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Quantum orbit method for the Connes-Landi-Matsumoto 3-sphere

Nicola Ciccoli

To Gianni Landi for his 60ieth birthday

Abstract. *We discuss the correspondence between the symplectic foliation of a Poisson structure on the* 3*-sphere and the unitary spectrum of its* C^**-algebraic quantization, known as Connes-Landi-Matsumoto* 3*-sphere. Quantization is obtained via symplectic groupoid quantization and this allows to understand various peculiarities of such correspondence. In the last section we discuss how this relates to quantization of Dirac structures (and foliations) and speculate on how to extend this correspondence to general locally abelian Poisson manifolds.*

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1. Introduction

In the words of David Vogan ([22]: a review of the wide ranging overview given in [12]) the orbit method is *a kind of damaged treasure map, offering cryptic hints about where to find some (but certainly not all) of the representations we seek to understand.* It is by now well known that, in a nutshell, orbit method can be seen as an instance of recognizing geometric data on a Poisson manifold in terms of algebraic properties of its quantization. The usual case of Lie groups fits into this framework by considering the universal enveloping algebra as an algebraic quantization of the linear Poisson structure on the dual Lie algebra $\mathfrak{g}^*$. For this reason the orbit method applied to quantum groups and their homogeneous spaces, although often labelled as *quantum orbit method* [17, 19], should, more correctly, be considered a non linear version of the classical orbit method.

In [5] we outlined a program to describe a orbit method type correspondence between symplectic leaves of a Poisson manifolds and the unitary dual of its quantized C^*-algebra. The quantization procedure to be taken into consideration is the geometric quantization applied to the symplectic groupoid, as first introduced in [23]; this procedure was further clarified in [11], where the role of the choice of a multiplicative Lagrangian polarization was more thoroughly explained. The purpose of this paper is to show how Connes-Landi-Matsumoto 3-sphere fits nicely into this framework once one takes into account some of its specific features. Though the Poisson integration and geometric quantization steps present no big obstacles (despite being often the hardest step in buiding the correspondence) the resulting groupoid C^*-algebra has a non T_0 topology on its set of orbits and therefore it is not type I, so that the unitary dual and the space of primitive ideals do not coincide.

We will show how the symplectic foliation on the semiclassical Poisson 3-sphere is still partly reflected in the primitive ideal structure of its quantization, establishing a homeomorphism between the leaf space and the primitive ideal space with its Jacobson topology. To this aim we give a detailed analysis of closed invariant subsets of the unit space of the quantization groupoid. In the last section we will indicate how the above results can be understood in the framework of Dirac quantization and foliation C^*-algebras.

The main motivation to analize in such detail this specific example lies in the fact that, in a sense, it is of a completely different nature from the one already considered in [5]. In the language of [13], this 3-sphere is a homogeneous space of a twisted Poisson structure on a compact group, while the complex projective space was analyzed referring to standard (Bruhat-Poisson) structure. It thus strengthen the conjecture that groupoid quantization could provide a unifying framework to treat all quotient Poisson homogeneous structures associated to compact Poisson-Lie groups.

2. Abelian Poisson structures on $\mathbb{S}^3$

In the paper [8], motivated by a construction of families of non commutative instantons, a non commutative algebra deforming the usual algebra of functions on $\mathbb{S}^4$ was introduced. This algebra was explicitely described by generators and relations. In particular, since one of the generators (corresponding to a coordinate w) is a central one, it is possible to quotient the algebra by $w = 0$ and obtain a deformation of the algebra of functions on a 3-sphere. This algebra can be completed to a C^*-algebra and in fact, as C^*-algebra, had been also earlier considered by [14]. We will refer to this non commutative C^*- algebra as the Connes-Landi-Matsumoto 3-sphere.

By an easy semiclassical limit on generators and relations one finds that the Connes-Landi 3-sphere can be seen as deformation of an explicit Poisson bracket on $\mathbb{S}^3$. This Poisson bracket admist an alternative description, better suited for our purposes.

Let $\mathbb{T}^2$ be the 2-dimensional torus endowed with a right invariant Poisson structure $\pi_\theta = \theta\partial_{\phi_1} \wedge \partial_{\phi_2}$ where $\theta \in \mathbb{R} \setminus \{0\}$. Let us denote with $\mathfrak{t}$ its Lie algebra and with $\sharp_\theta : \mathfrak{t}^* \to \mathfrak{t}$ the corresponding (bijective) sharp map.

