Rend. Istit. Mat. Univ. Trieste
Vol. 56 (2024), Art. No. 4, 7 pages
DOI: 10.13137/2464-8728/36464

Characterizations of strongly star-Rothberger spaces by means of sequential singletonic intersection property and selection principles

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Abstract. *In this paper, using families of closed sets and a few modifications to the finite intersection features, we characterize the Rothberger Space and the Star-Rothberger Space. We also provide the selection principles that, in a reversed approach, can represent the Rothberger spaces and the star-Rothberger Spaces*

Keywords: Rothberger property, Selection principles, finite intersection property..
MS Classification 2020: 54D20, 54D30.

1. Introduction and Preliminaries

Rothberger property and Menger property are the most fascinating sequential covering features for the topologists worldwide.

Definition 1.1. *A space X is said have Rothberger covering property [12] if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X, there exists a sequence $\{V_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}, V_n \in \mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}}(V_n) = X$.*

The term "Rothbergerness" or "Rothberger covering property" was first used by Rothberger in 1938 [12]. In literature, there are a lot of generalizations about Rothbergerness. The St-Rothbergerness, which Kočinac introduced in 1999 [9, 10], strikes us as the most intriguing.

If M is a subset of a set X and $\mathcal{U}$ is a collection of subsets of X,then the star of M with respect to $\mathcal{U}$ is the set $St(M,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap M \neq \emptyset\}$ [7].

In 1991, Douwen utilized the star operator for the first time to generalize the ideas of compactness and Lindelöfness. Then Kočinac [9] used it to generalize selection principles, Rothberger space, and Menger space. Some examples of current St-operator usage can be found in [1, 2, 3, 4, 5, 6, 13, 14].

Definition 1.2. *A space X is said to have star-Rothberger property [9] if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X, there exists a sequence $\{V_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}, V_n \in \mathcal{U}_n$ and $X = \bigcup_{n\in\mathbb{N}} St(V_n,\mathcal{U}_n)$.*

Although many scholars have examined these sequential covering properties in-depth [12, 13], there hasn't been much focus on how to express them using a family of closed sets. Recall that a collection $\mathcal{F}$ of subsets of a set X has the finite intersection property(FIP) if the intersection of any finite sub collection of $\mathcal{F}$ is non empty. A topological space is compact if and only if every collection of closed subsets meeting the FIP has a non empty intersection itself. The use of the FIP makes this alternative notion of compactness achievable [8].

In our research, we find such type of representations for Rothbergerness and St-Rothbergerness with a little variation in finite intersection property(FI property).

Throughout the paper, a space X denotes a topological space X equipped with the corresponding topology τ. For a space X we adopt the following symbols:

$\mathcal{O}$: the collection of all open covers of X.

$\mathcal{C}_X$: the collection of all family $\mathcal{F}$ of closed sets for which $\cap\mathcal{F} = \emptyset$.

$\mathcal{A}, \mathcal{B}$: represents collections of families of subsets of a space X.

Selection principles are other ways to describe sequential covering properties.

Definition 1.3. *The symbol $S_1(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(V_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $V_n \in \mathcal{U}_n$ and $\{V_n : n \in \mathbb{N}\}$ is also an element of $\mathcal{B}$ [15].*

Definition 1.4. *The symbol $S_1^*(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(V_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}, V_n \in \mathcal{U}_n$ and $\{St(V_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is also an element of $\mathcal{B}$ [11].*

It is important to keep in mind that the selection principle type of characterization for Rothberger and St-Rothberger is given by the expressions $S_1(\mathcal{O}, \mathcal{O})$ and $S_1^*(\mathcal{O}, \mathcal{O})$, respectively.

We look for new selection principles in our research that can describe Rothberger spaces and star Rothberger spaces using the family $\mathcal{C}_X$.

2. Main Results

Definition 2.1. *Let $\{\mathcal{F}_n : n \in \mathbb{N}\}$ be a sequence of family of subsets of X. This sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ is said to have sequential singletonic intersection property (SSI property) if for every sequence $\{E_n : n \in \mathbb{N}\}$ where $E_n \in \mathcal{F}_n$, we have $\cap_{n\in\mathbb{N}}(E_n) \neq \emptyset$*

Theorem 2.2. *The following conditions are equivalent.*

(i) X is a Rothberger space.

