MAXIMUM PRINCIPLES FOR LINEAR ELLIPTIC SYSTEMS (*)

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SOMMARIO. - *In questo lavoro si ottengono alcune condizioni sufficienti (necessarie) per la validità del principio di massimo per sistemi ellittici di tipo cooperativo e non.*

SUMMARY. - *In this paper we obtain some sufficient (necessary) conditions for the validity of the maximum principle for cooperative and non-cooperative elliptic systems.*

0. Introduction. In this paper we shall prove some results concerning the maximum principle for weakly-coupled elliptic systems of the form

$$(0.1)\qquad \begin{aligned} \mathcal{L}(D)U &= A(x)U + F \quad \text{in} \quad \Omega \subset \mathbf{R}^N \\ U &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

where $\Omega \subset \mathbf{R}^N (N \geq 1)$ is a smooth bounded domain, $\mathcal{L} = [L_1(D), L_2(D), \dots, L_n(D)]$ is a diagonal-matrix of second order elliptic operators, $A(x) = (a_{ij}(x))$ is a $n \times n$ coefficient matrix and F is a given n-vector function defined in Ω. By a maximum principle we mean the statement of the positiveness of the solution U of (0.1) (that is, $U \geq 0$ in Ω) when the given function $F \geq 0$ in Ω.

The classical maximum principle of Protter-Weinberger (see [8]) for weakly-coupled elliptic systems of the form (0.1) holds if the following conditions on $A(x)$ are satisfied:

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(i) $a_{ij}(x) \geq 0 \quad \forall i \neq j \quad x \in \Omega$

(ii) $\sum_{j=1}^{n} a_{ij}(x) \leq 0 \quad \forall i \quad x \in \Omega$.

A system (0.1) for which (i) above holds is said to be *cooperative.* It is clear that since (ii) is assumed, the above result cannot contain the maximum principle for the single equation. Indeed a maximum principle for $-\Delta u = a(x)u + f$ holds, for example, if $a(x) < \lambda_1$ in Ω. Here λ_1 denotes the first eigenvalue of $(-\Delta, H_0^1(\Omega))$. Hence, it is natural to ask if it is possible to refine the above mentioned result for systems by relaxing condition (ii) somehow. In this paper we show that this is actually possible. Let us briefly sketch the contents of the present paper.

In section 1 we treat a general class of cooperative systems like (0.1) with variable coefficients. We start by giving a proof of the above mentioned result, see Theorem 1.1. Our proof is completely different from the "parabolic" proof contained in [8]. Then we prove our Theorem 1.2, which is a noticeable improvement of the previous theorem. The case when the elliptic operators L_k are all equal to the same selfadjoint elliptic operator is considered. We then obtain the right generalization of the scalar case (single equation) stated above.

In Section 2 we consider the case when all the entries of the matrix A are constant and the operators L_k are all equal to a self adjoint operator L. We then obtain a necessary and sufficient condition for the existence of a maximum principle in terms of the characteristic polynomial of A.

In Section 3 we analyse a 2×2 noncooperative system. Using the idea of embedding such a system into 3×3 cooperative system we are able to prove maximum principles of noncooperative systems. In this way we recapture a result obtained before by the authors in [5]. We also treat fourth order elliptic operators, and in particular we obtain a maximum principle for the biharmonic operator under the so-called Navier boundary conditions.

Some of the results contained in this paper were announced at the "Conference on Reaction Diffusion equations" held in Edinburgh in June 1988.

1. Maximum principle for cooperative elliptic systems with variable coefficients. Let us first fix the notation and state some conditions that will be used throughout this section.

Consider the following set of second order elliptic operators with real valued coefficients defined in some bounded domain Ω in $\mathbf{R}^N$,

$$L_k(D) = -\sum_{ij} b_{ij}^k(x) D_i D_j + \sum_i b_i^k(x) D_i, \quad k = 1, \ldots, n$$

where $b_{ij}^k = b_{ji}^k$ and

$$\sum_{ij} b_{ij}^k(x)\xi_i\xi_j \geq \lambda(x)|\xi|^2, \quad x \in \Omega,\ \xi \in \mathbf{R}^N,\ \forall k$$

for some function $\lambda(x) > 0$, $x \in \Omega$. So each L_k is elliptic as defined in Gilbarg-Trudinger [6]. No regularity on the coefficients is needed. We shall assume that there is a constant $M > 0$ such that

$$\frac{|b_i^k(x)|}{\lambda(x)} \leq M, \quad x \in \Omega,\ \forall i,\ \forall k\,. \tag{1.1}$$

Let $A = [a_{kj}(x)]$ be a $n \times n$ matrix whose entries are real-valued functions defined in Ω. We assume that the off-diagonal entries are nonnegative (cooperativeness):

$$a_{kj}(x) \geq 0, \quad x \in \Omega, \quad k \neq j\,. \tag{1.2}$$

Let $F(x) = (f_1(x), \ldots, f_n(x))$ be a given n-vector whose components are real-valued functions defined in Ω.

Let us consider the elliptic system

$$L_k(D)u_k = \sum_j a_{kj}(x)u_j + f_k \quad k = 1, \ldots, n \tag{1.3$_k$}$$

or, in short

$$\mathcal{L}(D)U = AU + F \tag{1.3}$$

where $\mathcal{L}(D)$ denotes the diagonal operator-matrix $(L_1(D), \ldots, L_n(D))$ and U is the solution n-vector $(u_1(x), \ldots, u_n(x))$.

We say that an n-vector $V(x) = (v_1(x), \ldots, v_n(x))$ is ≥ 0 if all its components are nonnegative functions.

In this section we shall discuss the nonnegativeness of solutions of system (1.3).

More precisely, we investigate when $F \geq 0$ implies that a solution of the Dirichlet problem

$$(1.4) \qquad \begin{aligned} \mathcal{L}(D)U &= AU + F \quad \text{in} \quad \Omega \\ U &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

is such that $U \geq 0$ in Ω.

By a solution we mean a classical solution, that is a function U defined in $\bar{\Omega}$ which is continuous in $\bar{\Omega}$ and belongs to $C^2(\Omega)$. Here we do not discuss existence questions. We assume that a classical solution of (1.4) is given and we prove its nonnegativeness.

The first result is the following (see [8] for a different proof):

THEOREM 1.1. *Suppose in addition to the above assumptions that*

$$(1.5) \qquad a_k(x) \equiv \sum_j a_{kj}(x) \leq 0 \quad x \in \Omega, \quad k = 1, \ldots, n.$$

Then $F \geq 0$ in Ω implies that $U \geq 0$, being U a solution to (1.4).

REMARK 1.1. Condition (1.5) and the hypothesis that the off-diagonal entries of A are ≥ 0 imply that $a_{kk}(x) \leq 0 \; \forall x \in \Omega, k = 1, \ldots, n.$

We shall prove next that if the conclusion of Theorem 1.1 holds under the stronger assumption

$$(1.5) \qquad a_k(x) < 0, \text{ for } x \in \Omega \text{ and } k = 1, \ldots, n$$

then Theorem 1.1 is true.

