

## A Note on Gevrey Well-Posedness for the Operator $\partial_t^2 - a(t)\partial_x(b(t, x)\partial_x)$

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SUMMARY. - *We use a Littlewood-Paley decomposition to obtain a Gevrey-well-posedness result for a weakly hyperbolic equation in one space variable with coefficients depending also on  $x$ .*

### 1. Introduction

Let  $s \geq 1$  and denote by  $\gamma^{(s)}(\mathbb{R})$  the Gevrey class of index  $s$  i.e. the set of the  $C^\infty$  functions  $f$  on  $\mathbb{R}$  for which, for all compact  $K$ , there exist  $C_K$  and  $M_K > 0$  such that  $|f^{(n)}(x)| \leq C_K M_K^n (n!)^s$  for all  $n \in \mathbb{N}$  and for all  $x \in K$ . Let  $\tilde{\gamma}^{(s)}(\mathbb{R})$  be the subset of  $\gamma^{(s)}(\mathbb{R})$  given by a similar definition in which the constants  $C$  and  $M$  do not depend on  $K$ .

A well-known result due to Colombini, Jannelli and Spagnolo states that the Cauchy problem for the operator

$$\partial_t^2 - a(t)\partial_x^2 \tag{1}$$

is well-posed in  $\gamma^{(s)}(\mathbb{R})$  provided that

- $a(t) \geq 0$  for all  $t \in [0, T]$ ,
- $a \in C^{k, \alpha}([0, T])$ ,
- $s < 1 + (k + \alpha)/2$ .

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Moreover the limit value  $1 + (k + \alpha)/2$  cannot be improved (see [7]). As a corollary we have that if  $a \geq 0$  and  $a \in \mathcal{C}^\infty$  then the Cauchy problem for (1) is well-posed in every Gevrey class (but not in  $\mathcal{C}^\infty$ , as a counterexample in [9] shows).

For the operator

$$\partial_t^2 - \partial_x(a(t, x)\partial_x) \quad (2)$$

a result due to Nishitani establishes that the Cauchy problem is  $\gamma^{(s)}$ -well-posed under the hypotheses

- $a(t, x) \geq 0$  for all  $(t, x) \in [0, T] \times \mathbb{R}$ ,
- $a \in \mathcal{C}^{k, \alpha}([0, T], \tilde{\gamma}^{(s)}(\mathbb{R}))$ , i.e. for all  $j = 1, \dots, k$  there exist  $C_j, M_j > 0$  such that

$$|\partial_t^j \partial_x^n a(t, x)| \leq C_j M_j^n (n!)^s,$$

for all  $n$  and for all  $(t, x) \in [0, T] \times \mathbb{R}$ , and there exist  $C_k^*, M_k^* > 0$  such that

$$\frac{|\partial_t^k \partial_x^n a(t_1, x) - \partial_t^k \partial_x^n a(t_2, x)|}{|t_1 - t_2|^\alpha} \leq C_k^* (M_k^*)^n (n!)^s,$$

for all  $n$  and for all  $(t_1, x), (t_2, x) \in [0, T] \times \mathbb{R}$  with  $t_1 \neq t_2$ ,

- $s < \min\{1 + (k + \alpha)/2, 2\}$ ;

(see [11]). Recently the same author has considered the case of  $a \geq 0$  and  $a \in \mathcal{C}^\infty([0, T], \tilde{\gamma}^{(s)}(\mathbb{R}))$ , obtaining that the Cauchy problem for (2) is  $\gamma^{(s)}$ -well-posed for all  $s < 5/2$ . It is not known if in this case the limit value  $5/2$  is optimal (see [12]).

In this note we consider the operator

$$\partial_t^2 - a(t)\partial_x(b(t, x)\partial_x) \quad (3)$$

under the conditions:  $a(t) \geq 0$  for all  $t \in [0, T]$  and  $a \in \mathcal{C}^\infty([0, T])$ ,  $b(t, x) \geq \lambda_0 > 0$  for all  $(t, x) \in [0, T] \times \mathbb{R}$  and  $b \in \mathcal{C}^\infty([0, T], \tilde{\gamma}^{(s)}(\mathbb{R}))$ . We prove that the Cauchy problem for (3) is  $\gamma^{(\sigma)}$ -well-posed for all  $\sigma > s + 1$ .