Let us consider the isometric torus action of $\mathbb{T}^2$ on $\mathbb{S}^3$ given by considering so-called Hopf coordinates on the 3-sphere

$$\begin{cases} x_0 &= \cos\xi_1 \, \sin\eta \\ x_1 &= \sin\xi_1 \, \sin\eta \end{cases} \qquad \begin{cases} x_2 &= \cos\xi_2 \, \cos\eta \\ x_3 &= \sin\xi_2 \, \cos\eta \end{cases} \qquad \begin{cases} \eta \in \left[0, \frac{\pi}{2}\right] \\ \xi_i \in [0, 2\pi] \end{cases} \tag{1}$$

and letting $(\phi_1, \phi_2) \in \mathbb{T}^2$ act on them as:

$$(\phi_1, \phi_2) \cdot (\xi_1, \xi_2, \eta) = (\phi_1 + \xi_1, \phi_2 + \xi_2, \eta) \, .$$

When $\eta = 0$ (resp. $\eta = \frac{\pi}{2}$) we intend that $\xi_2 = 0$ (resp. $\xi_1 = 0$). The union of this two disconnected circle is called the Hopf link and the action is not free on them.

We will denote by $\rho : \mathfrak{t} \to \mathfrak{X}(\mathbb{S}^3)$ the corresponding infinitesimal action, letting $\rho_p : \mathfrak{t} \to \mathbb{T}_p\mathbb{S}^3$ its evaluation at a point $P \in \mathbb{S}^3$. In the following we denote $\partial_{\xi_i} = \rho(\partial_{\phi_i})$ and remark that the map ρ_p is injective at all points not belonging to the Hopf link. The dual map is $\rho^* : T^*\mathbb{S}^3 \to \mathfrak{t}^*$ can be understood as the associated moment map of the Hamiltonian lift of this torus action on $T^*\mathbb{S}^3$. In the following we will also need the map $R = \rho \circ \sharp_\theta : \mathfrak{t}^* \to \mathfrak{X}(\mathbb{S}^3)$ and the corresponding pull-back map $B : T^*\mathbb{S}^3 \to \mathrm{Diff}\,(\mathbb{S}^3)$:

$$B(p, \omega_p) = \exp[R(\rho^*(\omega_p)/2)] \, .$$

Let us endow the product Poisson manifold $\mathbb{S}^3 \times \mathbb{T}^2$, with the Poisson structure $\pi_\theta \oplus 0$. Due to its invariance properties such structure projects to the quotient with respect to the diagonal action of the torus:

$$\mathbb{S}^3 \times \mathbb{T}^2 \to \mathbb{S}^3 \times \mathbb{T}^2/\mathbb{T}^2 \simeq \mathbb{S}^3$$

so that the projection is a Poisson map. In Hopf coordinates this Poisson bivector is simply given by

$$\Pi_\theta = \rho^{\wedge 2}(\pi_\theta) = \theta(\partial_{\xi_1} \wedge \partial_{\xi_2}) \, , \tag{2}$$

This quotient Poisson bivector is the semiclassical limit of Connes-Landi 3-dimensional sphere (in [8, Section IV], in fact, a 4-dimensional sphere is considered having one Casimir generator t, our 3-dimensional sphere corresponds to the equatorial Poisson submanifold $t = 1/2$). In [26] a more general version of this procedure is considered; manifolds obtained through an invariant Poisson structure on an abelian group are called abelian Poisson manifolds in [11].

The symplectic foliation of this Poisson bivector is easily computed. The Hamiltonian distribution is tangent to each Hopf torus $\eta = \eta_0 \in]0, \frac{\pi}{2}[$ and therefore each such torus is a 2-dimensional symplectic leaf. When $\eta = 0, \pi/2$ the rank drops down to zero and therefore all points of the Hopf link

$$\left\{x_0^2 + x_1^2 = 1, x_2 = x_3 = 0\right\} \sqcup \left\{x_0 = x_1 = 0, x_2^2 + x_3^2 = 1\right\}$$

are 0-dimensional leaves. The symplectic foliation therefore corresponds to the usual Heegaard decomposition of the 3-sphere, with the singular tori decomposed into 0-dimensional leaves. The topology on the quotient space of leaves is thus equivalent to the topology on a quotient of the closed cylinder $[0, \frac{\pi}{2}] \times \mathbb{S}^1$ by identifying all inner parallel circles through $(\eta_0, \varphi) \simeq (\eta_0, \varphi')$.