(ii) For every sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of family of closed sets with sequential singletonic intersection property (SSI property), there exists a $n_0 \in \mathbb{N}$ such that $\bigcap \mathcal{F}_{n_0} = \bigcap_{F \in \mathcal{F}_{n_0}} F \neq \emptyset$.

Proof. $(i) \Rightarrow (ii)$. Let X be a Rothberger space; $\{\mathcal{F}_n : n \in \mathbb{N}\}$ be a family of closed sets having the sequential singletonic intersection property (SSI property) and let $\cap \mathcal{F}_n = \emptyset$, for all $n \in \mathbb{N}$.

We take, $\mathcal{G}_n = \{X \setminus F : F \in \mathcal{F}_n\}$, for all $n \in \mathbb{N}$. Therefore, for all $n \in \mathbb{N}$,

$$\begin{aligned} \cup \mathcal{G}_n &= \cup \{X \setminus F : F \in \mathcal{F}_n\} \\ &= X \setminus \cap_{F \in \mathcal{F}_n}(F) = X \setminus \cap \mathcal{F}_n = X \setminus \emptyset = X\,. \end{aligned}$$

Therefore $\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a sequence of open covers. But X is a Rothberger space. Therefore there exists a sequence $\{H_n : n \in \mathbb{N}\}$ such that $H_n \in \mathcal{G}_n$ for all $n \in \mathbb{N}$ and $\cup_{n\in\mathbb{N}}(H_n) = X$.

Now, we construct the sequence $\{E_n : n \in \mathbb{N}\}$ where $E_n = X \setminus H_n$ for all $n \in \mathbb{N}$. Clearly $E_n \in \mathcal{F}_n$ for all $n \in \mathbb{N}$ and

$$\cap_{n\in\mathbb{N}}(E_n) = \cap_{n\in\mathbb{N}}(\{X \setminus H_n\}) = X \setminus (\cup_{n\in\mathbb{N}}(H_n)) = X \setminus X = \emptyset\,.$$

Therefore $\{E_n : n \in \mathbb{N}\}$ is a sequence such that $E_n \in \mathcal{F}_n$ for all $n \in \mathbb{N}$ but $\cap_{n\in\mathbb{N}}(E_n) = \emptyset$, contradicts the fact that $\{\mathcal{F}_n : n \in \mathbb{N}\}$ has sequential singletonic intersection property (SSI property). Therefore there must exists a $n_0 \in \mathbb{N}$ such that $\mathcal{F}_{n_0} \neq \emptyset$.

$(ii) \Rightarrow (i)$. Let the condition (ii) holds and assume that $\{\mathcal{G}_n : n \in \mathbb{N}\}$ be a arbitrary sequence of open covers for a topological space X. Therefore $\cup \mathcal{G}_n = X$ for all $n \in \mathbb{N}$.

If we take, $\mathcal{F}_n = \{X \setminus G : G \in \mathcal{G}_n\}$ for all $n \in \mathbb{N}$, then $\bigcap \mathcal{F}_n = \emptyset$ for all $n \in \mathbb{N}$. Therefore $\{\mathcal{F}_n : n \in \mathbb{N}\}$ is a sequence of family of closed sets such that $\cap \mathcal{F}_n = \emptyset$ for all $n \in \mathbb{N}$.

So by contra-positivity of the statement (ii), $\{\mathcal{F}_n : n \in \mathbb{N}\}$ must not have the sequential singletonic intersection property (SSI property).

So, there exists a sequence $\{E_n : n \in \mathbb{N}\}$ such that $E_n \in \mathcal{F}_n$ and for all $n \in \mathbb{N}$ with $\cap_{n\in\mathbb{N}}(E_n) = \emptyset$. Suppose $H_n = X \setminus E_n$ for all $n \in \mathbb{N}$. Clearly $H_n \in \mathcal{G}_n$, for all $n \in \mathbb{N}$.