This result will be a corollary of a more general one: if

- $a(t) \geq 0$  for all  $t \in [0, T]$ ,
- $a \in \mathcal{C}^{k,\alpha}([0, T])$ ,
- $b(t, x) \geq \lambda_0 > 0$  for all  $(t, x) \in [0, T] \times \mathbb{R}$ ,
- $b \in \mathcal{C}^{k,\alpha}([0, T], \tilde{\gamma}^{(s)}(\mathbb{R}))$ ,
- $1 \leq s < (k + \alpha)/2$ ,

then the Cauchy problem for (3) is  $\gamma^{(\sigma)}$ -well-posed for all  $s + 1 < \sigma < 1 + (k + \alpha)/2$ .

The interest of this theorem is essentially in its proof which is based on the so-called approximate energy technique, introduced in [6] and [7]. The use of the Fourier transform with respect to the variable  $x$  is replaced here by the Littlewood-Paley decomposition, as in [8] (see for a similar approach [5] and [10]), and this one is adapted to the Gevrey classes. A difficulty in the estimate of a commutator brings to the difference of 1 between the index of the Gevrey regularity of the coefficient  $b$  and the index of the Gevrey regularity of the solution.

After the preparation of this paper was completed we were informed that similar results, with different proofs, have been obtained by Ascanelli in [1].

## 2. Littlewood-Paley decomposition

Let  $\Phi \in \bigcap_{s>1} \gamma^{(s)}(\mathbb{R})$  such that  $\Phi$  is decreasing and

$$\Phi(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ 0 & \text{if } x \geq 1. \end{cases}$$

We set  $\Phi_\nu(\xi) = \Phi\left(\frac{|\xi| - \nu^\rho}{(\nu+1)^\rho - \nu^\rho}\right)$  where  $\nu$  is a positive integer and  $\rho > 1$ . We define  $\phi_0(\xi) = \Phi_1(\xi)$  and  $\phi_\nu(\xi) = \Phi_{\nu+1}(\xi) - \Phi_\nu(\xi)$  for  $\nu \geq 1$ . Consequently

$$\begin{aligned} \text{supp } \phi_0 &\subseteq \{\xi \in \mathbb{R} : |\xi| \leq 2^\rho\}, \\ \text{supp } \phi_\nu &\subseteq \{\xi \in \mathbb{R} : \nu^\rho \leq |\xi| \leq (\nu + 2)^\rho\} \quad \text{for } \nu \geq 1. \end{aligned}$$

Moreover

$$\sum_{\nu=0}^k \phi_{\nu}(\xi) = \Phi_{k+1}(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq (k+1)^{\rho} \\ 0 & \text{if } |\xi| \geq (k+2)^{\rho}, \end{cases}$$

so that  $\sum_{\nu=0}^{\infty} \phi_{\nu}(\xi) = 1$  and  $\phi_{\nu}(\xi)\phi_{\mu}(\xi) = 0$  if  $|\nu - \mu| \geq 2$ . Remark finally that

$$|\Phi_{\nu}^{(k)}(\xi)| = \left( \frac{1}{(\nu+1)^{\rho} - \nu^{\rho}} \right)^k |\Phi^{(k)}\left( \frac{|\xi| - \nu^{\rho}}{(\nu+1)^{\rho} - \nu^{\rho}} \right)|. \quad (4)$$

Let  $u \in \mathcal{S}'(\mathbb{R})$ . We set

$$\begin{aligned} u_{\nu}(x) = \phi_{\nu}(D_x)u(x) &= \frac{1}{(2\pi)^{1/2}} \int e^{ix\xi} \phi_{\nu}(\xi) \hat{u}(\xi) d\xi \\ &= \frac{1}{(2\pi)^{1/2}} \int \hat{\phi}_{\nu}(y) u(x-y) dy. \end{aligned}$$