3. Groupoid quantization of $(\mathbb{S}^3, \Pi_\theta)$

The first ingredient needed in groupoid quantization of a Poisson manifold is its integrating symplectic groupoid ([9]). The symplectic groupoid Σ_θ integrating $(\mathbb{S}^3, \Pi_\theta)$ can be explicitely described, as done in [26, 27]. As a symplectic manifold Σ_θ is simply the cotangent bundle $T^*\mathbb{S}^3$, endowed with the exact Liouville symplectic form ω.

For what concerns the groupoid structure, it is linked to the action groupoid $\mathfrak{t}^* \ltimes \mathbb{S}^3$ where the action of $\mathfrak{t}^*$ on $\mathbb{S}^3$ is integrating the map R. More precisely, there is a well defined groupoid morphism over the identity:

$$\Theta : \Sigma_\theta \to \mathfrak{t}^* \ltimes_R \mathbb{S}^3 \, ; \qquad \Theta(p, \omega_p) = (\rho^*(\omega_p), p) \, .$$

It is important to remember, here, that the R-action groupoid has non trivial isotropy over points in the Hopf link.

This description of the groupoid structure is often modified to look more simmetrical; composing with an easy groupoid isomorphism the source and target maps of $T^*\mathbb{S}^3$ can be determined as:

$$s(p, \omega) = B(-\omega)(p), \qquad t(p, \xi) = B(\omega)(p) \tag{3}$$

and the partially defined product is then just addition of covectors, after pullback, i.e.

$$(p_1, \omega_1) \cdot (p_2, \omega_2) = B(\omega_1)^*(\omega_2) + B(-\omega_2)^*\omega_1$$

where it is assumed that the inverse in this groupoid is then simply $(p, \omega_p)^{-1} = (p, -\omega_p)$ and the identity embedding coincides with the zero section.

Let us now describe an example of real Lagrangian multiplicative polarization on this symplectic groupoid. As explained in [11] a real multiplicative Lagrangian polarization is identified by a choice of a coisotropic subspace F_0 in $\mathfrak{t}$ which is transverse to each isotropy subspace $\ker\rho_p$. The easiest possible choice, considered in Hawkins' paper, is the diagonal subspace $F_0 = (v, v) \subseteq \mathfrak{t}$, since the choice of the generators of the two components of the torus do not satisfy the transversality condition, each one of them vanishing on a great circle.

We will consider a more general case by fixing $\mathfrak{h}_{(a,b)} = \langle (a,b) \rangle$ with $ab \neq 0$. We will limit ourselves to rational pairs (a,b) so that $\mathfrak{h}_{(a,b)}$ integrates to a 1-dimensional subtorus $H_{(a,b)}$ of $\mathbb{T}^2$. Let then $\mathrm{pr} : \mathfrak{t}^* \to \mathfrak{h}^*_{(a,b)} \simeq \mathfrak{t}^*/\mathfrak{h}^{\perp}_{(a,b)}$ be the quotient map and let $\mu_{(a,b)} = \rho^* \circ \mathrm{pr}$. We can then consider the groupoid morphisms composition

$$\begin{array}{ccccc} T^*\mathbb{S}^3 & \xrightarrow{\Theta} & \mathfrak{t}^* \ltimes_R \mathbb{S}^3 & \xrightarrow{\Psi} & \mathfrak{h}^*_{(a,b)} \ltimes_{\overline{R}} \mathbb{S}^2 \\ \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\ \mathbb{S}^3 & \longrightarrow & \mathbb{S}^3 & \longrightarrow & \mathbb{S}^2 \end{array} \tag{4}$$

where $\mathbb{S}^2 \simeq \mathbb{S}^3/H_{(a,b)}$ is the space of $H_{(a,b)}$-orbits and the base map in the right square is the corresponding projection. The action $\overline{R}$ can be written as:

$$\overline{R}(\lambda, \mathcal{O}_p) = \mathcal{O}_{R(t,p)}$$

where $p \in \mathbb{S}^3$, $\mathcal{O}_p$ is its $H_{(a,b)}$-orbit and $t \in \mathfrak{t}$ is such that $\mathrm{pr}(t) = \lambda$. The term on the right hand side does not depend on any of the choices involved. In projected Hopf coordinates $(\overline{\xi}, \eta)$, where $\overline{\xi} = -b\xi_1 + a\xi_2$, we have that:

$$\lambda \cdot (\overline{\xi}, \eta) = (\overline{\xi} - (a^2 + b^2)\theta, \eta) \tag{5}$$

where $\lambda = [a d_e \phi_1 + b d_e \phi_2]$ is a canonical choice of basis in $\mathfrak{h}_{(a,b)}^*$. Therefore, by choosing a norm one generator in $\mathfrak{h}_{(a,b)}$ the action is simply an irrational rotation on equatorial circles $\eta = \text{const.}$ with two fixed points at the poles $\eta = 0, \pi/2$.

We will denote by $\Phi = \Psi \circ \Theta$ the composed Lie groupoid morphism at the level of total spaces. For any $(\lambda, \mathcal{O}_p) \in \mathfrak{h}^*_{(a,b)} \ltimes_{\overline{R}} \mathbb{S}^2$, then,

$$\Phi^{-1}(\lambda, \mathcal{O}_p) = \left\{ (p', \omega_{p'}) \in T^*\mathbb{S}^3 \,\middle|\, p' \in \mathcal{O}_p \,,\, \mu_{(a,b)}(\omega_{p'}) = \lambda \right\} \tag{6}$$

where $\mu_{(a,b)}(\omega_p) = \lambda$ if and only if $\rho^*(\omega_p)(h_{a,b}) = \lambda$ with $h_{(a,b)}$ equal to a fixed non trivial generator of $\mathfrak{h}_{(a,b)}$. It is therefore clear that any such fiber of Ψ has the form of a *twisted* conormal bundle $N^{\perp}_{\lambda}$ to an $H_{(a,b)}$-orbit (as defined in [2, Example 3.28]), having Liouville class $[\pi^*_{\mathcal{O}}\lambda]$. Here $\lambda \in \mathfrak{h}^*_{(a,b)}$ is identified to a closed 1-form along $\mathcal{O}$. It is worth remarking that the two exceptional fibers over poles verify:

$$\Phi^{-1}(\lambda, \mathcal{O}_N) = T^*_N\mathbb{S}^3 \,; \qquad \Phi^{-1}(\lambda, \mathcal{O}_S) = T^*_S\mathbb{S}^3 \,,$$

and are the only contractible ones.

The fibers of this Lagrangian fibration define, as said, a real multiplicative Lagrangian polarization as in [11] (where the case $F_0 = \mathfrak{h}_{((1,1)}$ is considered). The image of Φ is the groupoid parametrizing the set of Lagrangian leaves and is independent of (a,b). In fact, if we use the symmetrized version of the symplectic groupoid given in equation (3), and denote by $\overline{s}$ (resp. $\overline{t}$) the source (resp. target) of the quotient groupoid, then

$$\overline{s}(\lambda, \mathcal{O}_p) = \mathcal{O}_{B(\lambda/2)(p)}\,, \qquad \overline{t}(\lambda, \mathcal{O}_p) = \mathcal{O}_{B(-\lambda/2)(p)} \tag{7}$$

and for a pair of composable elements the product is

$$(\lambda, \mathcal{O}_p) \cdot (\lambda', \mathcal{O}_{B(\lambda)(p)}) = (\lambda + \lambda', \mathcal{O}_p)\,.$$

The explicit form of Liouville class given above allows to check immediately Bohr-Sommerfeld conditions: since every leaf different from the polar ones has the topology of a cylinder $\mathbb{S}^1 \times \mathbb{R}^2$ these conditions are non trivial. They select a subgroupoid on the action side by imposing $[\pi_{\mathcal{O}}^* \lambda] \in 2\pi\mathbb{Z}$ to be an integer cohomology class in $H^1(\Phi^{-1}(\lambda, \mathcal{O}))$.

