SPLITTINGS OF MANIFOLDS WITH BOUNDARY AND RELATED INVARIANTS (*)

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Sommario. - *Si costruiscono speciali decomposizioni in manici di una n-varietà PL compatta, connessa e con bordo non vuoto e si studiano alcuni invarianti topologici associati. Come conseguenza, si ottiene una caratterizzazione del nodo banale n-dimensionale in* $\mathbb{S}^{n+2}$ $(n \leq 2)$ *come l'unico n-nodo il cui complementare ha genere uno. Infine, si espone una semplice dimostrazione geometrica del teorema di non cancellazione per n-nodi PL in* $\mathbb{S}^{n+2}$, $n \leq 2$.

Summary. - *We construct special handle decompositions for a compact connected PL manifold with non empty boundary and study the associated topological invariants. As a consequence, we characterize the unknot in* $\mathbb{S}^{n+2}$ $(n \leq 2)$ *as the unique n-knot whose complement has genus one. Then we obtain a simple geometric proof of the non cancellation theorem for tame n-knots in* $\mathbb{S}^{n+2}$, $n \leq 2$.

1. Introduction.

Let M^n be a compact connected (orientable) triangulated n-manifold with non empty boundary ∂M. We construct special handle decompositions of M and define the concept of *regular splitting* of M. Then we describe regular Heegaard diagrams of M $(n = 3)$ and relate them to another known 3-manifold representation, named P-*graph theory* (see [19] and [23]). As a consequence, we obtain nice properties about (geometric)

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finite presentations of $\Pi_1(M)$ which arise from regular Heegaard diagrams of M. Then we extend the notion of Heegaard genus of a closed 3-manifold to the boundary case, also considering higher dimensions. The concept of genus yields a nice characterization of cubes with handles among bordered 3-manifolds. As a consequence, we also characterize the unknot in $\mathbb{S}^3$ as the unique 1-knot whose complement has genus one. This gives a simple alternative proof of the classical non cancellation theorem for 1-knots in $\mathbb{S}^3$ (see for example [20]). Then we extend these results to dimension four. More precisely, we characterize the unknot in $\mathbb{S}^4$ as the unique 2-knot whose complement has genus one and obtain a geometric proof of the non cancellation theorem for tame 2-knots in $\mathbb{S}^4$. Some examples complete the paper.

2. Handle Decompositions.

Throughout the paper we work in the piecewise linear category in the sense of [12] and [21]. For convenience, we assume that any considered (pseudo) manifold is orientable. Recall that a *cube with n handles* is a 3-manifold V which contains n pairwise disjoint properly embedded 2-cells such that the result of cutting V along them is a 3-cell. The integer n is called the *genus* of V, written $g(V)$.

Let M^3 be a connected compact (orientable) 3-manifold with non-empty boundary components $\partial_1 M, \partial_2 M, \dots, \partial_h M$. We define the concept of "regular" splitting of M as follows. A pair (V_1, V_2) of cubes with handles is said to be a *(regular) Heegaard splitting* of M if it satisfies the following properties:

1) $V_1 \cup V_2 = M$

2) $V_2 \cap \partial_i M$ is a closed 2-cell D_i for $i = 1, 2, \dots, h$

3) $V_1 \cap V_2 = \partial V_1 \cap \partial V_2 = \partial V_2 \setminus \cup_{i=1}^{h} \overset{\circ}{D}_i$

4) $\partial V_1 = \partial V_2 \# \partial_1 M \# \dots \# \partial_h M$.

As a consequence, we have the relation $g(V_1) = g(V_2) + \sum_{i=1}^{h} g(\partial_i M)$, where g also denotes the genus of a closed surface. The *genus* of the regular splitting (V_1, V_2) is defined to be $g(V_1)$. The *(regular) Heegaard genus* of M is the minimum n for which M admits regular splittings of genus n. Obviously this concept extends to the boundary case the usual Heegaard genus of a closed (orientable) 3-manifold as $g(V_1) = g(V_2)$ whenever ∂M is a 2-sphere.

The following existence theorem was first proved in [4], Proposition 4, for manifolds with connected boundary and successively extended to the general case in [11], Proposition 12; we shall present an alternative proof of the result, which follows closely a construction contained in [24] (section 8.3.6, pp. 260-261).

THEOREM 1. *Let M^3 be a compact connected (orientable) 3-manifold with non empty boundary components $\partial_1 M, \partial_2 M, \ldots, \partial_h M$. Then M admits a regular Heegaard splitting.*

