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A Deformed Bargmann Transform by an $SU(2)$ Matrix Parameter

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Abstract. *The Laguerre $2D$ polynomials depending on an arbitrary matrix Q in $SU(2)$ as a fixed parameter are used to construct a set of coherent states. The corresponding coherent state transforms constitute a deformation by matrix Q of a generalized Bargmann transform.*

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1. Introduction

The Bargmann transform, originally introduced by V. Bargmann [1], is a windowed Fourier transform [5]. It is closely connected to the Heisenberg group and has many applications in quantum optics as well as in signal processing and harmonic analysis on phase space [3]. This transform defined through

$$\mathfrak{B}_0\left[f\right](z) := \int_{\mathbb{R}} f(x) e^{-x^2+2xz-\frac{1}{2}z^2} dx, \quad z \in \mathbb{C},$$

maps isometrically the space $L^2(\mathbb{R}, dx)$ of square integrable functions on the real line onto the Bargmann-Fock space $\mathcal{F}(\mathbb{C})$ of entire complex-valued functions which are $e^{-|z|^2}d\mu$-square integrable, $d\mu$ being the Lebesgue measure on $\mathbb{C}$.

In [2] H-Y. Chen and J. Fan have constructed an integral transform, called there generalized Bargmann transform, by

$$\mathfrak{B}\left[\varphi\right](z,w) := \int_{\mathbb{C}} \exp\left(- zw + w\bar{\xi} + z\xi - \frac{1}{2}|\xi|^2 \right) \overline{\varphi(\xi)} d\mu(\xi) \tag{1}$$

as a transform of two-mode Fock space represented by a two-variable complex Laguerre polynomials, which naturally accompanies Einstein-Podolsky-Rosen entangled states of continuous variables.

Our aim here is to construct a kind of deformation $\mathfrak{B}^Q$ of (1) by means of an arbitrary parameter matrix Q belonging to the special unitary group $SU(2)$, such that for $Q = I$, being the identity matrix, the kernel of $\mathfrak{B}^I$ coincides with that of (1). Indeed, we define:

$$\mathfrak{B}^Q[\varphi](\mathfrak{z}) := \int_{\mathbb{C}} \exp\left(\mathfrak{z}Q^t\Xi(\xi) - \frac{1}{2}\mathfrak{z}\Lambda^t Q^t\mathfrak{z} - \frac{1}{2}|\xi|^2\right)\overline{\varphi(\xi)}d\mu(\xi), \qquad (2)$$

where φ belongs to a suitable class of functions, ${}^t\mathfrak{z}$ (resp. ${}^t\Xi(\xi)$) denotes the matrix transpose of $\mathfrak{z} = (z, w) \in \mathbb{C}^2$ (resp. $\Xi(\xi) = (\xi, \bar{\xi})$) and $\Lambda := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. This can be handled by adopting a coherent states method [7]. The physical meaning of the obtained deformed Bargmann transform (2) is encoded in the two-variable complex Laguerre polynomials depending on a matrix Q as introduced by A. Wünsche [10], and occurring in the quantum mechanics of a degenerate $2D$ harmonic oscillator.

The paper is organized as follows. In Section 2, we shall recall some needed facts on the Laguerre $2D$ polynomials. Section 3 deals with a formalism of generalized coherent states. This formalism is applied in Section 4 so as to define a matrix parameter family of generalized coherent states and to discuss the corresponding coherent state transforms.

2. The Laguerre 2D Polynomials

The Laguerre $2D$ polynomials $L^Q_{m,n}(\xi, \xi^*)$ defined in [10] are polynomials of the pair complex conjugated variables (ξ, ξ^*), which depend on an arbitrary fixed $2D$ matrix Q as parameter. In fact, we have

$$L^Q_{m,n}(\xi, \xi^*) = \exp\left(-\frac{\partial^2}{\partial\xi\partial\xi^*}\right)(\xi')^m({\xi'}^*)^n; \qquad m, n = 0, 1, 2, ..., \qquad (3)$$

where for $Q = \begin{pmatrix} \alpha & \beta \\ \gamma & \sigma \end{pmatrix}$ we have

$$\begin{pmatrix} \xi' \\ {\xi'}^* \end{pmatrix} = Q\begin{pmatrix} \xi \\ \xi^* \end{pmatrix} = \begin{pmatrix} \alpha\xi + \beta\xi^* \\ \gamma\xi + \sigma\xi^* \end{pmatrix}.$$