The following results can be easily proved: let  $l \geq 0$ ,  $\sigma > 0$ ,  $\tau > 0$ ,

$$\mathcal{H}^l(\mathbb{R}) = \{u \in \mathcal{S}'(\mathbb{R}) : (1 + |\xi|^2)^{l/2} \hat{u}(\xi) \in \mathcal{L}^2(\mathbb{R})\}$$

and

$$\mathcal{H}_{\sigma,\tau}(\mathbb{R}) = \{u \in \mathcal{S}'(\mathbb{R}) : \exp(\tau(1 + |\xi|^2)^{\frac{1}{2\sigma}}) \hat{u}(\xi) \in \mathcal{L}^2(\mathbb{R})\};$$

- $u \in \mathcal{H}^l(\mathbb{R})$  if and only if there exists  $\{c_{\nu}\}_{\nu} \in l^2$  such that  $\|u_{\nu}\|_{\mathcal{L}^2} \leq c_{\nu}(\nu+1)^{-\rho l}$  for all  $\nu \in \mathbb{N}$  (see [2]);
- $u \in \mathcal{H}_{\sigma,\tau}(\mathbb{R})$  for some  $\tau > 0$  if and only if there exist  $\{c_{\nu}\}_{\nu} \in l^2$  and  $\tau' > 0$  such that  $\|u_{\nu}\|_{\mathcal{L}^2} \leq c_{\nu} \exp(-\tau'(\nu+1)^{\rho/\sigma})$  for all  $\nu \in \mathbb{N}$  (see [13], [3], [4]).

Let us finally remark that if  $w \in \mathcal{L}^2(\mathbb{R})$  then

$$\nu^{\rho} \|w_{\nu}\|_{\mathcal{L}^2} \leq \|\partial_x w_{\nu}\|_{\mathcal{L}^2} \leq (\nu+2)^{\rho} \|w_{\nu}\|_{\mathcal{L}^2}. \quad (5)$$

### 3. Energy estimate

Let  $k \geq 2$ ,  $\alpha \in [0, 1]$ ,  $a \in \mathcal{C}^{k,\alpha}[0, T]$  and  $a \geq 0$ . Let  $2 \leq s + 1 < \sigma < 1 + (k + \alpha)/2$ ,  $b \in \mathcal{C}^{k,\alpha}([0, T], \tilde{\gamma}^{(s)}(\mathbb{R}))$  and  $b \geq \lambda_0 > 0$ . Suppose that  $u \in \mathcal{C}^2([0, T], \mathcal{H}_{\sigma,\tau}(\mathbb{R}))$ , where  $\tau > 0$ . Let

$$\partial_t^2 u(t, x) - a(t) \partial_x (b(t, x) \partial_x u(t, x)) = Lu(t, x). \quad (6)$$

Applying to both side of (6) the operator  $\phi_\nu(D_x)$  we obtain

$$\partial_t^2 u_\nu = a(t) \partial_x (b(t, x) \partial_x u_\nu) + (Lu)_\nu + a \partial_x ([\phi_\nu, b] \partial_x u),$$

where  $[\phi_\nu, b] \partial_x u$  denotes the commutator

$$\phi_\nu(D_x)(b(t, x) \partial_x u(t, x)) - b(t, x)(\phi_\nu(D_x) \partial_x u(t, x)).$$

Let  $\varepsilon_\nu \in (0, 1]$ ; we define

$$\begin{aligned} E_{\nu,\varepsilon_\nu} u(t) &= \|\partial_t u_\nu(t, \cdot)\|_{\mathcal{L}^2}^2 + \|u_\nu(t, \cdot)\|_{\mathcal{L}^2}^2 \\ &\quad + (a(t) + \varepsilon_\nu) \langle b(t, \cdot) \partial_x u_\nu, \partial_x u_\nu \rangle_{\mathcal{L}^2, \mathcal{L}^2}. \end{aligned}$$

