

A counterexample to a priori bounds under the Ahmad-Lazer-Paul condition

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ABSTRACT. *In the context of scalar second order ODEs at resonance, we construct a counterexample showing that, in general, the Ahmad-Lazer-Paul condition does not imply a priori bounds for T -periodic solutions.*

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1. Introduction

The forced scalar second order ODE

$$x'' + \lambda x + r(t, x) = 0, \quad (1)$$

where $r : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is T -periodic in the t -variable and bounded, is said to be at resonance if the linear equation $x'' + \lambda x = 0$ associated with (1) has a nontrivial T -periodic solution. As is well known, this occurs if and only if $\lambda = \lambda_k = (2k\pi/T)^2$ for some $k \in \mathbb{N} = \{0, 1, 2, \dots\}$ and prevents, in general, the T -periodic solvability of (1), which can be guaranteed only under additional assumptions on the nonlinear term r (see, e.g., [7, 15]). After the pioneering work [13] dealing with the separate case $r(t, x) = g(x) - e(t)$, two celebrated such assumptions turned out to be the so-called Landesman-Lazer (LL) and Ahmad-Lazer-Paul (ALP) conditions. Denoting by Σ_k the eigenspace associated with the eigenvalue λ_k (namely, the set of T -periodic solutions of the equation $x'' + \lambda_k x = 0$), they read as follows:

- (LL) for every $\varphi \in \Sigma_k \setminus \{0\}$,

$$\int_{\{\varphi > 0\}} \left(\liminf_{x \rightarrow +\infty} r(t, x) \right) \varphi(t) dt + \int_{\{\varphi < 0\}} \left(\limsup_{x \rightarrow -\infty} r(t, x) \right) \varphi(t) dt > 0,$$

where $\{\varphi \geq 0\} = \{t \in [0, T] \mid \varphi(t) \geq 0\}$;

- (ALP) setting $R(t, x) = \int_0^x r(t, s) ds$,

$$\lim_{\substack{\|\varphi\|_\infty \rightarrow +\infty \\ \varphi \in \Sigma_k}} \int_0^T R(t, \varphi(t)) dt = +\infty.$$

Such assumptions are the T -periodic versions of the ones given, respectively, in [12] and in [1] for the Dirichlet problem associated with an elliptic PDE. In the context of the T -periodic problem, they were originally provided, respectively, in [14] and in [16], see also [7] for an exhaustive treatment of the subject. While the Landesman-Lazer condition usually leads to a T -periodic solution of (1) through topological methods relying on a priori bounds and degree theory, the Ahmad-Lazer-Paul one allows to prove T -periodic solvability by variational methods, precisely via the Rabinowitz saddle point theorem.

The aim of this brief note is to highlight that this dichotomy is not only a matter of the chosen techniques but is substantial, in that the a priori bounds on the solutions may actually fail under (ALP). Indeed, we are going to prove the following result.

THEOREM 1.1. *For every $k \in \mathbb{N}$, there exists a bounded, continuous and T -periodic function $r : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying (ALP) and such that equation (1), with $\lambda = \lambda_k$, has a sequence of T -periodic solutions $(u_j)_j$ with*

$$\|u_j\|_\infty \rightarrow +\infty. \quad (2)$$

Of course, the function r appearing in the above statement cannot satisfy condition (LL) (see also Remark 2.2 below).

2. Proof of Theorem 1.1 and related remarks

We start with the easier case $k = 0$ (with corresponding eigenvalue $\lambda_0 = 0$), which will indeed provide the idea for the proof in the more interesting case $k > 0$. Here, Σ_0 is the 1-dimensional space made up by constant functions and (1) reads as

$$x'' + r(t, x) = 0. \quad (3)$$

The corresponding (ALP) condition is given by

$$\lim_{|x| \rightarrow +\infty} \int_0^T R(t, x) dt = +\infty$$

and a natural choice for r is

$$r(t, x) = r(x) = \operatorname{sgn}(x) |\sin(\pi x)|,$$

where we mean $\operatorname{sgn}(0) = 0$. Of course, such a function satisfies (ALP), but (3) has the unbounded sequence of T -periodic solutions $u_j(t) \equiv j$ ($j \in \mathbb{N}$). Notice that all such u_j 's are multiple of the eigenfunction corresponding to λ_0 , that is, $\varphi_0(t) \equiv 1$. This is the key point for the construction of the counterexample when $k \geq 1$.