Indeed, let $\alpha = \alpha(x_1)$ be a positive C^2 real-valued function which will be chosen later. let $v_k(x)$ $(k = 1, \ldots, n)$ be defined by $u_k(x) = \alpha(x_1)v_k(x)$. Through some calculations we come to

$$L_k u_k = \alpha L_k v_k - 2\alpha' \sum_i b^k_{1i} D_i v_k - \alpha'' b^k_{11} v_k + \alpha' b^k_1 v_k \,.$$

Thus the v_k's satisfy the system

$$L_k v_k - \frac{2\alpha'}{\alpha}\sum_i b_{i1}^k D_i v_k = \left(\frac{\alpha''}{\alpha} b_{11}^k - \frac{\alpha'}{\alpha} b_1^k\right) v_k + \sum_j a_{kj} v_j + \frac{f_k}{\alpha},$$

or

$$\hat{\mathcal{L}}(D)V = \hat{A}V + \frac{F}{\alpha} \tag{1.6}$$

where $\mathcal{L}$ and $\hat{\mathcal{L}}$ have the same principal part and b_i^k is replaced by

$$\tilde{b}_i^k = b_i^k - \frac{2\alpha'}{\alpha} b_{i1}^k .$$

Also A and $\tilde{A}$ have the same off-diagonal terms and a_{kk} is replaced by

$$\tilde{a}_{kk} = a_{kk} + \frac{\alpha''}{\alpha} b_{11}^k - \frac{\alpha'}{\alpha} b_1^k .$$

Now we choose the function $\alpha(x_1)$ in such a way that $\tilde{a}_{kk} < 0$ in Ω, and consequently $\tilde{a}_k < 0$ in Ω.

Using the ellipticity of L_k and condition (1.1) we have

$$\frac{\alpha''}{\alpha} b_{11}^k - \frac{\alpha'}{\alpha} b_1^k \le \frac{\alpha''}{\alpha}\lambda(x) - \frac{\alpha'}{\alpha} b_1^k \le \lambda(x)\left[\frac{\alpha''}{\alpha} - M\frac{\alpha'}{\alpha}\right] \tag{1.7}$$

supposing that $\alpha > 0$, $\alpha' \le 0$ and $\alpha'' \le 0$.

Our claim is attained by showing that an α as such can be chosen to make the right hand side of (1.7) strictly negative. Since Ω is bounded, we may assume that Ω is contained in the strip $\{x, : 0 < x_1 < K\}$ for some $K \ge 0$. Thus, choosing

$$\alpha(x_1) = -e^{\beta x_1} + e^{\beta K} + 1$$

where $\beta > M$, we see that α fulfills all the above requirements. Clearly $U \ge 0$ if and only if $V \ge 0$.

Proof of Theorem 1.1. By the previous remark we assume (1.5)' instead of (1.5). Using the notation in (1.5) we write system $(1.3)_k$ as

$$L_k(D)u_k = a_k(x)u_k + \sum_{j\neq k} a_{kj}(x)[u_j - u_k] + f_k(x) \tag*{$(1.8)_k$}$$

Suppose by contradiction that U is not positive. This means that at least one of its components is not positive. Without loss of generality, assume that it is u_1. Hence, there is $x^1 \in \Omega$ such that

$$u_1(x^1) = \min_{x\in\Omega} u_1(x) < 0 \ . \tag{1.9}$$

Since $L_1(D)u_1(x^1) \leq 0$, it follows that, for at least one $j \neq 1$, we have

$$a_{1j}(x^1)[u_j(x^1) - u_1(x^1)] < 0 \ .$$

Without loss of generality assume that this happens for $j = 2$. So,

$$u_2(x^1) < u_1(x^1) \tag{1.10}$$

and let $x^2 \in \Omega$ be such that

$$u_2(x^2) = \min_{x\in\Omega} u_2(x) < 0 \ . \tag{1.11}$$

Using the second equation in $(1.8)_k$ we infer that

$$a_{2j}(x^2)[u_j(x^2) - u_2(x^2)] < 0$$

for some $j \neq 2$. Clearly $j \neq 1$ in view of (1.9), (1.10) and (1.11). As in the previous case we may assume that

$$u_3(x^2) < u_2(x^2)$$

and we proceed as before using the third equation in $(1.8)_k$. Observe that $u_3(x^2) < u_1(x^2)$. Repeating the above argument after using the $(n-1)^{th}$ equation, we come to a point $x^n \in \Omega$ where

$$u_n(x^n) = \min_{x\in\Omega} u_n(x) < 0$$

and $u_n(x^n) < u_j(x^n)$ for $j \neq n$.

Finally, using the n^{th} equation in $(1.8)_k$ we come to a contradiction.

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Next we obtain a maximum principle for the solutions of (1.4) under a weaker hypothesis than (1.5).

DEFINITION 1.1. The operator $\mathcal{L} - A$, satisfies property (ψ) if there exists a C^2-function $\psi : \bar{\Omega} \to \mathbf{R}^n$ such that

(i) $\psi(x) > 0 \quad$ for $\quad x \in \bar{\Omega}$

(ii) $\mathcal{L}(\psi) \geq A\psi \quad$ in $\quad \Omega$.

REMARK 1.2. If property (ψ) holds for $\mathcal{L}(D) - A$ where $A \geq B$ (i.e. $a_{ij}(x) \geq b_{ij}(x)$ $i, j, \ldots, n$) then it also holds for $\mathcal{L}(D) - B$.

REMARK 1.3. If the coefficients of the matrix A are bounded i.e. $a_{ij} \in L^\infty(\Omega)$ then a sufficient condition in order that $\mathcal{L}(D) - A$ satisfies property (ψ) is the solvability of the following system

$$(*) \qquad \begin{cases} \mathcal{L}(\phi) = A_\infty \phi + F & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega \\ \phi > 0 & \text{in } \Omega \end{cases}$$

where $F = (F_1, \ldots, F_n)$ with $F_j = \sum_{j=1}^n a_{ij}^\infty$, and $a_{ij}^\infty = ||a_{ij}||_{L^\infty}$.

Indeed if $(*)$ has a solution then, the operator $\mathcal{L} - A_\infty$ satisfies property (ψ) with $\psi = \phi + 1$ (i.e. $\psi_i = \phi_i + 1$, $i = 1, \ldots, n$) and then by the preceding Remark $\mathcal{L} - A$ satisfies the same property.

REMARK 1.4. In the special case $L_1(D) = L_2(D) = L(D)$ with L self-adjoint (see [6]).

$$(1.12) \qquad L(D)u = -\sum_{ij} D_i(a_{ij}D_j u + b_i(.u)) + \sum_i b_i D_i u$$

where the coefficients are continuous functions defined in a domain $\Omega' \supset \bar{\Omega}$ we can give a simple sufficient condition as follows:

Let $\lambda_1(\Omega)$ be the first eigenvalue of $(L(D), H_0^1(\Omega))$. Suppose that

$$a_k(x) \leq \Lambda < \lambda_1(\Omega) \quad x \in \Omega\ .$$

Let $\hat{\Omega}$ be, such that $\Omega' \supset \hat{\Omega} \supset \bar{\Omega}$, and $\Lambda \leq \lambda_1(\hat{\Omega}) < \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ denotes the first eigenvalue of $(L(D), H_0^1(\hat{\Omega}))$. Let ϕ be a positive eigenfunction corresponding to $\lambda_1(\hat{\Omega})$. Then we see that the operators $\mathcal{L} - A$ satisfies property (ϕ) with $\psi = (\phi, \ldots, \phi)$.

THEOREM 1.2. *If the operators $\mathcal{L} - A$ satisfies property (ψ) then $F \geq 0$ in Ω implies $U \geq 0$, being U a solution to* (1.4).

COROLLARY 1.1. *Let $\mathcal{L}$ be as in Remark* 1.4. *Assume that $a_k(x) \leq \Lambda < \lambda_1(\Omega), \forall x \in \Omega$. Then $F \geq 0$ in Ω implies $U \geq 0$ in Ω, being U a solution of* (1.4).

REMARK 1.5. Theorem 1.2 above contains Theorem 1.1. Indeed, if $a_k(x) \leq 0$, then it is sufficient to consider $\psi = (1, 1, \ldots, 1)$.

Proof of Theorem 1.2. Let us introduce the functions v_k, $k = 1, \ldots, n$ by

$$u_k(x) = v_k(x)\psi_k(x) \quad x \in \Omega ,$$

and put $h_k(x) = L_k(\psi_k) - \sum_{j=1}^{n} a_{kj}\psi_j(x)$.