In the special case of Q being the identity matrix I, definition (3) provides explicitly

$$L^I_{m,n}(\xi, \xi^*) = (-1)^n n! \xi^{m-n} L^{(m-n)}_n(\xi\xi^*) = (-1)^m m! {\xi^*}^{n-m} L^{(n-m)}_m(\xi\xi^*),$$

where $L^{(\alpha)}_m(\cdot)$ denote the generalized Laguerre polynomials and $L^{(0)}_m(\cdot) = L_m(\cdot)$ are the ordinary Laguerre polynomials [4].

Note that for an arbitrary matrix Q the polynomials $L^Q_{m,n}(\xi,\xi^*)$ are still connected to the polynomials $L_{m,n} := L^I_{m,n}$ through the relation [10, p. 670]:

$$L^Q_{m,n}(\xi,\xi^*) = (\sqrt{\det Q})^{m+n}\sum_{j=0}^{m+n}\left(\frac{\beta}{\sqrt{\det Q}}\right)^{m-j}\left(\frac{\sigma}{\sqrt{\det Q}}\right)^{n-j} \times P_j^{(m-j,n-j)}\left(1+\frac{2\alpha\gamma}{\det Q}\right)L_{j,m+n-j}(\xi,\xi^*) \tag{4}$$

where $P_j^{(\alpha,\beta)}(\cdot)$ denotes the Jacobi polynomial [4]. It should be also noted that there is a relation between the two polynomials $L^Q_{m,n}(\cdot,\cdot)$ and $L_{p,s}(\cdot,\cdot)$. in the degenerate case of vanishing determinant of Q see [10, p. 671].

Beside the Laguerre $2D$ polynomials, Wünsche has introduced the Laguerre $2D$ functions as

$$\mathfrak{L}^Q_{m,n}(\xi,\xi^*) := e^{-\frac{1}{2}|\xi|^2}\frac{L^Q_{m,n}(\xi,\xi^*)}{\sqrt{\pi m! n!}} \tag{5}$$

and has established for general $2D$ matrix Q the following orthonormalization relations:

$$\int_{\mathbb{C}} \frac{i}{2}(d\xi\wedge d\xi^*)\mathfrak{L}^Q_{m,n}(\xi,\xi^*)\mathfrak{L}^{({}^tQ)^{-1}}_{k,l}(\xi^*,\xi) = \delta_{m,k}\delta_{n,l}, \tag{6}$$

where $\frac{i}{2}(d\xi\wedge d\xi^*) = d\mu(\xi)$ is the area element of the plane. Here ${}^tQ$ denotes the transposed matrix of Q and $\delta_{m,k}$ the Kronecker symbol. In addition, we have the completeness relation:

$$\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\mathfrak{L}^Q_{m,n}(\xi,\xi^*)\mathfrak{L}^{({}^tQ)^{-1}}_{m,n}(\zeta^*,\zeta) = \delta(\xi-\zeta,\xi^*-\zeta^*), \tag{7}$$

where $\delta(\xi,\xi^*) = \delta(\Re\xi)\delta(\Im\xi)$ denotes the two-dimensional delta function.

For our purpose, we fix Q in the special unitary group $SU(2)$, i.e., so that its inverse Q^{-1} be equal to the transpose of its conjugate. Thus, one can easily see from (5) and (6) that the Laguerre $2D$ polynomials satisfy the following property

$$\int_{\mathbb{C}} |L^Q_{m,n}(\xi,\xi^*)|^2 e^{-|\xi|^2}d\mu = \sqrt{\pi m! n!} \tag{8}$$

which means that the function $\xi \mapsto L^Q_{m,n}(\xi,\xi^*)$ belongs to the Hilbert space $L^2(\mathbb{C}; e^{-|\xi|^2}d\mu)$ of complex-valued Gaussian square integrable functions on $\mathbb{C}$. Consequently, the Laguerre $2D$ functions are elements of the Hilbert space