Proof. Let K be a simplicial triangulation of M and $Sd^r K$ the r-th barycentric subdivision of K. Let us denote by Γ_1 and Γ_2 the 1-skeleton and the dual 1-skeleton of K respectively. Recall that Γ_2 is the maximal 1-subcomplex of $Sd^1 K$ disjoint from Γ_1. We consider a derived simplicial neighbourhood H_i of Γ_i in $Sd^2 K$. Then the polyhedron underlying H_i, also named H_i, is a tubular neighbourhood of Γ_i in M. Obviously we have that $M = H_1 \cup H_2$ and $H_1 \cap H_2 = \partial H_1 \cap \partial H_2$. Furthermore, H_1 and H_2 are not identified along their whole boundaries as the points where ∂H_1 and ∂H_2 are not identified constitute ∂M. The pieces of $\partial_i M$ on ∂H_2 are 2-cells e_j, $j = 1, 2, \ldots, \alpha(i)$, arising from the middle of the faces in the triangulation of $\partial_i M$. By doing isotopies inside a collar of ∂M in M, we push the 2-cells e_j into the interior of a 2-cell f_i of $\partial_i M$ for any $i = 1, 2, \ldots, h$. Let C_i be a 3-cell such that $C_i \cap M = \partial C_i \cap \partial M = f_i$ and $\partial C_i \setminus \mathring{f}_i$ is the 2-cell D_i. Then the manifold $\widetilde{M} = M \cup \cup_{i=1}^h C_i$ is homeomorphic to M. Now $\widetilde{M}$ splits into two cubes with handles $V_2 = H_2 \cup \cup_{i=1}^h C_i$ and $V_1 = \mathrm{cl}(\widetilde{M} \setminus V_2) \cong \mathrm{cl}(M \setminus H_2) \cong H_1$. Here we have also denoted by the same symbol the image of H_i under the above mentioned isotopies. Finally the pair (V_1, V_2) satisfies the statement. ◊

By Theorem 1 we can analyze the bordered 3-manifolds in terms of the manner in which the pieces are attached and thus we reduce the study of these 3-manifolds to problems about 2-manifolds.

Suppose we have a (regular) Heegaard splitting (V_1, V_2) of a 3-manifold M with non empty boundary components $\partial_1 M, \partial_2 M, \ldots, \partial_h M$. Render V_2 simply connected by removing suitable meridian plates P_k, $k = 1, 2, \ldots, m$. More precisely, let $\{B_1, B_2, \ldots, B_m\}$ be any collection of pairwise disjoint properly embedded 2-cells in V_2 which cut V_2 into a 3-cell. The pairwise disjoint 1-spheres $\{J_1, J_2, \ldots, J_m\}$, $J_k = \partial B_k$, cut ∂V_2 into a 2-sphere with $2m$ holes. The plates P_k are precisely $B_k \times I \subset V_2$, where $I = [0,1]$. Since the pieces of ∂M on ∂V_2 are the 2-cells D_i, $i = 1, 2, \ldots, h$, we can place the plates P_k so that they do not meet ∂M by pushing their rims $\partial B_k \times I = J_k \times I$ away from the discs D_i where necessary.

Let V_2' be the result of cutting V_2 along $\cup_{k=1}^m B_k$. Then V_2' is a 3-cell as $g(V_2) = m$. Furthermore V_2' meets $\partial_i M$ along the 2-cell D_i. For any $i = 1, 2, \ldots, h-1$ cut a plate $P_i' = B_i' \times I$ from V_2' which has D_i as its top face and its rim $\partial B_i' \times I = J_i' \times I$ is an annulus common to ∂V_1 and ∂V_2.

We call the system $(V_1; J_1, J_2, \ldots, J_m, J_1', J_2', \ldots, J_{h-1}')$ a *(regular) Heegaard diagram* of M. We can recover M from a (regular) Heegaard diagram of it. Conversely, every set of disjoint simple closed curves on a cube V_1 with n handles determines a bordered 3-manifold M. Indeed, M is obtained by glueing plates to annular neighbourhoods of the curves.

Given a (regular) Heegaard diagram $(V_1; J_1, J_2, \ldots, J_m, J_1', J_2', \ldots, J_{h-1}')$ as above we can construct a presentation for $\Pi_1(M)$ as follows. Choose a free basis $\{x_1, x_2, \ldots, x_n\}$ for the free group $\Pi_1(V_1) \simeq \star_n \mathbb{Z}$, where $n = g(V_1)$. For $k = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, h-1$, let r_k and r_i' be words in $x_1, x_2, \ldots, x_n$ representing the elements of $\Pi_1(V_1)$ determined by J_k and J_i' respectively. These words are unique up to inversion and conjugation. By Van Kampen's theorem we have that

$$< x_1, x_2, \ldots, x_n; r_1, r_2, \ldots, r_m, r_1', r_2', \ldots, r_{h-1}' >$$

is a presentation for $\Pi_1(M)$.
In particular, we obtain the following result:

THEOREM 2. *Let M^3 be a compact connected (orientable) 3-manifold with non empty boundary components $\partial_1 M, \partial_2 M, \ldots, \partial_h M$. Then the fundamental group $\Pi_1(M)$ has a finite presentation of deficiency*

$$\sum_{i=1}^{h} g(\partial_i M) - h + 1 = 1 - \chi(M).$$

Proof. By Theorem 1, we have

$$\Pi_1(M) \cong < x_1, x_2, \ldots, x_n; r_1, r_2, \ldots, r_m, r_1', r_2', \ldots, r_{h-1}' >$$

where $n = g(V_1) = g(V_2) + \sum_{i=1}^{h} g(\partial_i M)$ and $m = g(V_2)$. Thus the deficiency d of the presentation is

$$d = n - m - (h-1) = \sum_{i=1}^{h} g(\partial_i M) - h + 1 \ .$$

Now let $D(M)$ be the closed 3-manifold which is the double of M. Then we have $\chi(D(M)) = 2\chi(M) - \chi(\partial M) = 0$, i.e.

$$2\chi(M) = \sum_{i=1}^{h} \chi(\partial_i M) = 2h - 2\sum_{i=1}^{h} g(\partial_i M).$$

This implies that $\chi(M) = h - \sum_{i=1}^{h} g(\partial_i M)$, hence $d = -\chi(M) + 1$ as requested ◊

Define:

1) $rk(M)$ the minimum rank of $\Pi_1(M)$;

2) $d(M)$ the minimum deficiency over all presentations of $\Pi_1(M)$.