Therefore,

$$\cup_{n\in\mathbb{N}}(H_n) = \cup_{n\in\mathbb{N}}(\{X \setminus E_n\}) = X \setminus (\cap_{n\in\mathbb{N}}(E_n)) = X \setminus \emptyset = X\,.$$

Therefore $\{H_n : n \in \mathbb{N}\}$ is a sequence such that $H_n \in \mathcal{G}_n$ for all $n \in \mathbb{N}$ and $\cup_{n\in\mathbb{N}}(H_n) = X$. Therefore X is a Rothberger space. □

Corollary 2.3. *$S_1(\mathcal{O}, \mathcal{O})$ and $S_1(\mathcal{C}_X, \mathcal{C}_X)$ are equivalent.*

Proof. Similar to the proof of above theorem. Hence omitted. □

Definition 2.4. *A sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of family of subsets of X is said to have modified sequential singletonic intersection property (MSSI property) if for all sequences $\{E_n : n \in \mathbb{N}\}$ and $\{\mathcal{H}_n : n \in \mathbb{N}\}$ such that $E_n \in \mathcal{F}_n$ and $\mathcal{H}_n \subseteq \mathcal{F}_n$, for all $n \in \mathbb{N}$ either*

$E_n \cup F = X$ for some $F \in \mathcal{H}_n$ and for all $n \in \mathbb{N}$ or

$$\bigcap_{n\in\mathbb{N}} \left(\bigcap \mathcal{H}_n\right) \neq \emptyset\,.$$

Theorem 2.5. *The following conditions are equivalent.*

(i) (X, τ) is St-Rothberger space.

(ii) If the sequence of closed sets $\{\mathcal{F}_n : n \in \mathbb{N}\}$ has modified sequential singletonic intersection property (MSSI property) then there exists a $n_0 \in \mathbb{N}$ such that $\bigcap \mathcal{F}_{n_0} = \bigcap_{F\in\mathcal{F}_{n_0}} F \neq \emptyset$.

Proof. $(i) \Rightarrow (ii)$. Let (X, τ) be a St-Rothberger space; $\{\mathcal{F}_n : n \in \mathbb{N}\}$ be a family of closed sets having modified sequential singletonic intersection property (MSSI property) and $\cap\mathcal{F}_n = \emptyset$, for all $n \in \mathbb{N}$.

Now we assume $\mathcal{G}_n = \{X \setminus F : F \in \mathcal{F}_n\}$ for all $n \in \mathbb{N}$.

Therefore, $\cup\mathcal{G}_n = \bigcup\{X \setminus F : F \in \mathcal{F}_n\} = X$, for all $n \in \mathbb{N}$.

Therefore, $\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a sequence of open covers. But(X, τ) is a St-Rothberger space therefore there exists a sequence $\{G'_n : n \in \mathbb{N}\}$ where $G'_n \in \mathcal{G}_n$ for all $n \in \mathbb{N}$ such that

$$\begin{aligned}
&\bigcup_{n\in\mathbb{N}} \{St(G'_n, \mathcal{G}_n)\} = X \\
&\implies \bigcup_{n\in\mathbb{N}} \bigcup \left\{G \in \mathcal{G}_n : G'_n \bigcap G \neq \emptyset\right\} = X \\
&\implies \bigcup_{n\in\mathbb{N}} \bigcup \left\{(X \setminus F) \in \mathcal{G}_n : G'_n \bigcap (X \setminus F) \neq \emptyset\right\} = X \\
&\implies \bigcup_{n\in\mathbb{N}} \bigcup \left\{(X \setminus F) : F \in \mathcal{F}_n \text{ and } G'_n \bigcap (X \setminus F) \neq \emptyset\right\} = X \\
&\implies X \setminus \bigcap_{n\in\mathbb{N}} \bigcap \left\{F \in \mathcal{F}_n : (X \setminus E_n) \bigcap (X \setminus F) \neq \emptyset\right\} = X, \\
&\qquad\qquad\qquad\qquad\qquad \text{where } G'_n = X \setminus E_n \text{ for all } n \in \mathbb{N} \\
&\implies \bigcap_{n\in\mathbb{N}} \bigcap \left\{F \in \mathcal{F}_n : X \setminus \left(E_n \bigcup F\right) \neq \emptyset\right\} = \emptyset \\
&\implies \bigcap_{n\in\mathbb{N}} \bigcap \left\{F \in \mathcal{F}_n : \left(E_n \bigcup F\right) \neq X\right\} = \emptyset.
\end{aligned}$$

Now let $\mathcal{H}_n = \{F \in \mathcal{F}_n : (E_n \bigcup F) \neq X\}$ for all $n \in \mathbb{N}$. Therefore, $\{E_n : n \in \mathbb{N}\}$ and $\{\mathcal{H}_n : n \in \mathbb{N}\}$ are two sequences such that $E_n \in \mathcal{F}_n$ $\mathcal{H}_n \subseteq \mathcal{F}_n$, for all $n \in \mathbb{N}$.