Differentiating  $E_{\nu,\varepsilon_\nu} u(t)$  with respect to the variable  $t$  and using (5) and (6) we have that

$$\begin{aligned} E'_{\nu,\varepsilon_\nu} u(t) &\leq (C(\varepsilon_\nu^{1/2}(\nu + 1)^\rho + 1) + \frac{|a'(t)|}{a(t) + \varepsilon_\nu}) E_{\nu,\varepsilon_\nu} u(t) \\ &\quad + 2a(t) |\langle [\phi_\nu, b] \partial_x u, \partial_x \partial_t u_\nu \rangle| + 2 |\langle (Lu)_\nu, \partial_t u_\nu \rangle|, \end{aligned}$$

where  $C$  depends only on  $\|b\|_\infty$  and  $\|\partial_t b\|_\infty$ . Since  $a \in \mathcal{C}^{k,\alpha}([0, T])$  and  $a \geq 0$ , from [7, Lemma 1], we obtain that there exists  $C'$  depending only on  $a$  such that, for all  $t \in [0, T]$ ,

$$\int_0^t \frac{|a'(s)|}{a(s) + \varepsilon_\nu} ds \leq C' \varepsilon_\nu^{-1/(k+\alpha)} \leq C' \varepsilon_\nu^{-1/(2\sigma-2)}.$$

We choose  $\rho = \sigma$ ,  $\varepsilon_\nu = (\nu + 1)^{-(2\sigma-2)}$  and we set

$$h_\nu(t) = \int_0^t C(\varepsilon_\nu^{1/2}(\nu + 1)^\rho + 1) + \frac{|a'(s)|}{a(s) + \varepsilon_\nu} ds.$$

We obtain, for all  $t \in [0, T]$  and for all  $\nu$ ,

$$0 \leq h_\nu(t) \leq C_1(\nu + 1), \quad (7)$$

where  $C_1$  depends only on  $a$  and  $b$ . Moreover a simple computation gives, for all  $t \in [0, T]$  and for all  $\nu$ ,

$$0 \leq h_{\nu+1}(t) - h_\nu(t) \leq C_2, \quad (8)$$

where again  $C_2$  does not depend on  $\nu$ .

We define

$$\mathcal{E}(t) = \sum_{\nu=0}^{\infty} e^{-h_\nu(t) - (2\beta t - \tau')(\nu+1)} E_{\nu, \varepsilon_\nu} u(t),$$

where the constant  $\beta$  will be fixed later on and  $\tau' > 0$  is such that there exists  $C' > 0$ , not depending on  $u$ , such that

$$\mathcal{E}(0) \leq C' (\|u(0, \cdot)\|_{H_{\sigma, \tau}}^2 + \|\partial_t u(0, \cdot)\|_{H_{\sigma, \tau}}^2).$$

Differentiating  $\mathcal{E}(t)$  we obtain

$$\begin{aligned} \mathcal{E}'(t) = & \sum_{\nu=0}^{\infty} \left[ (-C(\varepsilon_\nu^{1/2}(\nu+1)^\rho + 1) - \frac{|a'(t)|}{a(t) + \varepsilon_\nu} - 2\beta(\nu+1)) \right. \\ & \left. \cdot e^{-h_\nu(t) - (2\beta t - \tau')(\nu+1)} E_{\nu, \varepsilon_\nu} u(t) \right] \\ & + \sum_{\nu=0}^{\infty} e^{-h_\nu(t) - (2\beta t - \tau')(\nu+1)} E'_{\nu, \varepsilon_\nu} u(t), \end{aligned}$$

so that

$$\begin{aligned} \mathcal{E}'(t) \leq & \sum_{\nu=0}^{\infty} -2\beta(\nu+1) e^{-h_\nu(t) - (2\beta t - \tau')(\nu+1)} E_{\nu, \varepsilon_\nu} u(t) \\ & + \sum_{\nu=0}^{\infty} e^{-h_\nu(t) - (2\beta t - \tau')(\nu+1)} \left[ 2a(t) |\langle [\phi_\nu, b] \partial_x u, \partial_x \partial_t u_\nu \rangle| \right. \\ & \left. + 2 |\langle (Lu)_\nu, \partial_t u_\nu \rangle| \right]. \quad (9) \end{aligned}$$