In this case, take $0 \neq \varphi_k \in \Sigma_k$, with $\|\varphi_k\|_\infty = 1$ (for instance, one may take $\varphi_k(t) = \cos(\sqrt{\lambda_k}t)$) and, after having set $Z_k = \{t \in \mathbb{R} \mid \varphi_k(t) = 0\}$, define

$$r(t, x) = \begin{cases} \operatorname{sgn}(x) |\varphi_k(t)| \left| \sin\left(\frac{\pi x}{\varphi_k(t)}\right) \right| & \text{if } t \notin Z_k, \\ 0 & \text{if } t \in Z_k. \end{cases} \quad (4)$$

Notice that Z_k is discrete and made up by a countable number of points. Of course, r is T -periodic in the first variable and $|r(t, x)| \leq 1$ for every t, x ; moreover, it is immediate to check that for the choice

$$u_j(t) = j\varphi_k(t), \quad j \in \mathbb{N},$$

it holds

$$r(t, u_j(t)) \equiv 0.$$

Thus, u_j satisfies (1) - with $\lambda = \lambda_k$ - and (2). It remains to show that r is continuous and satisfies (ALP).

The continuity at the points (\bar{t}, \bar{x}) with $\bar{t} \notin Z_k$ is obvious; on the other hand, if $\bar{t} \in Z_k$ and $(t_n, x_n) \rightarrow (\bar{t}, \bar{x})$, then $r(t_n, x_n) \rightarrow 0$ since $\operatorname{sgn}(x_n) |\sin(\pi x_n / \varphi_k(t_n))|$ is bounded.

Finally, we show that r satisfies (ALP). To this end, we first notice that $R(t, x) \geq 0$ for every t, x and $R(t, x) = 0$ if and only if $t \in Z_k$ or $x = 0$; moreover, it is easy to check that

$$\lim_{|x| \rightarrow +\infty} R(t, x) = +\infty, \quad \text{for every } t \notin Z_k. \quad (5)$$

Then, we observe that

$$\varphi \in \Sigma_k \iff \varphi(t) = \gamma\varphi_k(t + \theta) \text{ for some } \gamma \geq 0, \theta \in [0, T],$$

so that $\|\varphi\|_\infty = \gamma$ does not depend on θ . Hence, the Ahmad-Lazer-Paul condition (ALP) can be equivalently written as

$$\lim_{\gamma \rightarrow +\infty} \int_0^T R(t, \gamma\varphi_k(t + \theta)) dt = +\infty, \quad \text{uniformly in } \theta \in [0, T]. \quad (6)$$

We then show that r satisfies (6). By contradiction, let us assume that there exist $M > 0$ and two sequences $(\gamma_n)_n, (\theta_n)_n$, with $\gamma_n \rightarrow +\infty$ and $\theta_n \rightarrow \bar{\theta} \in [0, T]$ such that

$$\int_0^T R(t, \gamma_n\varphi_k(t + \theta_n)) dt \leq M \quad (7)$$

for every n . Passing to the inferior limit at both sides and using Fatou's Lemma (recall that $R(t, x) \geq 0$), we obtain

$$\begin{aligned} & \int_{[0, T] \setminus (Z_k \cup \bar{Z})} \liminf_{n \rightarrow +\infty} R(t, \gamma_n \varphi_k(t + \theta_n)) dt \\ & \leq \int_0^T \liminf_{n \rightarrow +\infty} R(t, \gamma_n \varphi_k(t + \theta_n)) dt \leq M, \end{aligned}$$

where $\bar{Z} = \{t \in [0, T] \mid \varphi_k(t + \bar{\theta}) = 0\}$. For every $t \in [0, T] \setminus (Z_k \cup \bar{Z})$, we now have that $|\gamma_n \varphi_k(t + \theta_n)| \rightarrow +\infty$, so that, by (5), we deduce that

$$\liminf_{n \rightarrow +\infty} R(t, \gamma_n \varphi_k(t + \theta_n)) = +\infty \quad \text{for every } t \in [0, T] \setminus (Z_k \cup \bar{Z}).$$

This contradicts (7) and eventually concludes the proof of Theorem 1.1. \square

REMARK 2.1: By defining

$$r(t, x) = \begin{cases} \operatorname{sgn}(x) |\varphi_k(t)|^\alpha \left| \sin\left(\frac{\pi x}{\varphi_k(t)}\right) \right|^\alpha & \text{if } t \notin Z_k, \\ 0 & \text{if } t \in Z_k \end{cases}$$

for α sufficiently large, it is possible to construct similar counterexamples where the nonlinear term enjoys higher regularity.