The following facts are straightforward:

PROPOSITION 3. *Let M^3 be a compact connected (orientable) 3-manifold with non empty boundary components $\partial_1 M, \partial_2 M, \ldots,$ $\partial_h M$. Then we have:*

1) $g(M) \geq g(\partial M)$.
2) $g(M) \geq rk(M)$.
3) $0 \leq d(M) \leq g(\partial M) - h + 1 = 1 - \chi(M)$.
4) $d(M) + \beta_2(M) \leq \beta_1(M)$ where $\beta_i(M)$ is the i-th Betti number of M.
   In particular, if $d(M) > 0$, then $H_1(M)$ (and hence $\Pi_1(M)$) is an infinite group.
5) $g(M) = 0$ if and only if M is a punctured 3-cell, i.e. a manifold which becomes a 3-sphere by capping off each 2-sphere component of ∂M with a 3-cell.

Now we prove a nice characterization of cubes with handles among 3-manifolds with non empty connected boundary.

Theorem 4. *Let M^3 be a compact connected (orientable) 3-manifold with non empty connected boundary ∂M. Then M is a cube with n handles if and only if $g(M) = g(\partial M) = n$.*

Proof. The necessity is clear. For sufficiency, let (V_1, V_2) be a (regular) Heegaard splitting of M such that $g(M) = g(V_1) = g(V_2) + g(\partial M) = g(\partial M)$. By hypothesis, it follows that $g(V_2) = 0$, hence V_2 is a 3-cell. Furthermore V_2 meets V_1 in a 2-cell in the boundary of each as $\partial M \cap V_2 = D$, a 2-cell, and $V_1 \cap V_2 = \partial V_1 \cap \partial V_2 = \partial V_2 \setminus \mathring{D}$ is a 2-cell. Hence M is the 3-manifold obtained from the cube with handles V_1 by attaching a 3-cell along a 2-cell in their boundaries. Thus $M \cong_{PL} V_1$ as required ♢

Note that Theorem 4 gives a simple (non combinatorial) proof of the main theorem of [3]. Indeed the regular genus $\tilde{g}(M)$ of a 3-manifold M with boundary, used in [3], satisfies the relation $\tilde{g}(M) \geq g(M) \geq g(\partial M) = \tilde{g}(\partial M)$ as one can easily verify.

Corollary 5. *Let K be a tame knot in $\mathbb{S}^3$ and M the knot manifold of K, i.e. M is the closed complement of a regular neighbourhood of K in $\mathbb{S}^3$. Then K is the trivial knot if and only if $g(M) = g(\partial M) = 1$.*

Proposition 6. *Let K_i be a tame knot in $\mathbb{S}^3$, M_i the knot manifold of K_i, $i = 1, 2$, and K the composite knot $K_1 \# K_2$. If M is the knot manifold of K, then we have $g(M) = g(M_1) + g(M_2) - 1$.*

Proof. For composite knots it is convenient to use a new view of the knot manifold as described in [1] and [2], chp. 15, part B. One looks at the complement M_1 of a regular neighbourhood of the knot $K_1 \subset \mathbb{S}^3$ from the centre of a ball in the regular neighbourhood. Now M_1 looks like a cube with a knotted hole (for details see the quoted papers). Suppose W_2 is a regular neighbourhood of K_2 in $\mathbb{S}^3$ such that $M_1 \subset W_2$ and $M_1 \cap M_2 = \partial M_1 \cap \partial M_2$ is an annulus A, where $M_2 = \mathbb{S}^3 \setminus \mathring{W}_2$. Then $M_1 \cup_A M_2$ is just the knot complement of the composite knot $K = K_1 \# K_2$ if the annulus A is meridional with respect to K_1 and K_2. Let $(V_1^{(i)}, V_2^{(i)})$, $i = 1, 2$, be a minimal regular Heegaard splitting of M_i, i.e. $g(M_i) = g(V_2^{(i)}) + 1$. By isotopy there exist closed 2-cells $D_2^{(i)}$, $C_2^{(i)}$, B_2 which satisfy the following properties:

1) $V_2^{(i)} \cap \partial M_i = D_2^{(i)}$

2) $V_2^{(i)} \cap A = B_2 \subset D_2^{(i)}$

3) $V_2^{(i)} \cap (\partial M_i \setminus \mathring{A}) = C_2^{(i)} \subset D_2^{(i)}$

4) $B_2 \cup C_2^{(i)} = D_2^{(i)}$

5) $B_2 \cap C_2^{(i)} = \partial B_2 \cap \partial C_2^{(i)}$ is an 1-arc properly embedded in $D_2^{(i)}$.