Resuming all information thus acquired, symplectic groupoid quantization tells us that the groupoid quantization of the Poisson manifold $(\mathbb{S}^3; \pi_\theta)$ is the groupoid C^*-algebra:

$$\mathcal{C}^*_\theta(\mathbb{S}^3) = C^*(\mathbb{Z} \ltimes_\theta \mathbb{S}^2)\,,$$

with respect to an irrational rotation action of angle θ on each parallel. In the next section we will analyze it from a C^*-algebraic point of view.

It has to be remarked that a different approach to groupoid quantization for a family of toric manifolds, including the one considered here, was developed by Cadet in [4]. It would be interesting to connect the two approaches in a rigorous manner.

4. Primitive ideals of $\mathcal{C}^*_\theta(\mathbb{S}^3)$

Now that we have obtained a quantization of $(\mathbb{S}^3, \pi_\theta)$ as a groupoid C^*-algebra we are in a position to understand how orbits and isotropy of this groupoid are related to its primitive ideals. The bad piece of news is the fact that since the orbit space is not even a T_0 space (a consequence of the fact that irrational orbits are dense on parallel circles) the corresponding C^* algebra is not even postliminal and its unitary dual does not coincide with the topological space of primitive ideals ([7]). It, hence, will be $\mathrm{Prim}\, \mathcal{C}^*(\mathbb{Z} \ltimes_\theta \mathbb{S}^2)$ the target of the orbit correspondence. The good news is that our C^*-algebra is the transformation group C^*-algebra of an abelian group, $\mathbb{Z}$, acting on a compact Hausdorff space: and this is a fairly well understood situation.

From general C^*-algebra theory it is known that for any given point γ in the unit space and any representation $(\rho_\gamma, \mathcal{H}_\gamma)$ of the isotropy subgroup Σ_γ^γ it is possible to induce a representation of the whole groupoid on a suitable Hilbert space completion of $\mathcal{C}_c(\Sigma_\gamma^\gamma) \otimes \mathcal{H}_\gamma$.

For the specific case of transformation group C^* algebras, all irreducible unitary representations are induced by isotropy subgroups: such groupoid C^*

algebras are sometimes called EH-regular. Furthermore the induction correspondence allows to establish a topological description of the space of primitive ideals. From [25, Theorem 8.39] (see also [24]) one gets that the induction map is a homeomorphism between a certain quotient space of $\mathbb{S}^2 \times \mathbb{S}^1$ and $\mathrm{Prim}(\mathbb{Z} \ltimes \mathbb{S}^2)$. On the space $\mathbb{S}^2 \times \widehat{\mathbb{Z}} \simeq \mathbb{S}^2 \times \mathbb{S}^1$, with its natural product topology, in fact, the following equivalence relation can be considered:

$$(p, \phi) \simeq (q, \psi) \iff \begin{cases} \overline{\mathbb{Z} \cdot p} = \overline{\mathbb{Z} \cdot q} & \text{if } G_p = G_q = 0 \\ p = q & \text{if } G_p \neq 0 \neq G_q \end{cases}$$

where G_p stands for the isotropy group of p. Since all orbits on level sets of $\mathbb{S}^2$ are dense, by an easy comparison with the content of Section 2 we can therefore conclude the following.

PROPOSITION 4.1. *The primitive ideal space of* $C^*(\mathbb{S}^3_\theta)$ *is homeomorphic to the leaf space of the Poisson manifold* $(\mathbb{S}^3; \theta)$.

The same conclusion can also be derived by considering so-called Renault's disintegration theorem ([18]) which says that for any closed invariant subset X of the unit space there exists an exact sequence of C^*-algebras

$$0 \to \mathcal{C}^*(\Sigma\big|_{X^c}) \to \mathcal{C}^*(\Sigma) \to \mathcal{C}^*(\Sigma\big|_X) \to 0$$

