen that condition (1) is satisfied by taking $\alpha = \frac{1}{10}$, $\beta = \frac{4}{9}$.

2. PROOF OF THE TH. 1. Let $x_0 \in X$. We have

$$d(f(x_0), f^2(x_0)) \leq \alpha\, d(x_0, f(x_0)) + \beta (d(x_0, f(x_0)) + d(f(x_0), f^2(x_0))).$$

Hence

$$d(f(x_0), f^2(x_0)) \leq \frac{\alpha + \beta}{1 - \beta} d(x_0, f(x)),\ 0 < \frac{\alpha + \beta}{1 - \beta} < 1.$$

In the remaing part of the proof one can apply the usual Picard-Banach arguments.

3. Theorem 1 may be formulated in many different more general forms using ideas from the Theory of fixed points in metric spaces. Here are some of them

3.1. Let (X, d) be a complete metric space. The mapping $f: X \to X$, is said to be of the type $(\varepsilon, \alpha, \beta)$, if

$$p, q \in \{y \mid y \in X, d(x, y) < \varepsilon\}$$

implies

$$d(f(p), f(q)) \leq \alpha\, d(p, q) + \beta\,(d(p, f(p)) + d(q, f(q)))$$

for all $x \in X$.

We have

THEOREM 2. *Let (X, d) be a complete metric ε-chainable space (see Edelstein [2]) and let $f: X \to X$ be of the type $(\varepsilon, \alpha, \beta)$, $\alpha, \beta \in \mathbb{R}_+$, $\alpha + 2\beta < 1$.*

Then f has a unique fixed point.

PROOF. See Edelstein [2].

3.2. Another type of generalized metric spaces are those in which the metric may take the value ∞. In the same way as in [3] one establishes.

THEOREM 3. *Let (X, d) be a generalized complete metric space (Jung) and $f: X \to X$ a mapping such that the condition (1) is satisfied. If there exists $x_0 \in Y$ such that*

$$d(x_0, f(x_0)) < +\infty$$

then f has at least a fixed point.

4. In this section we give a convergence theorem generalizing some results of Nadler [5] and of Singh [6]. We have

THEOREM 4. Let

(i) *$f_n: X \to X$ be a mapping with fixed point p_n, $n = 1, 2, \ldots$*

(ii) *$(f_n)_{n \in N}$ converges uniformly to f where f is a mapping which satisfies condition (1). Let p be the unique fixed point of f.*

Then p_n converges to p.

PROOF. In view of conditions (i) and (ii) we have

$$\begin{aligned} d(p_n, p) = d(f_n(p_n), f(p)) &\leq d(f_n(p_n), f(p_n)) + d(f(p_n), f(p)) \\ &\leq d(f_n(p_n), f(p_n)) + \alpha\, d(p_n, p) + \beta\,(d(p_n, f(p_n)) + d(p, f(p))) \\ &\leq d(f_n(p_n), f(p_n)) + \alpha\, d(p_n, p) + \beta\, d(f_n(p_n), f(p_n)). \end{aligned}$$

Hence

$$(1 - \alpha)\, d(p_n, p) \leq (1 + \beta)\, d(f_n(p_n), f(p_n))$$

which completes the proof.

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