REMARK 2.2: It is clear that the function r defined in (4) does not satisfy any kind of Landesman-Lazer condition, since

$$\liminf_{x \rightarrow +\infty} r(t, x) = 0 = \limsup_{x \rightarrow -\infty} r(t, x),$$

so that the left-hand side in (LL) is identically equal to 0 (recall that as shown, e.g., in [8], (ALP) is weaker than (LL)). It can also be seen that r satisfies the so-called *potential Landesman-Lazer condition* [18], an intermediate condition between (LL) and (ALP), reading as

$$\int_{\{\varphi > 0\}} \left(\liminf_{x \rightarrow +\infty} \frac{R(t, x)}{x} \right) \varphi(t) dt + \int_{\{\varphi < 0\}} \left(\limsup_{x \rightarrow -\infty} \frac{R(t, x)}{x} \right) \varphi(t) dt > 0$$

for every $\varphi \in \Sigma_k \setminus \{0\}$, so that also in this case the a priori bounds do not work (indeed, the arguments used in [18] are variational).

REMARK 2.3: The previous discussion can be extended with no difficulty to the case of resonance with respect to the Dancer-Fučik spectrum, namely replacing (1) by

$$x'' + \mu x^+ - \nu x^- + r(t, x) = 0,$$

where $\pi/\sqrt{\mu} + \pi/\sqrt{\nu} = T/k$. In this case, the Ahmad-Lazer-Paul condition to be considered has been given in [2, 4, 11], while the Landesman-Lazer one dates back to [6].

REMARK 2.4: We observe that the provided counterexample can be extended to several different situations. For instance, with obvious modifications we can deal with Dirichlet and Neumann boundary conditions; in this case, the proof can be considerably simplified, since the eigenspaces are 1-dimensional and thus there is no dependence on θ in (6). More in general, one could adapt the discussion to the equation

$$(p(t)x')' + \lambda w(t)x + r(t, x) = 0,$$

where $p(t) > 0$ and $w(t) \geq 0$ for every $t \in (0, T)$ and suitable regularity/summability assumptions on p and q are fulfilled, together with boundary conditions of Sturm-Liouville type

$$\begin{cases} \alpha x(0) + \beta x'(0) = 0 \\ \gamma x(T) + \delta x'(T) = 0. \end{cases}$$

It is worth noticing that the search for radial solutions of boundary value problems associated with a second order elliptic PDE enters this setting. Again, the main point is here that there is a sequence of eigenvalues whose associated eigenspace is 1-dimensional and the nodal properties of the corresponding eigenfunctions are known. However, our counterexample becomes much more meaningful if both (LL) and (ALP) provide existence for the considered boundary value problem; from this point of view, existence results of Landesman-Lazer and Ahmad-Lazer-Paul type in this more general setting seem not completely established in literature, especially for what concerns condition (ALP) (cf. [5, 17] and the references therein).

REMARK 2.5: We finally mention that an alternative way to prove the existence of T -periodic solutions under the Landesman-Lazer condition is based on the Poincaré-Bohl theorem, as in [9]. Within this approach, condition (LL) plays a role in showing that large-norm (in the phase-plane) solutions of the Cauchy problems associated with (1) do not have integer rotation number (a property which has been highlighted in [3] and later improved in other contexts, e.g., [10]). Of course, such a property implies a priori bounds for T -periodic solutions¹, so that our example also shows that the Poincaré-Bohl approach fails under the sole Ahmad-Lazer-Paul condition.

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¹However, in general the converse does not hold, as is well seen already in the linear case: the equation $x'' + x = 2 \cos t$ is solved by the unbounded family of functions $\{x_A(t) = (A+t) \sin t\}_{A \in \mathbb{R}}$, having winding number equal to 1 on $[0, 2\pi]$, but does not have 2π -periodic solutions.

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