It follows that the pair $(V_1^{(1)} \cup_{A \setminus \mathring{B}_2} V_1^{(2)}, V_2^{(1)} \cup_{B_2} V_2^{(2)})$ is a regular Heegaard splitting of M. Then we have

$$g(M) \leq g(V_2^{(1)} \cup_{B_2} V_2^{(2)}) + 1 = g(V_2^{(1)}) + g(V_2^{(2)}) + 1 =$$
$$= g(M_1) + g(M_2) - 1.$$

Conversely, let (V_1, V_2) be a minimal regular Heegaard splitting of M, i.e. $g(M) = g(V_2) + 1$. By the general position theorem we can always assume that V_2 transversely intersects the annulus A in a finite number of disjoint closed 2-cells e_j. Then $V_2^{(i)} = V_2 \cap M_i$ and $V_1^{(i)} = V_1 \cap M_i$ are cubes with handles. Now we cut a plate $B_j^{(i)} \times I$ from $V_2^{(i)}$ which has the 2-cell e_j as its top face and its rim $\partial B_j^{(i)} \times I$ is an annulus in $\partial V_2^{(i)}$. Repeating this process yields a cube with handles, $\overline{V}_2^{(i)}$ say, which has the same genus of $V_2^{(i)}$. Moreover, attaching the plates $B_j^{(i)} \times I$ to $V_1^{(i)}$ gives a homeomorphic cube with handles, $\overline{V}_1^{(i)}$ say. By construction the pair $(\overline{V}_1^{(i)}, \overline{V}_2^{(i)})$ is a regular Heegaard splitting of M_i such that $g(\overline{V}_2^i) = g(V_2^{(i)})$. Finally we have $g(M) = g(V_2) + 1 \geq g(V_2^{(1)}) + g(V_2^{(2)}) + 1 = g(\overline{V}_2^{(1)}) + g(\overline{V}_2^{(2)}) + 1 \geq g(M_1) + g(M_2) - 1$. This proves the statement. ◊

Corollary 5 and Proposition 6 yield a simple alternative proof of the classical non cancellation theorem for 1-knots in $\mathbb{S}^3$ (see for example [2] and [20]).

Corollary 7. (The non cancellation theorem for 1-knots in $\mathbb{S}^3$). *The composite knot $K_1 \# K_2$ is trivial if and only if K_1 and K_2 are trivial.*

Proof. If $K_1 \# K_2$ is unknotted, then $g(M) = 1$. Because $g(M) = g(M_1) + g(M_2) - 1$ and $g(M_i) \geq g(\partial M_i) = 1$, it follows that $g(M_i) = 1$, $i = 1, 2$, and hence K_i is trivial by Corollary 5. ◊

Now we shall apply Theorem 5.2 of [15] and the additivity of the genus ([13] and [14]) to obtain the following result:

PROPOSITION 8. *Let M^3 be a compact connected orientable 3-manifold with nontrivial free fundamental group. If $g(M) = rk(M)$, then M is homeomorphic to a connected sum whose factors are cubes with handles and copies of $\mathbb{S}^1 \times \mathbb{S}^2$.*

Proof. Let $\partial_1 M$, $\partial_2 M, \dots, \partial_h M$ be the boundary components of M and let us denote the genus of $\partial_i M$ by g_i, $i = 1, 2, \dots, h$. By Theorem 5.2 and Corollary 5.3 of [15] the manifold M is a connected sum of type $\Sigma \# H_1 \# \cdots \# H_h \# \Lambda_1 \# \cdots \# \Lambda_s$, where H_i is a cube with g_i handles, Λ_j is a copy of $\mathbb{S}^1 \times \mathbb{S}^2$ and Σ is a homotopy 3-sphere. Furthermore, the following relation

$$s = rk(M) - \sum_{i=1}^{h} g_i = rk(M) - g(\partial M)$$

is verified. To prove the result we have to show that Σ is really a 3-sphere. Let (V_1, V_2) be a minimal regular Heegaard splitting of M, i. e. $g(M) = g(V_1) = g(V_2) + g(\partial M) = g(V_2) + rk(M) - s$. Then the hypothesis of the statement implies that $g(V_2) = s$. Let H_i' be a copy of H_i so that the union $H_i \cup H_i'$ is a connected sum of g_i factors of type $\mathbb{S}^1 \times \mathbb{S}^2$. Let M' be the closed orientable 3-manifold obtained from M by capping off each boundary component $\partial_i M = \partial H_i$ with H_i'. Then M' is homeomorphic to a connected sum $\Sigma \# p(\mathbb{S}^1 \times \mathbb{S}^2)$, where $p = s + g(\partial M)$. Haken's theorem on the additivity of the Heegaard genus in the closed case (see [13] and [14]) implies that

$$g(M') = g(\Sigma) + s + g(\partial M) = g(\Sigma) + g(V_2) + g(\partial M) = g(\Sigma) + g(M).$$

Because V_2 meets each boundary component $\partial_i M = \partial H_i$ in a 2-cell, the union $V_2' = V_2 \cup \bigcup_{i=1}^{h} H_i'$ is a cube with handles whose genus is

$$g(V_2') = g(V_2) + \sum_{i=1}^{h} g_i = g(V_2) + g(\partial M) = g(V_1) = g(M).$$

Thus the closed 3-manifold M' admits the Heegaard splitting (V_1, V_2'), in the usual sense, of genus $g(M)$. This implies that $g(M') \leq g(M)$ and hence $g(\Sigma)$ vanishes as $g(M') = g(\Sigma) + g(M)$. Thus Σ must be a genuine 3-sphere and the proof is complete. ◇

COROLLARY 9. *$g(M) = 1$ if and only if M^3 is either a punctured lens space (including $\mathbb{S}^1 \times \mathbb{S}^2$) or $M = \mathbb{S}^1 \times D^2$ (cube with 1-handle).*

Examples of genus two 3-manifolds with toroidal boundary components are given by the closed complements of small regular neighbourhoods of certain knots and links in $\mathbb{S}^3$ (see the next section).