Then

$$L_k u_k = \psi_k L_k v_k + v_k L_k \psi_k - 2\sum_{1j} b_{ij}^{(k)} D_i\psi_k D_j v_k$$

which implies

$$L_k v_k - \frac{2}{\psi_k}\sum_{ij} b_{ij}^{(k)} D_i\psi_k D_j v_k = \frac{v_k}{\psi_k}[-h_k - \sum_{j=1}^{n} a_{kj}\psi_j] +$$

$$+\sum_{j=1}^{n} \frac{a_{kj}v_j\psi_j}{\psi_k} + \frac{f_k}{\psi_k} .$$

Thus, system (1.1) is equivalent to

$$\tilde{L}_k v_k = \sum_{j=1} c_{kj}v_j + \frac{f_k}{\psi_k} \quad k = 1, \ldots, n \tag{1.13}$$

where $\tilde{L}_k$ and L_k have the same principal part and

$$\tilde{b}_i^k = b_i^k - \frac{2}{\psi_k}\sum_j b_{ij}^k D_j\psi_j .$$

$$c_{kj} = \begin{cases} -\frac{1}{\psi_k}[h_k + \sum_{i=1}^n a_{ki}\psi_i - a_{kk}\psi_k] \text{ if } j = k \\ \frac{a_{kj}\psi_j}{\psi_k} \ j \neq k \end{cases}$$

For system (1.13) we have

$$\tilde{c}_k = \sum_j c_{kj} = -\frac{h_k}{\psi_k} \leq 0$$

and so Theorem 1.1 may be applied to conclude the proof.

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REMARK 1.6. Assume that each L_k is strictly elliptic, that is $\lambda(x) \geq \lambda_0$ for some positive constant λ_0 and all $x \in \Omega$. Suppose that the a_k's are L^∞ functions and let $M > 0$ be such that $|a_k(x)| \leq M, k = 1, \ldots, n$ $x \in \Omega$. It follows from a result of Protter and Weinberger [8; p. 73] that if the domain is contained in a sufficiently narrow slab bounded by two parallel planes, or if M is sufficiently small, then $\mathcal{L} - A$ satisfies property (ψ). To see how narrow has the slab to be or how small M, look at the footnote on page 74 of [6].

REMARK 1.7 (a strong maximum principle). Assume that the diagonal entries a_{kk} are L^∞ functions, and that Ω is smooth. Then on the hypotheses of Theorem 1.2, if, for some k, f_k is positive on a set of positive measure, it follows that $u_k > 0$ in Ω and the outward normal derivative $\frac{\partial u_k}{\partial \nu} < 0$. To see that let M be a positive constant such that $M \geq |a_{kk}(x)|$ for all $x \in \Omega$ and apply the usual strong maximum principle for scalar equations to the equation

$$(L_k + M)u = \sum_{j \neq k} a_{kj}u_j + (a_{kk} + M)u_k + f_k .$$

REMARK 1.8. If $f : \Omega \to \mathbf{R}^n$ is a given C^1 function with cooperative Jacobian J_f and $f(0) = 0$, then it follows from Theorem 1.1 and the mean value theorem (we apply the mean value theorem componentwise) that if there exists a function $\psi : \Omega \to \mathbf{R}^n$ such that

(i) $\psi > 0$ in $\bar{\Omega}$

(ii) $\mathcal{L}(\psi) \geq f(\psi)$ in Ω

then for every $F : \Omega \to \mathbf{R}^n$ with $F \geq 0$ in Ω the solutions to

$$\begin{aligned} \mathcal{L}(u) &= f(U) + F && \text{in } \Omega \\ U &= 0 && \text{on } \partial\Omega \end{aligned}$$

are non negative.

2. A maximum principle for a cooperative system with constant coefficients.

In this section we study system (1.4) of the previous section in the case when

$$L_1(D) = L_2(D) = \ldots = L_n(D) = L$$

with L self-adjoint i.e. L is of the form

$$L(D) = -\sum_{ij} D_i(a_{ij}D_j + b_i) + \sum_i b_i D_i$$

the coefficients $a_{ij}, b_i \in C(\Omega)$ and A is a constant matrix with nonnegative off-diagonal entries. Namely

$$\text{(2.1)} \qquad \begin{aligned} \mathcal{L}(D)U &= AU + F, && \text{in } \Omega \\ U &= 0 && \text{on } \partial\Omega \end{aligned}$$

For such a system we prove the following result, which is a strong sharpening of Theorem 1.2. We shall denote by λ_1 the principal eigenvalue of $(L, H_0^1(\Omega))$.

THEOREM 2.1. *Let $p_1(\lambda), \ldots, p_n(\lambda)$ be the characteristic polynomials of the first n-principal minors of A, namely*

$$(a_{11}), \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \ldots A\,.$$

Then

$$\text{(2.2)} \qquad p_i(\lambda_1) > 0 \quad i = 1, \ldots, n$$

is a necessary and sufficient condition for

$$(2.3) \qquad F > 0 \Rightarrow U > 0 \ ,$$

where U is a solution to (3.1).

REMARK 2.1. $F > 0$ means $F_j(x) \geq 0$ and $F_j(x) \not\equiv 0$ for all j. $F \geq 0$ means $F_j(x) \geq 0$ for all j. Conditions (2.2) in the above theorem is sufficient for $F \geq 0 \Rightarrow U \geq 0$.

REMARK 2.2. We see as a consequence of Lemma 3.1 below, that condition (2.2) implies that the characteristic polynomial $p(\lambda)$ of any principal minor is positive at λ_1, i.e. $p(\lambda_1) > 0$. This fact dissipates a possible suspicion raised by the apparent asymmetry of condition (2.2).

To fix terminology used here we recall that a principal minor is a minor obtained from the original matrix by dropping lines and columns of the same order. For instance, the second and fourth lines and the second and fourth columns. All othe rminors are called secondary.

DEFINITION 2.1. A matrix of the form

$$M_n = \begin{bmatrix} a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & a_{22} & & -a_{2n} \\ \cdots & \cdots & & \cdots \\ -a_{n1} & -a_{n2} & & a_{nn} \end{bmatrix}$$

where $a_{ij} \geq 0$ for all $i, j = 1, \dots, n$ is called an M-matrix. Now we list some facts about these matrices and, for sake of completeness, we include the proofs.

LEMMA 2.1. *Suppose that all principal minors of M_n, including M_n itself, have positive determinants.*

Then the $(n+1) \times (n+1)$ matrix below has non positive determinant

$$z_{n+1} = \left| \begin{array}{c|c} z_1 & t_2 \ \cdots \ t_{n+1} \\ z_2 & \\ \vdots & M_n \\ z_{n+1} & \end{array} \right|$$

where $z_i \leq 0, i = 1, \ldots, n+1, t_i \leq 0$ *for* $i = 2, \ldots, n+1$. *Moreover, if* $z_i < 0$ *for all* i, *then* $\det Z_{n+1} < 0$.

Proof. By induction. The result is clearly true for 1×1 M-matrices. Now, assume the result is true for $(n-1) \times (n-1)$ M-matrices. Let M_n be a given $n \times n$ M-matrix satisfying the hypothesis of the Lemma. Then,

$$\det Z_{n+1} = z_1 \det M_n + \sum_{j=2}^{n} (-1)^{j+1} \det \hat{Z}_{1,j} \tag{2.4}$$

where $Z_{1,j}$ is the minor obtained from Z_{n+1} by omitting line j, $2 \leq j \leq n+1$, and the first column. The matrix $Z_{1,j}$ has its $(j-1)^{th}$ column formed by nonpositive numbers. We transpose this column with the first one and we obtain a matrix $\hat{Z}_{ij}$ of the type Z_n whose determinant is nonpositive by induction hypothesis. Since

$$\det Z_{1,j} = (-1)^{j-2} \det \hat{Z}_{1,j}$$

we get the result from (2.4).