#### 4. Estimate of the commutator

We set  $\psi_0 = \phi_0 + \phi_1$  and, for  $\nu \geq 1$ ,  $\psi_\nu = \phi_{\nu-1} + \phi_\nu + \phi_{\nu+1}$ . We have

$$\partial_x u = \sum_{\mu} \psi_{\mu}(D_x) \partial_x u_{\mu}$$

and consequently

$$[\phi_{\nu}, b] \partial_x u = \sum_{\mu} ([\phi_{\nu}, b] \psi_{\mu}) \partial_x u_{\mu}.$$

Using (5) we have that there exists  $C_0 > 0$  depending only on  $a$  and  $b$  such that

$$\begin{aligned} & \sum_{\nu=0}^{\infty} e^{-h_{\nu}(t)-(2\beta t-\tau')(\nu+1)} 2a(t) |\langle [\phi_{\nu}, b] \partial_x u, \partial_x \partial_t u_{\nu} \rangle| \\ & \leq C_0 \sum_{\nu, \mu}^{\infty} k_{\nu, \mu} \left[ e^{-\frac{h_{\mu}(t)}{2}-(\beta t-\frac{\tau'}{2})(\mu+1)} (\mu+1)^{\frac{1}{2}} (\lambda_0 a(t))^{\frac{1}{2}} \|\partial_x u_{\mu}\| \right. \\ & \quad \left. \cdot e^{-\frac{h_{\nu}(t)}{2}-(\beta t-\frac{\tau'}{2})(\nu+1)} (\nu+1)^{\frac{1}{2}} \|\partial_t u_{\nu}\| \right], \end{aligned} \quad (10)$$

where

$$k_{\nu, \mu} = e^{-\frac{h_{\nu}(t)-h_{\mu}(t)}{2}-(\beta t-\frac{\tau'}{2})(\nu-\mu)} (\mu+1)^{-\frac{1}{2}} (\nu+1)^{\rho-\frac{1}{2}} \|[\phi_{\nu}, b] \psi_{\mu}\|$$

and  $\|[\phi_{\nu}, b] \psi_{\mu}\|$  denotes the norm of  $[\phi_{\nu}, b] \psi_{\mu}$  as an operator from  $\mathcal{L}^2(\mathbb{R})$  to  $\mathcal{L}^2(\mathbb{R})$ . To evaluate this norm we use the following two lemmas.

LEMMA 4.1. *If  $|\mu - \nu| \leq 2$  then there exists  $C > 0$  such that*

$$\|[\phi_{\nu}, b] \psi_{\mu}\| \leq C(\nu+1)^{1-\rho}. \quad (11)$$

*Proof.* It is sufficient to evaluate the norm of  $[\phi_{\nu}, b]$  as an operator from  $\mathcal{L}^2(\mathbb{R})$  to  $\mathcal{L}^2(\mathbb{R})$ . We have

$$(\phi_{\nu}(D_x)(bu) - b\phi_{\nu}(D_x)(u))(t, x) = \int \hat{\phi}_{\nu}(x-y)(b(t, y) - b(t, x))u(y) dy.$$

Since  $|b(t, x) - b(t, y)| \leq C|x - y|$ , we deduce that  $\|[\phi_\nu, b]\|$  will be estimated by a constant times the quantity  $\sup_x \int |\hat{\phi}_\nu(x - y)||x - y| dy$ . Using the fact that

$$\phi_\nu(\xi) = \Phi\left(\frac{|\xi| - (\nu + 1)^\rho}{(\nu + 2)^\rho - (\nu + 1)^\rho}\right) - \Phi\left(\frac{|\xi| - \nu^\rho}{(\nu + 1)^\rho - \nu^\rho}\right)$$

it is easy to reach the conclusion.  $\square$

LEMMA 4.2. *If  $|\mu - \nu| \geq 3$  then for all  $s' > s$  there exist  $C, \delta > 0$  such that*

$$\|[\phi_\nu, b]\psi_\mu\| \leq C \exp(-\delta(\max\{\nu, \mu\})^{(\rho-1)/s'}). \quad (12)$$