Here $E_n \cup F \neq X$ for all $F \in \mathcal{H}_n$ and for all $n \in \mathbb{N}$ and $\bigcap_{n\in\mathbb{N}}(\bigcap \mathcal{H}_n) = \emptyset$. Which contradicts the fact that $\{\mathcal{F}_n : n \in \mathbb{N}\}$ has Modified sequential singletonic intersection property (MSSI property). Therefore there exists a $n_0 \in \mathbb{N}$ such that $\mathcal{F}_{n_0} \neq \emptyset$.

$(ii) \Rightarrow (i)$. Let condition (ii) holds and assume that $\{\mathcal{G}_n : n \in \mathbb{N}\}$ be a arbitrary sequence of open covers of a topological space (X, τ). Therefore $\cup\mathcal{G}_n = X$, for all $n \in \mathbb{N}$. Let $\mathcal{F}_n = \{X \setminus G : G \in \mathcal{G}_n\}$, for all $n \in \mathbb{N}$.

Therefore $\cap\mathcal{F}_n = \cap\{X \setminus G : G \in \mathcal{G}_n\} = \emptyset$ for all $n \in \mathbb{N}$.

So, $\{\mathcal{F}_n : n \in \mathbb{N}\}$ is a sequence of family of closed sets such that $\cap\mathcal{F}_n = \emptyset$, for all $n \in \mathbb{N}$. By contra positivity of the statement (ii), $\{\mathcal{F}_n : n \in \mathbb{N}\}$ must not have modified sequential singletonic intersection property (MSSI property).

So,there exist sequences $\{E_n : n \in \mathbb{N}\}$ and $\{\mathcal{H}_n : n \in \mathbb{N}\}$ such that $E_n \in \mathcal{F}_n$ and $\mathcal{H}_n \subseteq \mathcal{F}_n$, for all $n \in \mathbb{N}$ with $E_n \cup F \neq X$ for all $F \in \mathcal{H}_n$ and for all $n \in \mathbb{N}$ or

$$\bigcap_{n\in\mathbb{N}} \left(\bigcap \mathcal{H}_n\right) = \emptyset.$$

Consider the sequences $\{G'_n = X \setminus E_n : n \in \mathbb{N}\}$ and $\mathcal{M}_n = \{M = X \setminus F : F \in \mathcal{H}_n\}$. Therefore, $G'_n \in \mathcal{G}_n$ and $\mathcal{M}_n \subseteq \mathcal{G}_n$ for all $n \in \mathbb{N}$. Now,

$$\begin{aligned}
&\Longrightarrow E_n \cup F \neq X, \quad \text{for all } F \in \mathcal{H}_n \text{ and } E_n \in \mathcal{F}_n, \quad \text{for all } n \in \mathbb{N} \\
&\Longrightarrow X \setminus \{(E_n) \cup F\} \neq \emptyset, \quad \text{for all } F \in \mathcal{H}_n \text{ and } \quad E_n \in \mathcal{F}_n, \quad \text{for all } n \in \mathbb{N} \\
&\Longrightarrow (X \setminus E_n) \bigcap (X \setminus F) \neq \emptyset, \quad \text{for all } F \in \mathcal{H}_n \text{ and } E_n \in \mathcal{F}_n, \ \text{for all } n \in \mathbb{N} \\
&\Longrightarrow (G'_n) \bigcap M \neq \emptyset, \quad \text{for all } M \in \mathcal{M}_n\,, \quad \text{for all } n \in \mathbb{N}\,.
\end{aligned}$$