$L^2(\mathbb{C}; d\mu)$. Indeed, these functions can be viewed as unitary transforms of the normalized Laguerre $2D$ polynomials as

$$\mathfrak{L}^Q_{m,n}(\xi,\xi^*) := T^{-1}\left[L^Q_{m,n}(\xi,\xi^*)\right] \tag{9}$$

where T is the unitary map from $L^2(\mathbb{C}; d\mu)$ to $L^2(\mathbb{C}; e^{-|\xi|^2}d\mu)$ defined by

$$T\left[\phi\right](\zeta) := e^{\frac{1}{2}|\zeta|}\phi(\zeta)\,, \qquad \phi \in L^2(\mathbb{C}; d\mu)\,, \tag{10}$$

called a *ground state transformation.* These precisions are just to make sense when talking about the closure in $L^2(\mathbb{C}; d\mu)$ of the vector space spanned by all linear combinations of the Laguerre $2D$ functions.

REMARK 2.1. *The involved polynomials $L^I_{m,n}(\xi,\xi^*)$ in* (4)*, corresponding to the special case of the identity matrix $Q = I$, play an important role when studying representations of quasi-probabilities in quantum optics [8, 9]. Indeed, for $Q = I$ the identity* (4) *can be used to describe the transition from linear polarization to circular polarization or for a beam splitter to the splitting of a beam into two partial beams of equal intensity [6].*

3. Generalized Coherent States

In this section, we present a generalization of coherent states according to the procedure in [7]. For this, let (X, ν) be a measure space and $\mathcal{A} \subset L^2(X, \nu)$ a closed subspace of infinite dimension. Let $\{f_k\}_{k=0}^{\infty}$ be a given orthogonal basis of $\mathcal{A}$ satisfying

$$\omega(a) := \mathfrak{K}(a,a) := \sum_{k=0}^{\infty} \rho_k^{-1}|f_k(a)|^2 < +\infty; \qquad a \in X, \tag{11}$$

where $\rho_k := \|f_k\|^2_{L^2(X,\nu)}$ and

$$\mathfrak{K}(a,b) := \sum_{k=0}^{\infty} \rho_k^{-1} f_k(a)\overline{f_k(b)},\ a,b \in X, \tag{12}$$

is the reproducing kernel of the Hilbert space $\mathcal{A}$.

DEFINITION 3.1. *Let $\mathcal{H}$ be a infinite Hilbert space with an orthonormal basis $\{\psi_k\}_{k=0}^{\infty}$. The coherent states labeled by points $a \in X$ are defined as the ket-vectors $\mid \phi_a >\in \mathcal{H}$:*

$$\mid \phi_a > := (\omega(a))^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{f_k(a)}{\sqrt{\rho_k}}\psi_k. \tag{13}$$

Then, it is straightforward to show that $<\phi_a \mid \phi_a>= 1$.

Definition 3.2. *The coherent state transform corresponding to the set of coherent states* $(\mid \phi_a >)$ *is the isometric mapping* $W : \mathcal{H} \to \mathcal{A} \subset L^2(X,\nu)$ *defined by*

$$W[\psi](a) := (\omega(a))^{\frac{1}{2}} < \phi_a \mid \psi >_{\mathcal{H}}, a \in X. \tag{14}$$

Thus, for $\phi, \psi \in \mathcal{H}$, we have

$$\begin{aligned} < \phi \mid \psi >_{\mathcal{H}} &= < W[\phi] \mid W[\psi] >_{L^2(X,\nu)} \\ &= \int_X d\nu(a)\omega(a) < \phi \mid \phi_a >< \phi_a \mid \psi > . \end{aligned}$$

Thereby, we have a resolution of the identity of $\mathcal{H}$ which can be expressed in Dirac's bra-ket notation as

$$\mathbf{1}_{\mathcal{H}} = \int_X d\nu(a)\omega(a) \mid \phi_a >< \phi_a \mid, \tag{15}$$

where $\omega(a)$ appears as a weight function. The notation $\mid \phi_a >< \phi_a \mid$ means the rank one operator.