3. P-graphs.

Let M be a connected compact (orientable) 3-manifold with non empty boundary ∂M. In this section we relate the concept of (regular) Heegaard diagram of M to another known 3-manifold representation, named *P-graph theory* (see for example [19] and [23]). As a consequence, we obtain a nice property about the finite presentations of $\Pi_1(M)$, which arise from (regular) Heegaard diagrams of M. In order to do this, we recall some definitions and results about P-graphs, listed in the quoted papers. Let φ be a group presentation with n generators and m relators, $n \geq m$, i.e. $\varphi =< x_1, x_2, \ldots, x_n : r_1, r_2, \ldots, r_m >$. By $K\varphi$ we denote the *canonical 2-complex* associated to φ. Then $K\varphi$ is a 2-dimensional CW-complex with one vertex v and n 1-cells (resp. m 2-cells) corresponding to generators (resp. relators) of φ. Each 1-cell of $K\varphi$ will be labelled by the associated generator x_i of φ, for $i = 1, 2, \ldots, n$. Every presentation φ determines a unique P-graph $P\varphi$ obtained as the boundary of a regular neighbourhood of the vertex v in $K\varphi$. If $x_i \cap P\varphi = \{e_i^+, e_i^-\}$, then the points (*vertices*) on the boundary of regular neighbourhoods of e_i^+, e_i^- in $P\varphi$ will be denoted by e_{ij}^+, e_{ij}^- respectively ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, k_i$). Then we set $E_i^\varepsilon = \{e_{ij}^\varepsilon : j = 1, 2, \ldots, k_i\}$ and $E = \bigcup_{i,\varepsilon} E_i^\varepsilon$ for $\varepsilon = +$ or $-$.
Now let $B = B(\varphi)$ be the involutory permutation of E, defined by $B(e_{ij}^+) = e_{ij}^-$. If $P\varphi$ is embedded into the 2-sphere $\mathbb{S}^2$, then walking clockwise around each vertex of E_i^ε induces a permutation $C = C(\varphi)$ of E, whose orbits are the sets E_i^ε. An embedding $f : P\varphi \longrightarrow \mathbb{S}^2$ is said to be *faithful* if $B = CBC$. In this case, we say that φ *fits*.
A basic result of P-graph theory is the following representation theorem (see [18], [19] and [23]).

THEOREM 10. *Let M be a connected compact orientable 3-manifold (with or without boundary). Suppose φ is a finite presentation of $\Pi_1(M)$. Then φ fits if and only if $K\varphi$ is a spine of M, i.e. there exists an embedding $K\varphi \subset M$ such that $M \setminus K\varphi$ is homeomorphic to $\partial M \times [0, 1[$. Moreover, the manifold M is uniquely determined by the faithful embedding of $P\varphi$ in $\mathbb{S}^2$.*

Now we are going to construct a Heegaard diagram of M from a faithfully embedded P-graph $(P\varphi, f)$. We consider the disc $B_i^\varepsilon \subset \mathbb{S}^2$ with center e_i^ε and such that $E_i^\varepsilon \subset \partial B_i^\varepsilon$. Since $B = CBC$, there exists an orientation reversing homeomorphism $\psi_i : \partial B_i^+ \longrightarrow \partial B_i^-$ such that $\psi_i(e_i^+) = e_i^-$ for

$i = 1, 2, \ldots, n$. Let Σ denote the closed complement of $\bigcup_{i,\varepsilon} B_i^\varepsilon$ in $\mathbb{S}^2$. Then the quotient space obtained from Σ by identifying each ∂B_i^+ with ∂B_i^- via ψ_i is the closed orientable surface S of genus n, standardly embedded in the euclidean 3-space $\mathbb{R}^3$. Let $H = H(\varphi, f)$ denote the orientable cube with n handles, in $\mathbb{R}^3$, such that $\partial H = S$. Let $\gamma = \gamma(\varphi, f)$ be the set of simple disjoint closed curves in ∂H obtained from $f(P\varphi) \cap \Sigma$ via the natural projection $\pi : \Sigma \longrightarrow S$. Now the pair (H, γ) is a Heegaard diagram of M, called the diagram induced by $(P\varphi, f)$. This construction can be reversed as follows. Let (H, γ) be a (regular) Heegaard diagram of M and let φ denote the group presentation of $\Pi_1(M)$ arising from (H, γ). We construct a faithfully embedded P-graph $(P\varphi, f)$ such that the induced diagram $(H(\varphi, f), \gamma(\varphi, f))$ coincides with (H, γ). For this, it is convenient to take the usual representation of the diagram in the euclidean plane as shown in [22]. Let $\mathbb{S}^2$ be the 2-sphere, represented as the (x, y)-plane plus a point at infinity. For $i = 1, 2, \ldots, n$, let $e_i^+ \equiv (i, +1)$, $e_i^- \equiv (i, -1)$ and B_i^ε the 2-cell of radius $1/4$ and center at e_i^ε, where $\varepsilon = +$ or $-$. As usual, Σ denotes the bordered surface $\mathbb{S}^2 \setminus \bigcup_{i,\varepsilon} \overset{\circ}{B}{}_i^\varepsilon$. Let $\pi : \Sigma \longrightarrow \partial H$ be a map, one-to-one everywhere except that each point of $\pi(\partial\Sigma)$, has two points, one of ∂B_i^+ and one of ∂B_i^-, as inverse image. Let Σ^+ (resp. Σ^-) be the subset of Σ consisting of all the points with non negative (resp. non positive) ordinate, plus the point at infinity. By isotoping, if necessary, the curves of $\gamma \subset \partial H$, we can suppose that the following conditions are satisfied:

1) for each $j = 1, 2, \ldots, m$, $\pi^{-1}(\gamma_j)$ is the disjoint union of a finite set of arcs $\{\alpha_{jr}\}$, each meeting a circle only at its endpoints;

2) $\alpha_{jr} \cap \Sigma^\varepsilon$ is either empty or the disjoint union of a finite set of arcs $\{\beta_{jrs}^\varepsilon\}$, none of which meets the x-axis (plus ∞) at an inner point.

Each circle ∂B_i^ε is split by the endpoints of the arcs β_{jrs}^ε into the union of a finite set of arcs with ends e_{ik}^ε. We can consider the pseudo-graph $G = (V, E)$ (multiple edges and loops may occur) where:

1) $V = \{e_i^\varepsilon, e_{ik}^\varepsilon\}_{ik\varepsilon}$ is the vertex-set;

2) two vertices v, $w \in V$ are joined by an edge in E if either they are the endpoints of the same arc α_{jr} or $\{v, w\} = \{e_i^\varepsilon, e_{ik}^\varepsilon\}$.

The pseudo-graph G is the desired P-graph $P\varphi$ associated to φ. Moreover, G is faithfully embedded in $\mathbb{S}^2$ and the induced diagram coincides with (H, γ).

Thus Theorem 4.1 of [19] applies to obtain the following result:

THEOREM 11. *Let M be a connected compact orientable 3-manifold (with or without boundary), (H,γ) a (regular) Heegaard diagram of M and φ the finite presentation of $\pi_1(M)$ arising from the diagram. Suppose that x is an arbitrary generator of φ and that $\{x^{m_1}, x^{m_2}, \ldots, x^{m_s}\}$ is the set of x-syllabes in the relators of φ. Then there exist relatively prime integers m_x, p_x such that the absolute value $|m_t|$ of m_t, $t = 1, 2, \ldots, s$, belongs to the set $\{m_x, p_x, m_x + p_x\}$.*

Now we illustrate our constructions showing Heegaard diagrams and faithfully embedded P-graphs of certain classical knot and link complements.
Let us consider the figure-eight knot (see for example [20]) in $\mathbb{S}^3$, shown in figure 1.

Fig. 1 - The figure-eight knot K.

We prove the following result:

PROPOSITION 12. *Let* φ *be the finite presentation*

$$< x, y : xyxy^{-1}x^{-1}yxyx^{-1}y^{-1} > \ .$$

Then the complement of the figure-eight knot is the unique orientable prime 3-manifold with connected boundary which has the canonical 2-complex $K\varphi$ *as spine.*

Proof. Let us denote the oriented 1-cells of $K\varphi$ by x, y and the unique 2-cell of $K\varphi$ by c. Then there exists an attaching map $\partial B^2 \longrightarrow x \vee y$ (one point union) given by the relator of φ. The set E consists of exactly 20 elements, two for each occurrence of a generator in the relator of φ. Suppose we denote these elements by $e^+_{1,1}$, $e^+_{1,2}, \ldots, e^+_{1,5}$, $e^+_{2,6}$, $e^+_{2,7}, \ldots, e^+_{2,10}$, $e^-_{1,1}$, $e^-_{1,2}, \ldots,$ $e^-_{1,5}$, $e^-_{2,6}$, $e^-_{2,7}, \ldots,$ $e^-_{2,10}$ which is more convenient to identify with $1, 2, \ldots, 5, 6, 7, \ldots, 10, \bar{1}, \bar{2}, \ldots, \bar{5}, \bar{6}, \bar{7}, \ldots, \bar{10}$. Assume this numbering chosen so that an appropriate closed curve parallel to and near ∂c intersects 1, $\bar{1}$, 6, $\bar{6}$, 2, $\bar{2}$, $\bar{7}$,7, $\bar{3}$, 3, 8, $\bar{8}$, 4, $\bar{4}$, 9, $\bar{9}$, $\bar{5}$, 5, $\bar{10}$, 10 in this order. Then we have the involutory permutation

$$B = B(\varphi) = (1\ \bar{1})(2\ \bar{2})(3\ \bar{3})(4\ \bar{4})(5\ \bar{5})(6\ \bar{6})(7\ \bar{7})(8\ \bar{8})(9\ \bar{9})(10\ \bar{10}).$$

Now the P-graph $P\varphi$, determined by φ, is embedded in the 2-sphere $\mathbb{S}^2$, as shown in figure 2.

Then walking clockwise around each vertex of E_i^ε, $i = 1, 2$, induces the permutation

$$C = C(\varphi) = (1\,3\,4\,5\,2)(6\,7\,9\,8\,10)(\bar{2}\,\bar{5}\,\bar{4}\,\bar{3}\,\bar{1})(\bar{10}\,\bar{8}\,\bar{9}\,\bar{7}\,\bar{6}).$$

Obviously the presentation φ fits, i.e. the embedding of $P\varphi$ in $\mathbb{S}^2$ satisfies the relation $B = CBC$ as one can easily verify. Now we apply Theorem 10. The unicity of the manifold follows from the Whitten rigidity theorem, (see [9], [25] and [26]). ◊

Fig. 2 - A P-graph of the complement of the figure-eight knot.