*Proof.* We follow the proof of [8, Prop. 4.5]. We have

$$\begin{aligned} [\phi_\nu, b]w(x) &= \int \hat{\phi}_\nu(x - y)(b(t, y) - b(t, x))w(y) dy \\ &= \int (k_1(x, y) + k_2(x, y))u(y) dy, \end{aligned}$$

where

$$\begin{aligned} k_1(x, y) &= \hat{\phi}_\nu(x - y)(\partial_x b(t, x)(y - x) + \cdots + \frac{\partial_x^N b(t, x)}{N!}(y - x)^N), \\ k_2(x, y) &= \hat{\phi}_\nu(x - y) \frac{\partial_x^{N+1} b(t, \theta)}{(N + 1)!} (y - x)^{N+1} \end{aligned}$$

with  $\theta$  between  $x$  and  $y$ . The functions  $k_1$  and  $k_2$  are the kernels of the operators  $K_1$  and  $K_2$  respectively, so that  $[\phi_\nu, b] = K_1 + K_2$ .

Since  $|\nu - \mu| \geq 3$  and consequently  $\phi_\nu(\xi)\psi_\mu(\xi) = 0$  for all  $\xi \in \mathbb{R}$  then  $K_1\psi_\mu = 0$ . We evaluate  $\|K_2\|$ . We have to estimate  $\sup_x \int |k_2(x, y)| dy$  and  $\sup_y \int |k_2(x, y)| dx$ . From (4) and the Gevrey regularity of  $\Phi$  we have that

$$\|\phi_\nu^{(N)}\|_\infty \leq CM^N \nu^{(1-\rho)N} (N!)^{1+s'-s},$$

and consequently

$$\int |\phi_\nu^{(N)}(\xi)| d\xi \leq C' M^N \nu^{(1-\rho)(N-1)} (N!)^{1+s'-s}.$$



We remark now that

$$\begin{aligned} |z|^{N+3}|\hat{\phi}_\nu(z)| &\leq C|(\phi_\nu^{(N+3)})^\wedge(z)| \leq C' \int |\phi_\nu^{(N+3)}(\xi)| d\xi \\ &\leq C'' M^{N+3} \nu^{(1-\rho)(N+2)} ((N+3)!)^{1+s'-s}, \end{aligned}$$

and then

$$\int |z|^{N+1}|\hat{\phi}_\nu(z)| dz \leq CM^{N+3} \nu^{(1-\rho)(N+2)} ((N+3)!)^{1+s'-s}.$$

On the other hand

$$\left| \frac{\partial_x^{N+1} b(t, \theta)}{(N+1)!} \right| \leq CM^{N+1} ((N+1)!)^{s-1}.$$

Putting together we obtain

$$\begin{aligned} \max\left\{ \sup_x \int |k_2(x, y)| dy, \sup_y \int |k_2(x, y)| dx \right\} \\ \leq CM^{N+1} \nu^{(1-\rho)(N+1)} ((N+1)!)^{s'}. \end{aligned}$$

Since  $[\phi_\nu, b]\psi_\mu = \phi_\nu[b, \psi_\mu]$  we deduce that there exist  $C$  and  $M > 0$  such that for all  $N \in \mathbb{N}$

$$\|[\phi_\nu, b]\psi_\mu\| \leq CM^{N+1} (\max\{\nu, \mu\})^{(1-\rho)(N+1)} ((N+1)!)^{s'},$$

and (12) follows. □

Our goal is now to evaluate the quantity

$$\sup_\nu \sum_\mu k_{\nu, \mu} + \sup_\mu \sum_\nu k_{\nu, \mu}.$$

We set  $k_{\nu, \mu} = k'_{\nu, \mu} + k''_{\nu, \mu}$  where  $k'_{\nu, \mu} = k_{\nu, \mu}$  if  $|\nu - \mu| \leq 2$  and  $k'_{\nu, \mu} = 0$  otherwise, while  $k''_{\nu, \mu} = k_{\nu, \mu}$  if  $|\nu - \mu| \geq 3$  and  $k''_{\nu, \mu} = 0$  otherwise. We remark now that, for  $\nu \geq 2$ ,

$$\sum_\mu k'_{\nu, \mu} = k_{\nu, \nu-2} + k_{\nu, \nu-1} + k_{\nu, \nu} + k_{\nu, \nu+1} + k_{\nu, \nu+2},$$