Remark 3.3. *Note that the formula* (11) *can be considered as a generalization of the series expansion of the canonical coherent states :*

$$\mid \phi_z > := e^{-\frac{1}{2}|z|^2} \sum_{k=0}^{+\infty} \frac{z^k}{\sqrt{k!}} \psi_k, z \in \mathbb{C}, \tag{16}$$

where $\{\psi_k\}_{k=0}^{+\infty}$ *denotes an orthonormal basis of eigenstates of the quantum harmonic oscillator, consisting of Gaussian-Hermite functions in* $L^2(\mathbb{R}, dx)$*. In this case, the space* $\mathcal{A}$ *is nothing but the Bargmann-Fock space* $\mathfrak{F}(\mathbb{C})$ *and* $\omega(z) = \pi^{-1} e^{|z|^2}, z \in \mathbb{C}$.

4. A Coherent State Transform Associated with Laguerre 2*D* Functions

We are now going to attach to Laguerre $2D$ polynomials with a fixed matrix parameter $Q \in SU(2)$ a set of coherent states by using the formalism described in Section 3. This can be handled by considering the following points:

- $(X, \nu) = (\mathbb{C}^2, e^{-|z|^2 - |w|^2} d\mu)$, $d\mu(z, w)$ is the Lebesgue measure on $\mathbb{C}^2$.

- $\mathcal{A} := \mathfrak{F}(\mathbb{C}^2) \subset L^2(\mathbb{C}^2, e^{-|z|^2}d\mu)$ denotes the Bargmann-Fock space of entire functions $\varphi : \mathbb{C}^2 \to \mathbb{C}$ with finite norm square

$$||\varphi||^2 := \int\limits_{\mathbb{C}^2} \varphi(z,w)\overline{\varphi(z,w)}e^{-|z|^2-|w|^2}d\mu(z,w) < +\infty. \qquad (17)$$

  Its reproducing kernel is known to be given by $K((z_1,w_1),(z_2,w_2)) = \pi^{-2}\exp(z_1\overline{z}_2 + w_1\overline{w}_2)$ so that

$$\omega(z,w) = K((z,w),(z,w)) = \pi^{-2}e^{|z|^2+|w|^2}. \qquad (18)$$

- $\{f_{m,n}\}_{m,n=0}^{+\infty}$ is an orthogonal basis of $\mathcal{A}$ given by

$$f_{m,n}(z,w) := z^m w^n; \qquad m,n = 0,1,2,\cdots \qquad (19)$$

  whose the norm is given by $\rho_{m,n} = \|f_{m,n}\|^2 = \pi m!n!$.

- $Q \in SU(2)$ is a fixed matrix parameter and $\mathcal{H}_Q(\mathbb{C})$ denotes the Hilbert subspace of $L^2(\mathbb{C}, d\mu)$ obtained as the closure of vector space $span(\mathfrak{L}^Q_{m,n})$ spanned by all linear combinations of the Laguerre $2D$ functions $\mathfrak{L}^Q_{m,n}$ in (5).

Definition 4.1. *The vectors* $(\Phi_{\mathfrak{z},Q})$ *of the Hilbert space* $\mathcal{H}_Q(\mathbb{C})$ *labelled by elements* $\mathfrak{z} = (z,w) \in \mathbb{C}^2$ *and defined formally through* (13) *by*

$$\Phi_{\mathfrak{z},Q} \equiv |(z,w), Q> := (\omega(z,w))^{-\frac{1}{2}} \sum_{m,n=0}^{+\infty} \frac{f_{m,n}(z,w)}{\sqrt{\rho_{m,n}}}\mathfrak{L}^Q_{m,n}, \qquad (20)$$

are called generalized coherent states.