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LEMMA 2.2. *Let* M_n *be an* M*-matrix and suppose that its* n*-first principal minors have positive determinants. Then*

(i) all other principal minors have positive determinants

(ii) $(-1)^{i+j} \det M_{ij} \geq 0$ *when* M_{ij} *is the secondary minor determined from* M_n *by dropping the* i^{th} *line and the* j^{th} *column,* $i \neq j$.

Proof. By induction. For $n = 2$ we obtain

$$a_{11}a_{22} - a_{12}a_{21} > 0 \quad a_{11} > 0$$

by hypothesis, which implies $a_{22} > 0$. Now, we assume the result true for $(n-1) \times (n-1)$ M-matrices, and let M_n be a $n \times n$ M-matrix satisfying the hypothesis of the lemma. Let us prove (ii). Assume $i < j$. The case $i > j$ can be proved similarly. Observe that the i^{th} column of M_{ij} is formed by non-positive numbers: we transpose it with the first column. The $(j-i)^{th}$ line of M_{ij} is formed by nonpositive numbers: we transpose it with the first line. We then obtain a matrix $\hat{M}_{ij}$ of the type Z_{n-1}. So, by Lemma 2.1, det $\hat{M}_{ij} \leq 0$. On the other hand

$$\det M_{ij} = (-1)^{i-1}(-1)^{j-2} \det \hat{M}_{ij} .$$

So the result follows. Now let us prove (i). In view of the induction hypothesis it remains to prove that the principal minors containing a_{nn} have positive determinants. To do that it suffices to prove that $a_{nn} > 0$. For that matter we write

$$\det M_n = a_{nn} \det M_{n-1} - \sum_{j=2}^{n} a_{nj}(-1)^{n+j} \det M_{nj} .$$

By part (ii) each term ($j = 2, \ldots, n$) of the above summation is ≥ 0. Since det $M_n > 0$ and det $M_{n-1} > 0$ it follows that $a_{nn} > 0$.

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LEMMA 2.3. *Let M be an M-matrix. If all solutions X of*

$$MX = Y \tag{2.5}$$

for $Y > 0$ are such that $X > 0$, then all principal minors of M have positive determinant. (Here $X > 0$ means $X_j > 0$ for each j).

Proof. The i^{th} equation in (2.5) gives

$$a_{ii}x_i = \sum_{i \neq j} a_{ij}x_j + y_1 .$$

Since $a_{ij} \geq 0$ for all i, j and $x_i > 0$, $y_i > 0$ it follows $a_{ii} > 0$. Now we proceed by induction. Assume the result has been proved to a system (2.5) with $n-1$ equations.

Let us prove it for a system with n equations. From such a system we can write n system with $n-1$ equations and $n-1$ unknown using the following procedure: drop the i^{th} equation and write the other $n-1$ equations in the form

$$a_{jj}x_j - \sum_{k \neq i} a_{jk}x_k = a_{ji}x_i + y_j . \tag{2.6}$$

The matrix of system (2.6) is an M-matrix. So by the induction hypothesis all $j \times j$ ($j \leq n-1$) principal minors of M have positive determinant. It remains to show that det $M > 0$. We first observe that det $M \neq 0$. Indeed, if det $M = 0$, then there would exist $X_0 \neq 0$, such that $MX_0 = 0$. Then if $X > 0$ is a solution to (2.5) for a given $Y > 0$, then $X + tX_0$ for any $t \in \mathbf{R}$ is also a solution. Clearly for a convenient choice, at t we can get $X + tX_0$ not > 0. Now, by Cramer's rule

$$(2.7) \qquad x_1 \det M = \det \begin{vmatrix} y_1 & -a_{12} & \dots & -a_{1n} \\ y_2 & a_{22} & & -a_{2n} \\ \dots & & & \\ y_n & -a_{n2} & & a_{nn} \end{vmatrix}$$

By Lemma 2.1 it follows that the determinant in the right hand side of (2.7) is positive. Consequently det $M > 0$.

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To prove Theorem 2.1 we need some preliminaries.

Let E be a Banach space ordered by a closed convex cone K. An operator $P : E \to E$ is said to be positive if $P(K) \subset K$. A set $\mathcal{P}$ of positive bounded linear operators in E is said to be *positive-numbers-like* if

(i) $I \in \mathcal{P}$

(ii) $P_1P_2 = P_2P_1 \forall P_1, P_2 \in \mathcal{P}$

(iii) $||P_1 + P_2|| = ||P_1|| + ||P_2||, \quad ||P_1P_2|| = ||P_1||\,||P_2|| \forall \mathcal{P}, \mathcal{P}_2 \in \mathcal{P}$,

where $||P||$ denotes the norm of the operators P, and I is the identity operator.

REMARK 2.3. Let $\mathcal{P}$ be a positive-numbers-like set of operators. Then, if and $\mathcal{P} \in \mathcal{P}$ and $||P|| < 1$, it follows that $(I - P)^{-1}$ is a positive bounded lienar operator and

$$||(I - P)^{-1}|| = (I - ||P||)^{-1} .$$

Indeed,

$$|| \lim_n \sum_{j=1}^{n} P^j || = \lim_n || \sum_{j=1}^{n} P^j || = \lim_n \sum_{j=1}^{n} ||P||^j .$$

REMARK 2.4. Let $\mathcal{P}$ be as in the previous remark. Then the set formed by all polynomials

$$\sum_{|\alpha|\leq m} a_\alpha P_1^{\alpha_1} \dots P_k^{\alpha_K} \dots, \quad a_\alpha \geq 0,$$

for any set of operators $P_1, \dots, P_k$ in $\mathcal{P}$, is a positive-numbers-like set of operators.

REMARK 2.5. Let $\mathcal{P}$ be as in Remark 3 above. Then, the set made up by operators of form $(I - P_1)^{-1}P_2$, with $P_1, P_2 \in \mathcal{P}$ and $||P_1|| < 1$, is also a positive-numbers-like set of operators.

REMARK 2.6. A set $\mathcal{P}$ of positive bounded linear operators with the property that there exists $\phi \in K$ such that $P\phi = ||P||\phi$ for all $P \in \mathcal{P}$, is a positive-numbers-like set of operators.

EXAMPLE 2.1. Let $E = L^2(\Omega)$ where Ω is a bounded domain in $\mathbf{R}^N$. Let K be the cone of a.e. non-negative L^2 -functions. Let L and λ_1 be as defined in the beginning of this section.

For each real number $a < \lambda_1$, consider the operator $B_a : L^2(\Omega) \to L^2(\Omega)$ defined by $B_a u = v$ where

$$\begin{cases} (L-a)v &= u \quad \text{in } \Omega \\ \qquad v &= 0 \quad \text{on } \partial\Omega \end{cases}.$$

The norm of B_a is $(\lambda_1 - a)^{-1}$ and $B_a\phi = (\lambda_1 - a)^{-1}\phi$, where ϕ is the first eigenfunction of $(L, H_0^1(\Omega))$.

The set

$$\mathcal{P} = \{I\} \cup \{B_a : a < \lambda_1\}$$

is a positive-numbers-like set of operators in $L^2(\Omega)$.

Now we prove a result on the positivity of an inverse matrix of positive-numbers-like operators. Consider the following matrix

$$\mathbf{P} = \begin{bmatrix} I & -P_{12} & -P_{13} & \dots & -P_{1n} \\ -P_{21} & I & -P_{23} & & -P_{2n} \\ -P_{31} & -P_{32} & I\dots & & -P_{3n} \\ \dots & \dots & \dots & & \dots \\ -P_{n1} & -P_{n2} & -P_{n3} & & I \end{bmatrix} \tag{2.8}$$

where P_{ij} are positive-numbers-like operators in a Banach space E with a cone K. We consider the equation

$$\mathbf{P}U = F \tag{2.9}$$

where $U = (u_1, \ldots, u_n)$ and $F = (f_1, \ldots, f_n)$ with $u_j, f_j \in E$.