The Heegaard diagram (full outside) of the complement of the figure-eight knot, induced from the above-mentioned faithfully embedded P-graph, is shown in figure 3.

Fig. 3 - A Heegaard diagram of the knot complement of the figure-eight knot.

Let us consider the link $L \subset \mathbb{S}^3$ with two components shown in figure 4.

Fig. 4 - A link L with two components J, K.

As before, one can prove the following result:

Proposition 13. *Let φ be the finite presentation*

$$< x, y : xyx^{-1}yx^{-1}y^{-1}xy^{-1} > \quad .$$

Then the complement of the link L is the unique orientable prime 3-manifold with two toroidal boundary components, which has the canonical 2-complex $K\varphi$ as spine.

The faithfully embedded P-graph $P\varphi$, induced by φ, is shown in figure 5.

Walking clockwise around each vertex of E_i^{ε}, $i = 1, 2$, yields the permutation

$$C = C(\varphi) = (1\,3\,4\,2)(5\,8\,6\,7)(\bar{2}\,\bar{4}\,\bar{3}\,\bar{1})(\bar{7}\,\bar{6}\,\bar{8}\,\bar{5}).$$

Because the permutation $B = B(\varphi)$ is given by

$$B = (1\ \bar{1})(2\ \bar{2})(3\ \bar{3})(4\ \bar{4})(5\ \bar{5})(6\ \bar{6})(7\ \bar{7})(8\ \bar{8}),$$

one can easily verify that the relation $B = CBC$ holds. The unicity of the manifold follows from the fact that the above $C = C(\varphi)$ is the unique permutation for which φ fits. Finally the Heegaard diagram, induced by the P-graph of figure 5, is shown in figure 6.

Fig. 5 - A P-graph of the knot space of L.

Fig. 6 - A Heegaard diagram of the knot space of L.

4. Results in Higher Dimension.

In this section we partially extend some results, proved for bordered 3-manifolds, to higher dimension. As a consequence, we obtain a simple geometric proof of the non cancellation theorem for tame 2-knots embedded into the 4-sphere $\mathbb{S}^4$.

Let M^n be a compact connected (PL) n-manifold with h boundary components $\partial_1 M, \partial_2 M, \ldots, \partial_h M$. A *handle* of dimension n and index p (briefly a *p-handle*) H^p is a homeomorph of $D^p \times D^{n-p}$ $(0 \leq p \leq n)$, D^j being a closed j-cell.
Given a p-handle $H = D^p \times D^{n-p}$, let us consider a (PL) homeomorphism $\psi : \partial D^p \times D^{n-p} \longrightarrow \partial M$. Then $M \cup_\psi H$ is the manifold obtained from M by attaching a p-handle H via ψ. Attaching disjoint 1-handles to a closed n-cell yields an *n-cube with handles* (compare section 2 for $n = 3$), also named *n-handlebody*.

A *handle decomposition* of M is a presentation

$$M = H_0 \cup H_1 \cup \ldots \cup H_t \ ,$$

where H_0 is a closed n-cell and H_i is a handle attached to $M_{i-1} = \bigcup\{H_j : j \leq i-1\}$. It is well-known that any (PL) n-manifold with non void boundary admits a handle decomposition with one 0-handle and no n-handles ([21]).

Let K be a simplicial triangulation of M. Let us denote by Γ_1 and Γ_2 the $(n-2)$-skeleton and the dual 1-skeleton of K respectively. Now one can directly repeat the arguments developed in the proof of Theorem 1 to obtain the following natural extension.

Proposition 14. *Let M^n be a compact connected (orientable) n-manifold with h boundary components $\partial_1 M, \partial_2 M, \ldots, \partial_h M$. Then there exists a pair (V_1, V_2) of bordered connected n-manifolds satisfying the following properties:*

1) $V_1 \cup V_2 = M$,

2) $V_2 \cap \partial_i M$ is a closed $(n-1)$-cell D_i, for $i = 1, 2, \ldots, h$;

3) $V_1 \cap V_2 = \partial V_1 \cap \partial V_2 = \partial V_2 \setminus \bigcup_{i=1}^{h} \overset{\circ}{D}_i$;

4) V_1 admits a handle decomposition with handles of index $\leq n-2$;

5) V_2 is an n-dimensional handlebody;

6) $\partial V_1 = \partial V_2 \# \partial_1 M \# \ldots \# \partial_h M$.

According to section 2, any pair (V_1, V_2) with the properties of Proposition 14 is called a *(regular) splitting* of M. From now on, we suppose that M is a compact connected orientable 4-manifold with h boundary components. The *genus* of a splitting (V_1, V_2) of M is defined to be the Heegaard genus of the closed orientable 3-manifold ∂V_1. As usual, the *genus* of M^4 is the minimum m for which M admits splittings of genus m. By [11] it follows that $g(M^4) \geq g(\partial M)$ since $g(M) = g(\partial V_1) = g(\partial V_2) + g(\partial M)$ for any splitting (V_1, V_2) of minimal genus. We also observe that the genus $g(M^4)$ equals the following expression

$$\alpha_1(M^4) - h + 1 + \sum_{i=1}^{h} g(\partial_i M^4)$$

where $\alpha_1(M^4)$ is the minimum number of 1-handles in V_2 among all regular splittings (V_1, V_2) of M^4 and $g(\partial_i M^4)$ is the Heegaard genus of $\partial_i M^4$. For instance, suppose that M^4 is a compact connected orientable 4-manifold with non empty connected boundary ∂M. Then $g(M) = g(\partial M)$ if and only if $\alpha_1(M^4) = 0$, i. e. V_2 is a 4-cell and M^4 is homeomorphic to V_1. In particular, if M^4 is a cube with n handles, then $g(M) = g(\partial M) = n$.