And

$$\begin{aligned}
\bigcup_{n\in\mathbb{N}} St(G'_n, \mathcal{G}_n) &= \bigcup_{n\in\mathbb{N}} \bigcup \{G \in \mathcal{G} : G \cap G'_n \neq \emptyset\} \\
&\supseteq \bigcup_{n\in\mathbb{N}} \bigcup \{M \in \mathcal{M} : M \cap G'_n \neq \emptyset\} = \bigcup_{n\in\mathbb{N}} \bigcup \mathcal{M}_n \\
&= \bigcup_{n\in\mathbb{N}} \bigcup \{X \setminus F : F \in \mathcal{H}_n\} = X \setminus \bigcap_{n\in\mathbb{N}} \bigcap \mathcal{H}_n = X \setminus \emptyset = X\,.
\end{aligned}$$

Therefore, $\bigcup_{n\in\mathbb{N}} St(G'_n, \mathcal{G}_n) = X$. So,$(X, \tau)$ is star-Rothberger space. □

EXAMPLE 2.6. There exists a T_0-space in which selection principles $S_1^*(\mathcal{O}, \mathcal{O})$ and $S_1^*(\mathcal{C}_X, \mathcal{C}_X)$ are not equivalent.

Let $X = [a, b)$, where $a, b \in \mathbb{R}$ and $b - a = 1$. $\tau = \{\emptyset, X\} \cup \{[a, \alpha) : \alpha \in [a, b]\}$. So (X, τ) forms a T_0 space.

Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be an arbitrary sequence of open covers. Clearly, there exits an $U_n \in \mathcal{U}_n$ such that $0 \in U_n$, for all $n \in \mathbb{N}$ then $St(U_n, \mathcal{U}_n) = \bigcup \mathcal{U}_n = X$, for all $n \in \mathbb{N}$. But X is open. Thus $\{St(U_n, \mathcal{U}_n) : n \in N\} = \{X\} \in \mathcal{O}$. Thus the space X follows the Selection Principle $S_1^*(\mathcal{O}, \mathcal{O})$.

Now consider a sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of family of closed sets such that $\mathcal{F}_n = \{F_{n_m} : m \in N\}$, for all $n \in \mathbb{N}$ and $F_{n_m} = [b - \frac{1}{m}, b)$.

Clearly $\bigcap \mathcal{F}_n = \bigcap_{m=1}^{\infty} F_{n_m} = \bigcap_{m=1}^{\infty} [b - \frac{1}{m}, b) = \emptyset$, $\mathcal{F}_n \in \mathcal{C}_X$, for all $n \in N$. Now for every selection $F_n \in \mathcal{F}_n$,

$$St(F_n, \mathcal{F}_n) = [a, b) = X\,, \quad \{St(F_n, \mathcal{F}_n) : n \in \mathbb{N}\} = \{X\}$$
$$\text{but } \bigcap \{St(F_n, \mathcal{F}_n) : n \in \mathbb{N}\} = X \neq \emptyset\,, \quad \{St(F_n, \mathcal{F}_n) : n \in \mathbb{N}\} \notin \mathcal{C}_X\,.$$

So, the space X doesn't follow the selection principle $S_1^*(\mathcal{C}_X, \mathcal{C}_X)$. So, $S_1^*(\mathcal{O}, \mathcal{O})$ and $S_1^*(\mathcal{C}_X, \mathcal{C}_X)$ are not equivalent.

Now we want a selection principle that can act on $\mathcal{C}_X$ and produce an equivalent condition to the selection Principle $S_1^*(\mathcal{O}, \mathcal{O})$.

Definition 2.7. *The symbol $S_{1,S}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exist sequences $(E_n : n \in \mathbb{N})$ and $(\mathcal{H}_n : n \in \mathbb{N})$ such that $E_n \in \mathcal{U}_n$ and $\mathcal{H}_n \subset \mathcal{U}_n$ for each $n \in \mathbb{N}$ such that $(E_n) \cup F \neq X$ for any $F \in \mathcal{H}_n$ and $\{\bigcap \mathcal{H}_n : n \in \mathbb{N}\} \in \mathcal{B}$.*

Corollary 2.8. *$S_1^*(\mathcal{O}, \mathcal{O})$ and $S_{1,S}(\mathcal{C}_X, \mathcal{C}_X)$ are equivalent.*

Proof. The proof is similar to the proof of Theorem 2.5, hence omitted. □

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Received May 16, 2024
Revised July 22, 2024
Accepted July 30, 2024