We now state conditions on the operators P_{ij} which ensure that a solution $U \in K^n$ when $F \in K^n$. For that matter consider the following matrix of scalars associated with the matrix $\mathbf{P}$,

$$||\mathbf{P}|| = \begin{bmatrix} 1 & -||P_{12}|| & -||P_{13}|| & \ldots & -||P_{1n}|| \\ -||P_{21}|| & 1 & -||P_{23}|| & \ldots & -||P_{2n}|| \\ -||P_{31} & -||P_{32}|| & 1 & \ldots & -||P_{3n}|| \\ -||P_{n1}|| & -||P_{n2}||\ldots & -||P_{n3}|| & \ldots & 1 \end{bmatrix} \tag{2.10}$$

we have the following result.

THEOREM 2.2. *Let* $\mathbf{P}$ *be a matrix as in (2.8). Suppose that the* n *first principal minors of the matrix* $||\mathbf{P}||$ *defined in (2.10) have positive determinants.*

Then $U \in K^n$ if $F \in K^n$, for all solutions U of (2.9).

REMARK 2.7 By Lemma 3.1 it follows that all principal minors of $||\mathbf{P}||$ have positive determinants.

Proof. By induction. For $n = 2$ we have

$$u_1 - P_{12}u_2 = f_1$$

$$-P_{21}u_1 + u_2 = f_2\ .$$

Applying P_{21} to the first equation and adding the result to the second equation we obtain

$$(I - P_{12}P_{21})u_2 = P_{21}f_1 + f_2\ . \tag{2.11}$$

By hypothesis $||P_{12}||\ ||P_2|| < 1$, which implies, in view of Remark 2.3, that $(I - P_{12}P_{21})^{-1}$ is a positive operator.

So $u_2 \in K$ and $u_1 = f_1 + P_{12}u_2 \in K$.

Next, assume that the result is true for systems with $j \leq n-1$ equations; let us prove that it is true for $j = n$.

Applying $P_{i1}, 2 \leq i \leq n$, to the first equation and adding the i^{th} equation we obtain a system with $n-1$ equation, whose matrix is

$$\begin{bmatrix} I - P_{12}P_{21} & -P_{23} - P_{13}P_{21} \dots & -P_{2n} - P_{1n}P_{21} \\ -P_{32} - P_{12}P_{31} & I - P_{13}P_{31} \dots & -P_{3n} - P_{1n}P_{31} \\ \dots & \dots & \\ -P_{n2} - P_{12}P_{n1} & -P_{n3} - P_{13}P_{n1} \dots & I - P_{1n}P_{n1} \end{bmatrix}$$

and the right hand side is in K^n.

$$(f_2 + p_{12}f_1, \dots, f_n + p_{1n}f_1) \in K^n .$$

It follows from the hypothesis through Remark 2.3, that

$$Q_{ij} = (I - P_{1j}P_{j1})^{-1}, \quad j = 2, \dots, n$$

are positive operators.

So we are reduced to a system with $n-1$ equations

$$\mathbf{P}_1 \begin{bmatrix} u_2 \\ \\ \\ u_n \end{bmatrix} = \begin{bmatrix} g_2 \\ \cdot \\ \cdot \\ \cdot \\ g_n \end{bmatrix} \tag{2.12}$$

where

$$g_j = P_{j1}(f_j + P_{1j}f_1)$$

and the matrix $\mathbf{P}_1$ is given by

$$\begin{bmatrix} I & -Q_{12}(P_{23}+P_{13}P_{21})\dots & -Q_{12}(P_{2n}+P_{1n}P_{21}) \\ -Q_{13}(P_{32}+P_{12}P_{31}) & I & -Q_{13}(P_{3n}+P_{1n}P_{31}) \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ -Q_{1n}(P_{n2}+P_{12}P_{n1}) & Q_{1n}(P_{n3}+P_{13}P_{n1})\dots & I \end{bmatrix}$$

This matrix is like matrix $\mathbf{P}$. So, by the induction hypothesis, the positiveness of $u_2, \dots, u_n$ will follow, if one proves that the principal minors of the matrix $||\mathbf{P}_1||$, defined analogously to $||\mathbf{P}||$, have positive determinants. But this is the same as to show that this is true for the matrix below

$$\mathbf{C} = \begin{bmatrix} 1 - p_{12}p_{21} & -p_{23} - p_{13}p_{21} \cdots & -p_{2n} - p_{1n}p_{21} \\ -p_{32} - p_{12}p_{31} & 1 - p_{13}p_{31} \cdots & -p_{3n} - p_{1n}p_{31} \\ \cdots & \cdots & \cdots \\ -p_{n2} - p_{12}p_{n1} & -p_{n3} - p_{13}p_{n1} & 1 - p_{1n}p_{n1} \end{bmatrix}$$

where $p_{ij} = ||P_{ij}||$.

However, this is equivalent to show that the n first principal minors of the matrix

$$\mathbf{C}_1 = \begin{bmatrix} 1 & -p_{12} & \;\; -p_{-1} \\ 0 & 1 \ldots\ldots\ldots\ldots & \\ 0 & & \\ \cdot & & \\ & \mathbf{C} & \\ 0 & & \\ 0 & & \\ 0 & 1 & \end{bmatrix}$$

are positive.

Observe the matrix $\mathbf{C}_1$ is obtained from the matrix $||\mathbf{P}||$ by multiplying the first line by p_{i1} and adding the i^{th} line to it.

This procedure does not change the determinants of the first n principal minors of the matrix $\mathbf{C}_1$.

The proof is complete.

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Proof of Theorem 2.1.

Necessity.

Multiplying each equation in system (2.1) by ϕ, the first positive eigenfunction of $(L, H_0^1(\Omega))$, and integrating by parts we obtain the following linear algebraic system

$$(\lambda_1 I - A)X = Y \qquad (2.13)$$

where $X = (x_1, \ldots, x_n)$, $x_i = \int_\Omega u_i\phi_i$ and $Y = (y_1, \ldots, y_n)$, $y_i = \int_\Omega f_i\phi$.

The matrix $\lambda_1 I - A$ is an M-matrix. So, an application of Lemma (2.1) concludes the proof.

Sufficiency.

System (2.1) can be written as

$$u_1 - a_{12}B_1u_2 \ldots - a_{1n}B_1u_n = B_1f_1$$

$$-a_{21}B_2u_1 + u_2 \ldots - a2_nB_2u_n = B_2f_2$$

$$-a_{n1}B_nu_1 - a_{n2}B_nu_2 \ldots + u_n = B_nf_n$$

where $B_i = B_{a_{ii}}$ (see the definition of B_a in example 2.1). Recall that the hypothesis of the theorem implies $a_{ii} < \lambda_i \forall i = 1, \ldots, n$.

The matrix of the above system is like **P** in Theorem 2.2. So to use Theorem 2.2 in the present situation it suffices to verify that the matrix of scalars

$$\begin{bmatrix} 1 & -a_{12}(\lambda_1 - a_{11})^{-1} \ldots & -a_{1n}(\lambda_1 - a_n)^{-1} \\ -a_{21}(\lambda_1 - a_{22})^{-1} & 1 & -a_{2n}(\lambda_1 - a_{22})^{-1} \\ -a_{n1}(\lambda_1 - a_{nn})^{-1} & -a_{n2}(\lambda_1 - a_{nn})^{-1} & 1 \end{bmatrix}$$

has its n first principal minors with positive determinants. But by hypothesis this is the case since these minors are

$$1, (\lambda_1 - a_{11})^{-1}(\lambda_1 - a_{22})^{-1}p_2(\lambda), \ldots, (\lambda_1 - a_{11})^{-1} \ldots (\lambda_1 - a_{nn})^{-1}p_n(\lambda).$$

This concludes the proof.