Proposition 4.2. *The wave functions of the states in* (20) *admit the following closed form*

$$\Phi_{\mathfrak{z},Q}(\xi) = e^{-\frac{1}{2}(|\mathfrak{z}|^2+|\xi|^2)}\exp\big(\mathfrak{z}Q^t\Xi(\xi) - \frac{1}{2}\mathfrak{z}\Lambda^t Q^t\mathfrak{z}\big), \qquad (21)$$

where ${}^t\mathfrak{z}$ *(resp.* ${}^t\Xi(\xi)$*) denotes the matrix transpose of* $\mathfrak{z} = (z,w)$ *(resp.* $\Xi(\xi) = (\xi,\xi^*)$*),* $|\mathfrak{z}|^2 = |z|^2 + |w|^2$ *its square modulus and* Λ *denotes the first Pauli spin matrix* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Proof. By definition (20), the associated wave functions read

$$\begin{aligned}\Phi_{\mathfrak{z},Q}(\xi) &:= \ <\xi \mid \mathfrak{z}, Q> = <\xi \mid (z,w), Q> \\ &:= (\omega(z,w))^{-\frac{1}{2}} \sum_{m,n=0}^{+\infty} \frac{f_{m,n}(z,w)}{\sqrt{\rho_{m,n}}} \mathfrak{L}^{Q}_{m,n}(\xi,\xi^*) \\ &= (\pi^{-2} e^{|z|^2+|w|^2})^{-\frac{1}{2}} \sum_{m,n=0}^{+\infty} \frac{z^m w^n}{\sqrt{\pi m! n!}} e^{-\frac{1}{2}|\xi|^2} \frac{L^{Q}_{m,n}(\xi,\xi^*)}{\sqrt{\pi m! n!}} \\ &= e^{-\frac{1}{2}(|z|^2+|w|^2)} e^{-\frac{1}{2}|\xi|^2} \sum_{m,n=0}^{+\infty} \frac{z^m w^n}{m! n!} L^{Q}_{m,n}(\xi,\xi^*).\end{aligned}$$

Now, making use of the generating function for the Laguerre $2D$ polynomials [10, p. 675]:

$$\sum_{m,n=0}^{+\infty} \frac{z^m w^n}{m! n!} L^{Q}_{m,n}(\xi,\xi^*) = \exp\left[(z,w)Q\begin{pmatrix}\xi\\ \xi^*\end{pmatrix} - \frac{1}{2}(z,w)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^{t} Q\begin{pmatrix}z\\ w\end{pmatrix}\right],$$

one gets the announced result. □

The constructed generalized coherent states give rise to a transform according to Definition 3.2. Thus, the coherent state transform (CST) associated to $\Phi_{\mathfrak{z},Q}$; $Q \in SU(2)$, is the unitary map $\mathfrak{B}^Q$ from the Hilbert space $\mathcal{H}_Q(\mathbb{C}) \subset L^2(\mathbb{C}, d\mu)$ into the Bargmann-Fock space $\mathfrak{F}(\mathbb{C}^2)$ defined by

$$\mathfrak{B}^Q[\varphi](\mathfrak{z}) := (\omega(\mathfrak{z}))^{\frac{1}{2}} \langle \Phi_{\mathfrak{z},Q}, \varphi\rangle_{L^2(\mathbb{C},d\mu)}, \qquad \varphi \in \mathcal{H}_Q(\mathbb{C}), \tag{22}$$

Being motivated by this construction, we state the following definition

Definition 4.3. *The coherent state transform $\mathfrak{B}^Q$ whose integral representation is given by*

$$\mathfrak{B}^Q[\varphi](\mathfrak{z}) = \int_{\mathbb{C}} \exp\left(\mathfrak{z} Q^t \Xi(\xi) - \frac{1}{2}\mathfrak{z}\Lambda^t Q^t \mathfrak{z} - \frac{1}{2}|\xi|^2\right)\overline{\varphi(\xi)} d\mu(\xi) \tag{23}$$

will be called a deformed Bargmann transform by the $SU(2)$ matrix parameter Q.

Remark 4.4. *For the particular case of $Q = I$ being the identity matrix, the CST in* (23) *reduces further to*

$$\mathfrak{B}^I[\varphi](z,w) = \int_{\mathbb{C}} \exp\left(-zw + w\xi^* + z\xi - \frac{1}{2}|\xi|^2\right)\overline{\varphi(\xi)} d\mu(\xi)$$

which has the same integral kernel as the transform considered in [2].

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