and using (8) and (11) we have, for instance,

$$\begin{aligned} k_{\nu, \nu-1} &= e^{-\frac{h_{\nu}(t)-h_{\nu-1}(t)}{2}-(\beta t-\frac{\tau'}{2})} \nu^{-\frac{1}{2}} (\nu+1)^{\rho-\frac{1}{2}} \|\phi_{\nu}, b\| \psi_{\mu} \| \\ &\leq e^{C_2+\tau'+2\beta t} \nu^{-\frac{1}{2}} (\nu+1)^{\rho-\frac{1}{2}} C (\nu+1)^{1-\rho} \\ &\leq K e^{2\beta t}, \end{aligned}$$

where  $K$  does not depend on  $\beta$ . Computing the other terms in a similar way we obtain

$$\sup_{\nu} \sum_{\mu} k'_{\nu, \mu} + \sup_{\mu} \sum_{\nu} k'_{\nu, \mu} \leq K' e^{2\beta t},$$

where  $K'$  does not depend on  $\beta$ . Let us consider  $\sum_{\mu} k''_{\nu, \mu}$ . Using (7) and (12), in which we choose  $s < s' < \sigma - 1$  and consequently  $(\rho - 1)/s' > 1$ , we have

$$\begin{aligned} &\sum_{\mu} k''_{\nu, \mu} \\ &= \sum_{\mu, |\nu-\mu| \geq 3} e^{-\frac{h_{\nu}(t)-h_{\mu}(t)}{2}-(\beta t-\frac{\tau'}{2})(\nu-\mu)} (\mu+1)^{-\frac{1}{2}} (\nu+1)^{\rho-\frac{1}{2}} \|\phi_{\nu}, b\| \psi_{\mu} \| \\ &\leq \sum_{\mu, |\nu-\mu| \geq 3} e^{\frac{C_1}{2}(\nu+\mu)-(\beta t-\frac{\tau'}{2})(\nu-\mu)} (\mu+1)^{-\frac{1}{2}} (\nu+1)^{\rho-\frac{1}{2}} \|\phi_{\nu}, b\| \psi_{\mu} \| \\ &\leq e^{\frac{C_1+\tau'-2\beta t}{2}\nu} (\nu+1)^{\rho-\frac{1}{2}} C e^{-\delta\nu^{(\rho-1)/s'}} \sum_{\mu=0}^{\nu-3} e^{\frac{C_1-\tau'+2\beta t}{2}\mu} (\mu+1)^{-\frac{1}{2}} \\ &\quad + e^{\frac{C_1+\tau'-2\beta t}{2}\nu} (\nu+1)^{\rho-\frac{1}{2}} \sum_{\mu=\nu+3}^{\infty} e^{\frac{C_1-\tau'+2\beta t}{2}\mu} (\mu+1)^{-\frac{1}{2}} C e^{-\delta\mu^{(\rho-1)/s'}} \\ &\leq C (\nu+1)^{\rho-\frac{1}{2}} e^{(C_1+\tau')\nu-\delta\nu^{(\rho-1)/s'}} \\ &\quad + C \sum_{\mu=\nu+3}^{\infty} e^{(C_1+\tau'+2\beta t)\mu} (\mu+1)^{\rho} e^{-\delta\mu^{(\rho-1)/s'}}. \end{aligned}$$

An elementary computation shows that given positive constants  $\alpha$ ,  $\epsilon$  and  $\delta$ , there exist  $c_1$ ,  $c_2$  and  $c_3 > 0$  such that

$$\sum_{k=n}^{\infty} k^{\rho} e^{\alpha k - \delta k^{1+\epsilon}} \leq c_1 \alpha^{1/\epsilon} e^{c_2 \alpha^{1+1/\epsilon}} + c_3 e^{-\frac{\delta}{2} n^{1+\epsilon}}$$

for all  $n \in \mathbb{N}$ , so that

$$\begin{aligned} \sum_{\mu} k''_{\nu,\mu} \leq C & \left[ (\nu + 1)^{\rho - \frac{1}{2}} e^{(C_1 + \tau')\nu - \delta\nu^{(\rho-1)/s'}} \right. \\ & + c_1 (C_1 + \tau' + 2\beta t)^{\frac{s'}{\rho-1-s'}} e^{c_2(C_1 + \tau' + 2\beta t)\frac{\rho-1}{\rho-1-s'}} \\ & \left. + c_3 e^{-\frac{\delta}{2}\nu^{(\rho-1)/s'}} \right], \end{aligned}$$