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REMARK 2.8. A useful consequence of Theorem 2.1 is that it furnishes a sufficient condition for the validity of condition (ψ) in the case of systems with non-constant coefficients, see Remark 1.3.

We conclude this Section by proving a positivity result connected with the classical eigenvalue problem for systems. See [7] for general results in this direction.

We assume that $\mathcal{L}$ is as in (2.1) and A is a constant $n \times n$ cooperative matrix.

THEOREM 2.3. *Let $p_1(\lambda), \ldots, p_n(\lambda)$ be the characteristic polynomials of the first n principal minors of A.*

Let

$$v^* := \max\{v : p_j(v) = 0, \quad j = 1, \ldots, n\},$$

and assume that $a_{kk} > 0$ for at least one $k (1 \leq k \leq n)$.

Then $v^* > 0$ and $\mu_1(A) := \frac{\lambda_1}{v^*}$ is an eigenvalue of

$$(2.14) \qquad \begin{cases} \mathcal{L}\emptyset & = \mu A \emptyset \quad \text{in } \Omega \\ \emptyset & = 0 \quad \text{on } \partial\Omega \end{cases}$$

with non negative eigenfunction $\overline{\emptyset}$.

Moreover, if $\mu < \mu_1(A)$, then $F \geq 0$ in Ω implies $U \geq 0$ in Ω, being U a solution of

$$(2.15) \qquad \begin{cases} \mathcal{L}U & = \mu A U + F \quad \text{in } \Omega \\ U & = 0 \quad \text{on } \partial\Omega \end{cases}$$

Proof. Let us denote by $p_{1,\mu}, \ldots, p_{n,\mu}$ the characteristic polynomials of the first n principal minors of μA.

Clearly we have

$$p_{j,\mu}(\lambda) = \mu^j p_j(\frac{\lambda}{\mu}) \quad j = 1, \ldots, n.$$

Now we apply Theorem 2.1 to the equation (2.15). If $\mu < \frac{\lambda_1}{v^*}$ then $p_{j,\mu}(\lambda) > 0$ for all $j = 1, \ldots, n$.

So Theorem 2.1 implies that a maximum principle holds for (2.15). Next suppose that $\mu = \frac{\lambda_1}{v^*}$. Then for some $j, p_{j,\mu}(\lambda_1) = 0$. This means that λ_1 is an eigenvalue of the matrix

$$\mu A_j = \mu \begin{pmatrix} a_{11} & \ldots & a_{1j} \\ a_{j1} & \ldots & a_{j1} \end{pmatrix}$$

Let $e \in \mathbf{R}^J$ be a corresponding eigenvector

$$\mu A_j e = \lambda_1 e.$$

We claim that the coordinates $e_i (i = 1, \ldots, j)$ of e are all positive. Indeed this follows from Perron-Frobenius theorem applied to the matrix $\mu(A_j + mI)$ where m is a positive constant such that $m > |a_{ii}|$, $i = 1, \ldots, J$. To complete the proof take $\overline{\emptyset} = (\phi_1 e_1, \ldots, \phi_1 e_j, \ldots, 0 \ldots 0)$. It is easy to see that $\mathcal{L}\overline{\emptyset} = \mu A \overline{\emptyset}$.

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3. Embedding of a 2×2 noncooperative system into a 3×3 cooperative system.

In this section, we shall use the results of the two previous sections to obtain maximum principles for a class of noncooperative systems with variable coefficients, as well as stronger results for such systems in the case of constant coefficients. Since we restrict ourselves to a 2×2 system with the same differential operator in the left side, we use the simpler notation below

$$Lu = au + bv + f \tag{3.1a}$$

$$Lv = cu + dv + g\,, \tag{3.1b}$$

where a, b, c, d, f and g are given functions defined in Ω.

Let us start with the simpler case when $b \leq 0$ and $c \leq 0$ in Ω. Although the system is noncooperative in this case, it can be changed into a cooperative system for the unknowns u and $-v$. Namely

$$Lu = au - b(-v) + f$$

$$L(-v) = (-c)u + d(-v) - g\,.$$

So using Corollary 1.1 we obtain the following result.

THEOREM 3.1. *Suppose that L is self-adjoint and λ_1 is the first eigenvalue of $(L(D), H_0^1(\Omega))$. Assume also that*

(i) $b(x), c(x) \leq 0$ in Ω

(ii) $\max\{\sup_{x\in\Omega}[a(x) - b(x)]; \sup_{x\in\Omega}[-c(x) + d(x)]\} < \lambda_1$.

Then, if $f \geq 0$ and $g \leq 0$ in Ω, it follows that $u \geq 0$, $v \leq 0$ in Ω, being (u, v) the solution to (3.1).

Next we consider the more challenging case which corresponds to $b(x) \leq 0$, $b(x) \not\equiv 0$, $c(x) \geq 0$ in Ω. Our main purpose is to find out conditons on the coefficients of the system (3.1) which will give us a maximum principle. The idea now is first to embed the given 2×2 (non-cooperative) system into a 3×3 (cooperative) system. Then we apply the results already established for cooperative systems in order to obtain information about the given system. We introduce a new unknown $w = \varepsilon u + \delta v$, where $\varepsilon \not\equiv 0$, $\delta \not\equiv 0$ are real parameters to be determined as we proceed. So we have an additional equation

$$(3.1c) \qquad -\Delta w = (a\varepsilon + c\delta) u + (b\varepsilon + d\delta) v + \varepsilon f + \delta g \,.$$

Since $-\varepsilon u - \delta v + w = 0$ we may rewrite system (3.1) as follows

$$\begin{aligned} -\Delta u &= (a - r\varepsilon) u + (b - r\delta) v + rw + f \\ -\Delta v &= cu + dv + g \\ -\Delta w &= (a\varepsilon + c\delta - s\varepsilon) u + (b\varepsilon + d\delta - s\delta) v + sw + \varepsilon f + \delta g \,, \end{aligned}$$

where the real-valued functions $r(x)$ and $s(x)$ are to be determined later. In order to get cooperativeness, we observe initially that in the first equation we should have $r \geq 0$ and $b - r\delta \geq 0$ in view of the hypothesis $b \leq 0$ and $h \not\equiv 0$, we infer that $\delta < 0$ and $r \geq 0$, $r \not\equiv 0$. On the other hand, aiming at a maximum principle we should have $\varepsilon f + \delta g \geq 0$, which implies that $\varepsilon > 0$. We may without loss of generality take $\varepsilon = 1$. So we obtain the following system:

$$\begin{aligned} &(3.2a) \quad -\Delta u = (a - r) u + (b - r\delta) v + rw + f \\ &(3.2b) \quad -\Delta v = cu + dv + g \\ &(3.2c) \quad -\Delta w = (a - s + c\delta) u + (b + d\delta - s\delta) v + sw + f + \delta g \end{aligned}$$

which will be cooperative if

(3.3)

$$r(x) \geq 0; \quad b(x) - r(x)\delta \geq 0; \quad \frac{b(x) + d(x)\delta}{\delta} \leq s(x) \leq a(x) + c(x)\delta$$

for all x in Ω. These conditions can be satisfied by a proper choice of δ $r(x)$ and $s(x)$ provided that there is a $\delta < 0$ such that

$$c(x)\delta^2 + (a(x) - d(x))\delta - b(x) \leq 0, \quad \forall x \in \Omega . \tag{3.4}$$

We summarize the foregoing arguments in the

PROPOSITION 3.1. *The noncooperative system* (3.1) $[b(x) \leq 0,$ $b(x) \not\equiv 0,\ c(x) \geq 0]$ *can be embedded into a cooperative system (3.2) if there exists a* $\delta < 0$ *such that* (3.4) *holds.*

REMARK 3.1. Embedding in the previous proposition means that: if (u, v, w) is a solution to (3.2), then (u, v) is a solution of (3.1), and moreover any solution of (3.1) is obtained in this way.