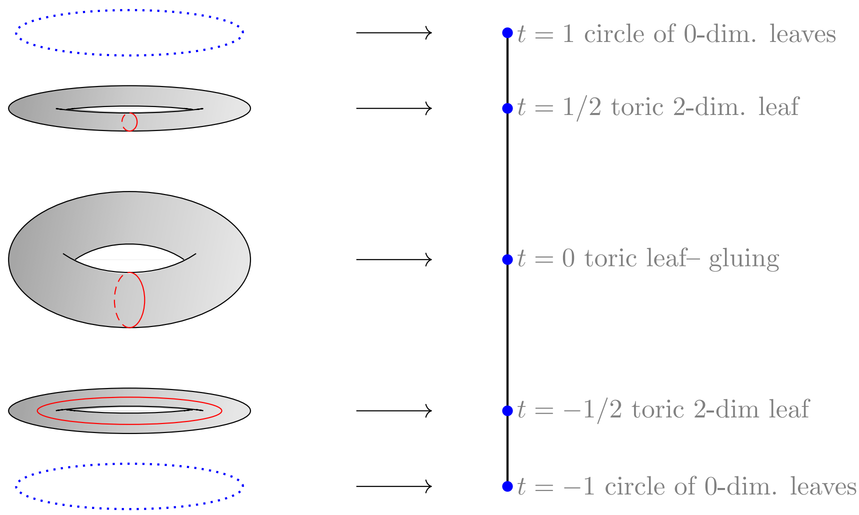
and furthermore any irreducible unitary representation factors through such a C^*-algebra $\mathcal{C}^*(\Sigma\big|_X)$. By considering, then the closure $X_t = \overline{\mathbb{Z} \cdot p}$ of orbits of points of height t one immediatly has:

- $\mathcal{C}^*(\Sigma\big|_{X_t}) \simeq \mathcal{C}^*(\mathbb{Z} \ltimes_\theta \mathbb{S}^1) \simeq \mathcal{C}^*(\mathbb{T}^2_\theta)$ which has one-point primitive ideal space;
- For every $t \in [-1, 1]$ there is a decomposition:

$$0 \to \mathcal{C}^*(\Sigma\big|_{X_t^c}) \to \mathcal{C}^*(\Sigma) \to \mathcal{C}^*(\mathbb{T}^2_\theta) \to 0$$

- There are two $\mathbb{S}^1$ families of characters corresponding to isotropy of North and South pole $X_{\pm 1}$ (Hopf link);
- Jacobson topology on $\mathrm{Prim}(C^*(\mathbb{Z} \ltimes \mathbb{S}^2))$ is of the form $\mathbb{S}^1 \times [-1, 1]/\simeq$ where the equivalence relation is given by

$$(\varphi, t) \simeq (\psi, s) \iff t = s \neq 0, 1\,.$$



Here the interval $[-1, 1]$ just plays the role of the T_0-ization of the space of $\mathbb{Z}$-orbits in $\mathbb{S}^2$. This quotient space is immediately identified with the corresponding space of leaves of the underlying Poisson manifold described at the end of Section 2. The above decomposition of the CLM 3-sphere C^*-algebra can be considered a non commutative analogue of the Hopf fibration and as such was earlier introduced in [15].

5. A different point of view: Dirac structures

In this section we will give some ideas on how to generalize results in [20] to the case of CLM 3-sphere. Details are postponed to future work where we plan to deal with the more general case of locally abelian Poisson manifolds which fits into the same scheme.

Let us consider a right invariant Dirac structure Γ on the torus such that $\Gamma \cap T\mathbb{T}^2 = \langle -\theta\partial_{\phi_1} + \partial_{\phi_2} \rangle$. The distribution thus determined, written also as $\mathcal{C}_\Gamma$ in what follows, is also called the *characteristic distribution* of Γ.

A canonical choice of Dirac structure on the product manifold $\mathbb{T}^2 \times \mathbb{S}^3$ is then $\Gamma \oplus T^*\mathbb{S}^3$. Since such subbundle is transversal to all isotropy groups of the $\mathbb{T}^2$-action on $\mathbb{S}^3$, it projects to a well defined Dirac structure $\widehat{\Gamma}$ on $\mathbb{S}^3$, according to [10, Proposition 4]. Explicitely the projected subbundle $\widehat{\Gamma}$ is the Dirac structure generated in Hopf coordinates by $-\theta\partial_{\xi_1} + \partial_{\xi_2}, d\xi_1 + \theta d\xi_2$ and $d\eta$. Let $\mathcal{C}_\theta$ be its characteristic distribution.