Given a regular splitting (V_1, V_2) of a compact connected orientable 4-manifold M^4, let $V_1 = H^0 \cup \lambda H^1 \cup \mu H^2$ and let $\psi_j : (\partial D^2 \times D^2)_j \longrightarrow \partial(H^0 \cup \lambda H^1) \simeq \#_\lambda \mathbb{S}^1 \times \mathbb{S}^2$ be the attaching map of the j-th handle of index 2. We consider the set γ of simple closed curves $\gamma_j = \psi_j(\partial D^2 \times 0)$. Then the pair $(\#\lambda(\mathbb{S}^1 \times \mathbb{S}^2), \gamma)$ is a *Heegaard diagram* of the bordered orientable 4-manifold M in the sense of [17]. This extends the results of the quoted paper to the boundary case. Now we are going to study some application about knot theory.

PROPOSITION 15. *Let K be a tame (PL or smooth) 2-knot in the 4-sphere $\mathbb{S}^4$. Let $M \subset \mathbb{S}^4$ be the knot manifold of K. Then K is unknotted if and only if $g(M) = 1$.*

Proof. If K is trivial, then $M \simeq_{PL} D^3 \times \mathbb{S}^1$, hence $g(M) = 1$. Conversely, let (V_1, V_2) be a regular splitting of M of minimal genus. By [13] and [14] it follows that $g(M) = g(\partial V_1) = g(\partial V_2) + g(\partial M)$. Because $\partial M \simeq \partial(\mathbb{S}^2 \times D^2) \simeq \mathbb{S}^2 \times \mathbb{S}^1$ and $g(\mathbb{S}^2 \times S^1) = 1$, we have $g(M) = g(\partial V_2) + 1$. Hence $g(M) = 1$ implies that $g(\partial V_2) = 0$, i.e. V_2 is a 4-cell as V_2 is a handlebody. Thus $M \simeq_{PL} V_1$. Because $H_1(M) \cong \mathbb{Z}$ and $H_2(M) \cong 0$, the Mayer-Vietoris sequence of the pair $(H^0 \cup \lambda H^1, \mu H^2)$, where $V_1 = H^0 \cup \lambda H^1 \cup \mu H^2$, yields $\lambda = 1$, hence $\pi_1(V_1) \simeq \pi_1(M) \simeq \mathbb{Z}$. By [8] the manifold M is homotopy equivalent to $\mathbb{S}^1 \times D^3$. Thus the results of [6], [7], [10] and [16] get that M is (TOP) homeomorphic to $\mathbb{S}^1 \times D^3$. Hence K is trivial. $\diamondsuit$

PROPOSITION 16. *Let K_i be a tame 2-knot in the 4-sphere $\mathbb{S}^4$, $i = 1, 2$, and M_i the knot manifold of K_i. If M is the knot manifold of the connected sum $K_1 \# K_2$, then we have $g(M) = g(M_1) + g(M_2) - 1$.*

Proof. By definition of connected sum there exists a tame 3-sphere $\Sigma \subset \mathbb{S}^4$ which divides $\mathbb{S}^4$ into two 4-balls B_1, B_2 containing K_1, K_2 respectively. Furthermore, $K_1 \cap K_2$ is a closed 2-cell C, tamely embedded in Σ, and $K = K_1 \# K_2$ is just the union of K_1, K_2 minus the interior of C. Let W be a regular neighbourhood of the unknotted 1-sphere ∂C in Σ and let W' denote the closed complement of W in Σ. Then the pair (W, W') of solid tori represents the standard genus one splitting of Σ. If we set $K_i' = K_i \setminus \mathring{C}$, $i = 1, 2$, then the composite knot K is $K_1' \cup K_2'$ and its knot manifold M is $M_1' \cup M_2'$, where M_i' denotes the closed complement of a small regular neighbourhood of K_i' in B_i, $i = 1, 2$. Moreover, the intersection of M_1' with

$M_2^{'}$ is just the solid torus $W^{'}$. Thus, according to notation of Proposition 6, there exists a 3-dimensional annulus $A = \mathbb{S}^1 \times D^2 \cong_{PL} W^{'}$ such that $M = M_1 \cup_A M_2$. Furthermore, A is properly embedded essential annulus in ∂M, i.e. the inclusion induced homomorphism $\Pi_1(A) \longrightarrow \Pi_1(\partial M)$ is monic. Now we can repeat the arguments discussed in the proof of Proposition 6 to obtain the result. ♢

The next result gives a partial solution to a problem stated in [5].

COROLLARY 17. (The non cancellation theorem for 2-knots in $\mathbb{S}^4$.) *Suppose a connected sum $K = K_1 \# K_2$ of two tame 2-knots is unknotted in $\mathbb{S}^4$. Then both K_1 and K_2 are themselves unknotted.*

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