REMARK 3.2. The existence of δ as required in the above proposition is equivalent to the following conditions on the entries of the matrix A:

$$[a(x) - d(x)]^2 + 4b(x)c(x) \geq 0, \quad \forall x \in \Omega . \tag{3.5}$$

$$d(x) < a(x), \quad \forall x \in \Omega, \tag{3.6}$$

$$\sup_{x\in\Omega} \delta_-(x) \leq \inf_{x\in\Omega} \delta_+(x) , \tag{3.7}$$

where $\delta_-(x)$ and $\delta_+(x)$ are the two roots of the quadratic (3.4). By (3.5) and (3.6) these roots are < 0.

REMARK 3.3. If the entries of the matrix A are constants, then the existence of δ as stated above is equivalent to $(a - d)^2 + 4bc \geq 0$ (which is also equivalent to saying that the matrix A has real eigenvalues) and $d < a + (-bc)^{1/2}$.

Now we obtain a maximum principle for system (3.1) by applying Corollary 1.1 to system (3.2). We should then require that δ, r and s be chosen to satisfy the additional conditions

$$a(x) + b(x) - r(x)\delta < \lambda_1, \quad c(x) + d(x) < \lambda_1 \tag{3.8}$$

$$a(x) + c(x)\delta + b(x) + d(x)\delta - s(x)\delta < \lambda_1 , \tag{3.9}$$

for all x in Ω. The less restrictive choice of s in (3.9) is obtained form (3.3), namely $s(x) = [b(x) + d(x)\delta]/\delta$. And we get from (3.9)

$$(3.10) \qquad a(x) + c(x)\delta < \lambda_1 .$$

In this way we see that the best choice of δ to attend (3.10) is

$$\delta_- = \sup_{x+\Omega} \delta_-(x)$$

where $\delta_-(x)$ is defined right after (3.7). Next we choose $r(x) = \frac{b(x)}{\delta_-}$ and the first requirement in (3.8) is attained if $a(x) < \lambda_1$, for all x in Ω. Summarizing, we have the following maximum principle

THEOREM 3.2. *Consider system* (3.1) *with* $b(x) \leq 0, b(x) \not\equiv 0$ *and* $c(x) \geq 0$ *in* Ω. *Assume condition* (3.4) *of Proposition 3.1. Suppose in addition that*

$$(3.11) \qquad \begin{array}{l} a(x) < \lambda_1; \\ c(x) + d(x) < \lambda_1; \quad a(x) + c(x)\delta_- < \lambda - 1, \quad \forall x \in \Omega . \end{array}$$

Then, if $f(x) \geq 0, g(x) \geq 0$ *and* $f(x) + \delta_- g(x) \geq 0$ *in* Ω, *it follows* $u \geq 0, v \geq 0$ *in* Ω.

For recent results on the maximum principle for non cooperative systems with variable coefficients see [3].

THE MAXIMUM PRINCIPLE OF [5]. In [5] the authors of the present paper considered the following system in Ω

$$(3.12a) \qquad -\Delta u = \lambda u - v + f$$

$$(3.12b) \qquad -\Delta v = \delta u - \gamma v ,$$

subject to Dirichlet boundary conditions on $\partial\Omega$. Here δ, γ are positive constants. We showed that $f \geq 0$ in Ω implies $u, v \geq 0$ in Ω, provided

$$(3.13) \qquad \sqrt{\delta} < \gamma + \lambda_1 ,$$

$$-\gamma + 2\sqrt{\delta} \le \lambda < \lambda_1 + \frac{\delta}{\lambda_1 + \gamma} . \tag{3.14}$$

We remark that this result is not a consequence of Theorem 3.2. Indeed, although the condition (3.4) gives the first half of the inequality in (3.14), the conditions (3.11) in the notation of the system (3.12) are

$$\lambda < \lambda_1, \quad \delta - \gamma < \lambda_1 ,$$

where the first inequality is more restrictive than (3.14), and the second is not comparable with (3.13) for general values of δ.

However we can derive the maximum principle in [5] from our Theorem 2.1. In fact, in the case when the coefficients of (3.1) are constant, the parameters r and s in the system (3.2) can be chosen also as constants. So in this case, with the further assumption that $L = -\Delta$, a maximum principle for (3.2) holds if (1) the system is cooperative, i.e.

$$r > 0, \quad b - r\delta \ge 0 , \tag{3.16}$$

$$\frac{b + d\delta}{\delta} \le s \le a + c\delta, \quad \text{and} \tag{3.17}$$

(ii) the conditions of Theorem 2.1 are verified, namely

$$a - r < \lambda_1 . \tag{3.18}$$

$$\begin{vmatrix} \lambda_1 - a + r & -b + r\delta \\ -c & \lambda_1 - d \end{vmatrix} > 0 \tag{3.19}$$

$$\begin{vmatrix} \lambda_1 - a + r & -b + r\delta & -r \\ -c & \lambda_1 - d & 0 \\ -a + s - c\delta & -b - d\delta + s\delta & \lambda_{1-s} \end{vmatrix} > 0 \tag{3.20}$$

Now we proceed to choose the parameters δ, r and s in such a way that conditions (3.16)–(3.20) are satisfied. First choose $r = b/\delta$. Therefore condition (3.16) is satisfied, and (3.18) and (3.19) are verified if

$$a - \frac{b}{\delta} < \lambda_1, \quad d < \lambda_1 . \tag{3.21}$$

Also, condition (3.20) can be written as

$$\left(\lambda_1 - s + \frac{b}{\delta}\right)\left(\lambda_1 - a - \frac{bc}{\lambda_1 - d}\right) > 0 \, .$$

Now we choose $s < \lambda_1 + b/\delta$. [Such a choice can be done attending (3.17), because $(b + d\delta)/\delta < \lambda_1 + b/\delta$]. So the above inequality becomes

$$a < \lambda_1 - \frac{bc}{\lambda_1 - d} \, . \tag{3.22}$$

Finally it remains to choose δ. Take as δ the smallest root of

$$c\delta^2 + (a - d)\delta - b = 0 \, .$$

And then check that

$$\frac{-bc}{\lambda_1 - d} \leq \frac{b}{\delta}, \text{ provided } \sqrt{-bc} < \lambda_1 - d \, .$$

Summarizing, we conclude that the parameters δ, r and s can be chosen to attend conditions (3.16)-(3.20) if

$$d + 2\sqrt{-bc} \leq a \tag{3.23}$$

$$d < \lambda_1, \quad a < \lambda_1 - \frac{bc}{\lambda_1 - d} \tag{3.24}$$

$$\sqrt{-bc} < \lambda_1 - d \, . \tag{3.25}$$

THEOREM 3.3. *Consider system* (3.1) *with constant matrix* A *and* $L = -\Delta$. *Assume conditions* (3.23)-(3.25). *Then* $f \geq 0, g = 0$ *in* Ω *implies* $u, v \geq 0$ *in* Ω.

REMARK 3.4. The above theorem gives the maximum principle [5]. For others results connected with [5] see [3,4,10].

On the biharmonic equation. Let us consider the fourth order elliptic equation

$$(3.26) \qquad (\Delta + a(x))(\Delta + b(x))v = \mu v + f \text{ in } \Omega \,,$$

subject to the so-called Navier boundary conditions, namely

$$(3.27) \qquad v = 0, \quad \Delta v = 0 \quad \text{on } \partial\Omega \,.$$

We discuss now conditions on the functions a and b and the real parameter μ that will lead to a maximum principle for the problem above. For that matter we introduce a new dependent variable $u = -(\Delta + b)v$, and transform the above equation into the system below

$$-\Delta u = au + \mu v + f$$

$$-\Delta v = u + bv$$

subject to Dirichlet boundary conditions $u = v = 0$ on $\partial\Omega$. Then applying previous results we have the following statements.