Let us fix a subalgebra $\mathfrak{h}_{(a,b)}$, with $(a, b) \in \mathbb{Q}^2 \setminus (0, 0)$ (and therefore a Lagrangian multiplicative polarization) as in Section 3. This is equivalent to the choice of a 1-dimensional rational complement of $\mathcal{C}_\theta$ inside $\mathfrak{t}^2$. If $H_{(a,b)}$ is the subtorus integrating $\mathfrak{h}_{(a,b)}$ then $\mathbb{S}^2 \simeq \mathbb{S}^3/H_{(a,b)}$ corresponds to the choice of a complete transversal to the characteristic foliation on $\mathbb{S}^3$. In general the Dirac structure $\widehat{\Gamma}$ should induce a Poisson structure on this quotient, but by dimensional reasons, since the rank of this Poisson structure is bounded above by $\dim\mathcal{C}(\Gamma) = 1$, it turns out to be trivial.

The action groupoid on the 3-sphere when restricted and projected to $\mathbb{S}^2$ is nothing but the transformation groupoid $\mathbb{Z} \ltimes_\theta \mathbb{S}^2$, with action given by irational rotations on parallels. From this point of view the crossed product $C*$-algebra can both be seen as the result of groupoid quantization, as explained in Section 3, and as the result of Dirac quantization of $\widehat{\Gamma}$ on $\mathbb{S}^3$, as proposed in [20].

Exactly the same proof as in [20], in the easy case in which the transverse Poisson structure on $\mathbb{S}^2$ is zero, allows to conclude that the Morita equivalence class of the quantization does not depend on the choice of transversal, i.e. it does not depend on the choice of the Lagrangian polarization, as long as it is determined by a rational complement of $\mathcal{C}(\Gamma)$ in $\mathfrak{t}^2$. Furthermore [20, Theorem 4.1.] still holds in this context, i.e. there is a well defined $O(2, 2; \mathbb{Z})$ action on the set of ($\mathbb{T}^2$-invariant) Dirac structures on $\mathbb{S}^3$ leaving the Morita equivalence class of the quantization unaltered.

6. Conclusions

As briefly mentioned before, Connes-Landi 3-sphere is but a specific example of a whole class of Poisson manifolds, called *locally abelian* in [11], which depend on the choice of an invariant Poisson structure on $\mathbb{T}^k$ (determined by an antisymmetric matrix $\underline{\theta} \in \mathfrak{so}(k; \mathbb{R})$) and a torus action $\mathbb{T}^k \times M \to M$ on a manifold M. The symplectic groupoid quantization for Poisson manifolds of this type is quite well understood ([11]) and an analysis of the general case, from the point of view of orbit method, is under preparation.

The discrepancy between the unitary dual and the primitive ideal space, which holds also in these cases, raises the question of which kind of Poisson data, richer than the plain symplectic foliation, can detect irreps of the C^*-algebra. We give indications on how a generalization of the concept of *rigged orbits* (see [1] for the classical case) find a natural interpretation in the setting of groupoid quantization. A more detailed program on how to develop this approach can be found in [6].

The example we consider here is somewhat complementary to the one in [5]; there the symplectic integration and geometric quantization (with singularities appearing in the Lagrangian distribution, as explained in [3]) are the main obstacles. Since the resulting groupoid C^*-algebra however has a nicely behaved T_0 orbit space the correspondence between groupoid orbits and unitary dual is easily understood.

The fact that such different cases both verifies a quantum orbit correspondence is a good indicator that such a correspondence should hold under pretty general conditions. Let us remark that the general quantum orbit method as proved via quantum groups in [17] covers the case of those Poisson homogeneous spaces which are quotient by a Poisson subgroup of the standard compact Poisson-Lie group. Since the Poisson CLM-sphere can be seen as a quotient of a non standard Poisson-Lie group structure by a coisotropic subgroup [21] it is tempting to foresee that the class of Poisson manifolds for which such result can hold should include all coisotropic quotients of Poisson-Lie compact groups.

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Author's address:

Nicola Ciccoli
Dipartimento di Matematica e Informatica
Università degli studi di Perugia
Via Vanvitelli 1
06123 Perugia, Italy
E-mail: `nicola.ciccoli@unipg.it`

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