and finally

$$\sum_{\mu} k''_{\nu,\mu} \leq K_1 (1 + (C_1 + \tau + 2\beta t)^{\frac{s'}{\rho-1-s'}}) \exp(K_2 (C_1 + \tau + 2\beta t)^{\frac{\rho-1}{\rho-1-s'}})$$

where  $K_1$  and  $K_2$  do not depend on  $\beta$ . The case of  $\sum_{\nu} k''_{\nu,\mu}$  is similar.

From these estimates we deduce that there exists  $K > 0$

$$\sup_{\nu} \sum_{\mu} k_{\nu,\mu} + \sup_{\mu} \sum_{\nu} k_{\nu,\mu} \leq K(1 + f(\beta t)), \quad (13)$$

where  $f$  is a continuous function with  $f(0) = 0$ . We fix  $\beta = 2C_0K$  and we choose  $T' \in (0, T]$  in such a way that for all  $t \in [0, T']$  is  $|1 + f(\beta t)| \leq 2$ . From (10) and (13) we obtain that

$$\begin{aligned} & \sum_{\nu=0}^{\infty} e^{-h_{\nu}(t) - (2\beta t - \tau')(\nu+1)} 2a(t) | \langle [\phi_{\nu}, b] \partial_x u, \partial_x \partial_t u_{\nu} \rangle | \\ & \leq \beta \left[ \sum_{\nu=0}^{\infty} e^{-h_{\nu}(t) - (2\beta t - \tau')(\nu+1)} (\nu + 1) \lambda_0 a(t) \|\partial_x u_{\nu}\|^2 \right]^{1/2} \\ & \quad \cdot \left[ \sum_{\nu=0}^{\infty} e^{-h_{\nu}(t) - (2\beta t - \tau')(\nu+1)} (\nu + 1) \|\partial_t u_{\nu}\|^2 \right]^{1/2}, \end{aligned}$$

for all  $t \in [0, T']$ . From (9)

$$\mathcal{E}'(t) \leq 2 \sum_{\nu=0}^{\infty} e^{-h_{\nu}(t) - (2\beta t - \tau')(\nu+1)} | \langle (Lu)_{\nu}, \partial_t u_{\nu} \rangle |$$

and consequently

$$\mathcal{E}'(t) \leq 2 \left[ \sum_{\nu=0}^{\infty} e^{-h_{\nu}(t) - (2\beta t - \tau')(\nu+1)} \|(Lu)_{\nu}\|^2 \right]^{\frac{1}{2}} (\mathcal{E}(t))^{\frac{1}{2}}$$

for all  $t \in [0, T']$ . On the other hand, possibly taking a smaller  $T'$  we have that there exists  $C''$  and  $\tau'' > 0$  such that

$$\mathcal{E}(t) \geq C''(\|u(t, \cdot)\|_{H_{\sigma, \tau''}}^2 + \|\partial_t u(t, \cdot)\|_{H_{\sigma, \tau''}}^2),$$

for all  $t \in [0, T']$ . Finally

$$\begin{aligned} & \sup_{t \in [0, T']} \{ \|u(t, \cdot)\|_{H_{\sigma, \tau''}} + \|\partial_t u(t, \cdot)\|_{H_{\sigma, \tau''}} \} \\ & \leq C(\|u(t, \cdot)\|_{H_{\sigma, \tau}} + \|\partial_t u(t, \cdot)\|_{H_{\sigma, \tau}}) + \int_0^{T'} \|Lu(s, \cdot)\|_{H_{\sigma, \tau''}} ds. \end{aligned}$$

This energy estimate implies the well-posedness of the Cauchy problem in the interval  $[0, T']$ . Remarking that the value of  $T'$  depends only on the coefficients  $a$  and  $b$ , with a finite number of step of length  $T'$ , also  $T$  can be reached. This concludes the proof of our result.

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