A) If $\mu \geq 0$, the system is cooperative and ther is a maximum principle for (3.26)-(3.27) if $a(x) + \mu < \lambda_1$ and $1 + b(x) < \lambda_1$, for all x in Ω.

B) If $\mu \geq 0$ and a and b are constants, there is a maximum principle if

$$a < \lambda_1 \text{ and } (\lambda_1 - a)(\lambda_1 - b) > \mu \,.$$

C) If $\mu < 0$, the system above is noncooperative and we apply Theorem 3.2 to obtain a maximum principle for (3.26)-(3.27) provided

$$a(x) \geq b(x) + 2\sqrt{-\mu}, \; a(x) < \lambda_1 \text{ and } 1 + b(x) < \lambda_1 \text{ for all } x \text{ in } \Omega \,.$$

D) If $\mu < 0$ and the coefficients a and b are constant, we apply Theorem 3.3 and the conditions are

$$a \geq b + 2\sqrt{-\mu}, \; b < \lambda_1, \; a < \lambda_1 - \frac{\mu}{\lambda_1 - b}, \; \sqrt{-\mu} < \lambda_1 - b \,.$$

E) Suppose now that $a = b = 0$. So in (3.26) we have precisely the biharmonic operator Δ^2. Then from B) above we have a maximum principle provided

$$0 \leq \mu < \lambda_1^2 .$$

On the other hand if $\mu < 0$, D) above does not give any information about the biharmonic operator. Nevertheless we have some partial results if the coefficients are constant and $\partial\Omega$ has some regularity property. Let us consider the simplest case

$$\begin{cases} \Delta^2 v + \delta^2 v & = f & \text{in } \Omega \\ \qquad\quad v & = \Delta v = 0 & \text{on } \partial\Omega \end{cases} \tag{3.28}$$

where $\delta \in \mathbf{R}$ and f is a given positive function defined on Ω. Observe that the biharmonic operator subject to Navier boundary conditions has eigenvalues λ_K^2, $k = 1, 2, \ldots$, where λ_K are the eigenvalues of $(-\Delta, H_0^1(\Omega)$.

By putting $B = (-\Delta)^{-1}$ and supposing that $\delta^2 < \lambda_1$, it follows that (3.28) is equivalent to

$$v = (I + \delta^2 B^2)^{-1} B^2(f) . \tag{3.29}$$

Since $(I + \delta^2 B^2)^{-1} = T \cdot (I + \delta^2 B^2)$ where $T = (I - \delta^4 B^4)^{-1}$, it follows that, in order to prove that $v \geq 0$ in Ω, it is sufficient to prove that the operator $B^2 - \delta^2 B^4$ is positive, hence in particular that $B - \delta^2 B^3$ is positive.

PROPOSITION 3.2. *Let $\Omega \subset \mathbf{R}^N$ be a bounded $C^{1,1}$ domain. Then there exists a constant $K > 0$ such that*

(i) $K < \lambda_1^2$
(ii) if $\delta^2 \leq K$ then $f \geq 0 \Rightarrow v \geq 0$ in Ω
(iii) $K \leq 1 + \frac{\lambda_1 \lambda_2}{\lambda_1^2 + \lambda_2^2}$.

Proof. We shall denote by $G(x, y)$ the Green function of $-\Delta$ subject to Dirichlet boundary conditions, that is $G(\cdot, y)(y \in \Omega)$ is the solution to the problem

$$-\Delta_x G(x, y) = \delta_y(x) \quad x \in \Omega \tag{3.30}$$

$$G(x,y) = 0 \quad x \in \partial\Omega \; .$$

Let $\lambda \in \mathbf{R}^+$ and consider

$$g_\lambda(x) = (Bf)(x) - \lambda(B^3 f)(x) \quad x \in \Omega \; .$$

We see that the solution of (3.28) is $v = (TB)g_{\delta^2}$ where $T = (I - \delta^4 B^4)^{-1}$.

So in order to show that v is positive, it is sufficient to prove that there exists $\lambda \in \mathbf{R}^+$ such that $g_\lambda(x) \geq 0$ $(x \in \Omega)$ (this will imply that $g_{\delta^2}(x) \geq 0$ if $\delta^2 \leq \lambda$). By using the integral representation we see that

$$g_\lambda(x) = \int_\Omega G(x,y)f(y)dy - \lambda \int_{\Omega^3} G(x,b)G(a,b)G(a,y)f(y)\, da\, db\, dy \, .$$

Hence, in order to show that $g_\lambda \geq 0$ it is sufficient to prove that the function

$$H(x,y) = \left(\int_{\Omega^2} G(x,b)G(a,b)G(a,y)da\; db\right) (G(x,y))^{-1}$$

is uniformly bounded in Ω.

Indeed if the last statement is true, i.e.

$$H(x,y) \leq \overline{K} \quad \forall x,y \in \Omega \tag{3.31}$$

then if we take $\lambda \leq (\overline{K})^{-1}$ we are done. In fact if $\lambda \leq (\overline{K})^{-1}$ then

$$(\overline{K})^{-1} \int_{\Omega^2} G(x,b)G(a,b)G(a,y)\; da\; db \leq G(x,y)$$

which implies (since $f \geq 0$) that

$$\lambda \int_{\Omega^3} G(x,b)G(a,b)G(a,y)f(y)da\; db\; dy \leq \int_\Omega G(x,y)f(y)dy$$

i.e. $g_\lambda(x) > 0$ in Ω.

To prove that (3.31) holds we can use the results of Zhao [11] in a special situation.

Namely by [11] we know that if $\Omega \subset \mathbf{R}^N$ is $C^{1,1}$ domain then there exists $A > 0$ such that:

$$\forall x,y,z \in \Omega \quad \frac{G(x,z)G(y,z)}{G(x,y)} \leq A[\,|x-z|^{2-N} + |y-z|^{2-N}] \; ,$$

hence,

$$H(x,y) = \int_{\Omega^2} G(x,b)G(a,b)G(a,y)G(x,y)^{-1} da\, db =$$

$$= \int_{\Omega^2} G(x,b)G(y,b)G(x,y)^{-1} \cdot G(a,b) \cdot G(a,y) \cdot G(y,b)^{-1} da\, db \leq$$

$$\leq A^2 M(x,y)$$

where

$$M(x,y) = \int_{\Omega^2} (|x-b|^{2-N} + |y-b|^{2-N})(|a-b|^{2-N} + |a-y|^{2-N}) da\, db\,,$$

Since $M(\cdot,\cdot)$ is continuous we conclude that

$$\sup_{x,y\in\Omega} H(x,y) \leq A^2 \max_{x,y\in\overline{\Omega}} M(x,y) = K$$

This concludes the proof of (ii).

To prove (i) it is sufficient to restrict our attention to $\langle\phi_1\rangle$ (the eigenspace generate by ϕ_1). Indeed if $B - KB^3$ is a positive operator then $B\phi_1 - KB^3\phi_1 > 0$ i.e. $K < \lambda_1^2$. The same remark can be used to prove (iii), in fact (ii) is the necessary condition on $B - KB^3$ in order to be positive on $\langle\phi_1, \phi_2\rangle$.

REMARK 3.5. Proposition above generalizes results of Sweers [9] and Bonnet [1] which hold respectively if $\Omega \subset \mathbf{R}^N$ is ball and if Ω is of class C^2. We observe that in general we are not able to find the "best constant" K. Some results in this direction are contained in [2].

REMARK 3.6. It is possible to prove by using the results of Zhao [11] in the full generality that a maximum principle still holds for equation (3.26) in a bounded $C^{1,1}$ domain, without requiring that the coefficients satisfy condition C).

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