PERIODIC SOLUTIONS FOR DIFFERENTIAL SYSTEMS OF RAYLEIGH TYPE (*)

by FABIO ZANOLIN (a Trieste) (**)

SOMMARIO. *Si dimostra un teorema relativo all'esistenza di soluzioni periodiche per i sistemi del tipo* $x'' + F(x') + G(x) = h(\cdot, x, x')$ *in presenza di smorzamento non lineare che permette di evitare fenomeni di risonanza. Tutti i risultati sono dimostrati nell'ambito della teoria del grado topologico di Leray-Schauder.*

SUMMARY. *We prove a theorem concerning the existence of the periodic solutions for the systems of the type* $x'' + F(x') + G(x) = h(\cdot, x, x')$ *with a nonlinear damping which does not permit resonance phenomena. All the result are proved by using the Leray-Schauder topological degree theory in its classical form.*

Introduction

In this paper we present a theorem on the existence of the periodic solutions for the differential systems of the Rayleigh type:

$$x'' + F(x') + G(x) = h(t, x, x')$$

where h is a bounded continuous vector function, periodic in the first variable and F and G are conservative vector fields, with G a continuously differentiable mapping with upper bounded derivative. We shall also assume that the nonlinear damping $F(x')$ cannot become too small when x' is sufficiently large, namely we shall suppose that

$$\liminf_{|x'| \to +\infty} (F(x') \mid x') / |x'| > \sup |h(\ldots)|.$$

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(**) Indirizzo dell'Autore: Istituto di Matematica dell'Università degli Studi - Piazzale Europa, 1 - 34100 Trieste.

We remark that this assumption on F, which was used by *Ascari* in [1] in order to get the boundedness of the solutions for the scalar equation $x'' + F(x') + x = p(t)$, enable us to consider all the linear dampings cx' with $c \neq 0$, as *Reissig* in [10] and all the dampings of the form $\beta \mid x' \mid^{\rho} x'$, $\beta > 0$, $\rho > 0$, as *Caristi-Invernizzi* in [2]. Moreover, we note that the nonlinearity of F permits us to avoid the resonance phenomena which appear when there is no damping.

We end the paper with some remarks about the possibility to lighten the restrictions on the function G for the scalar case, in order to get an improvement of some results in [1], [4], [6] and [12].

All the theorems are proved by using the classical setting of the Leray-Schauder continuation principle ([8], [13]), namely finding a priori bounds for the solutions of the equation

$$x'' + \lambda F(x') + (1 - \lambda) \alpha x + \lambda G(x) = \lambda h(t, x, x')$$

where $\lambda \in [0,1]$ and α is a real number different from the real eigenvalues of the second order linear scalar equation $x'' + ax = 0$ with periodic boundary conditions.

Notations, terminology and nonlinear functional setting.

We use the symbol $(\cdot \mid \cdot)$ for the euclidean scalar product in the space R^m and $\mid \cdot \mid$ for the corresponding norm. We say that x: $R \to R^m$ is a T-periodic function ($T > 0$) if $x(t + T) = x(t)$ for every $t \in R$ and pose $\omega = 2\pi/T$. If x: $R \to R^m$ is a continuous T-periodic function, we set $\mid x \mid_\infty = \max_{[0,T]} \mid x(t) \mid$, $\mid x \mid_q = (\int_0^T \mid x(t) \mid^q dt)^{1/q}$, $1 \leq q < \infty$. Let C_T^k $(k = 0, 1, 2)$ be the Banach space of all the T-periodic continuous functions $R \to R^m$, of class C^k, equipped with the norm $\max \{ \mid x^{(i)} \mid_\infty ,\ i = 0, ..., k\}$. If $u, v \in C_T^0$, we set $< u \mid v >_2 = \int_0^T (u(t) \mid v(t))\, dt$, the L^2-scalar product of u and v. Obviously, with the above notations, we have $\mid u \mid_2^2 = < u \mid u >_2$. If F: $R^d \to R^k$ is a differentiable mapping, $F'(x)$ denotes the (Fréchet) differential of F at x.

Henceforth we shall suppose that F, G: $R^m \to R^m$ are two continuous mappings and h: $R \times R^m \times R^m \to R^m$ is continuous T-periodic in the first variable and bounded by some positive constant H, namely: $\mid h(t, r, s) \mid \leq H$ for every $t \in [0, T]$, r, $s \in R^m$.

Let us consider the differential system of Rayleigh type:

$$x''(t) + F(x'(t)) + G(x(t)) = h(t, x(t), x'(t)). \tag{1}$$

If α is a real number, $\alpha \neq \omega^2 n^2$ $(n = 0, 1, 2, \ldots)$ then, for any $u \in C^0_T$ there exists exactly one function $\mathcal{L}\, u \in C^2_T$ such that $(\mathcal{L}\, u)'' + \alpha\, (\mathcal{L}\, u) = u$. Moreover, $u \mapsto \mathcal{L}\, u$ defines a completely continuous linear operator $\mathcal{L}$ from C^0_T into C^1_T. Let us consider now the following (nonlinear) operator from C^1_T into C^0_T, $\mathcal{N}$: $x \to h\,(\cdot, x, x') - F\,(x') + \alpha\, x - G\,(x)$. It is easy to prove that $\mathcal{N}$ is continuous and carries bounded sets (of C^1_T) into bounded sets (of C^0_T). Hence $\mathcal{L}\mathcal{N}$ is a continuous operator from C^1_T into itself which is compact on bounded sets ($\mathcal{L}\mathcal{N}$ is completely continuous). Thus, in the light of the Leray-Schauder topological degree theory, the existence of a fixed point for $\mathcal{L}\mathcal{N}$, that is the existence of a T-periodic solution of (1), is ensured as soon as we are able to prove the existence of an a priori bound (in terms of the C^1_T-norm) for the fixed points of $\lambda\, \mathcal{L}\, \mathcal{N}$, $\lambda \in (0, 1]$. Observe that the equation $x = \lambda\, \mathcal{L}\, \mathcal{N}\, x \,(= \mathcal{L}\,(\lambda\, \mathcal{N})\, x)$ in C^1_T is equivalent to system:

$$x''\,(t) + \lambda\, F\,(x'\,(t)) + (1 - \lambda)\, \alpha\, x\,(t) + \lambda\, G\,(x\,(t)) = \lambda\, h\,(\ldots) \qquad (2)$$

with $x \in C^2_T$, $\lambda \in (0, 1]$, $\alpha \neq \omega^2 n^2$.

The main result

THEOREM 1. Let $f : \mathrm{R}^m \to \mathrm{R}$ *be a* C^1*-map,* $F = \operatorname{grad} f$.

Let $g : \mathrm{R}^m \to \mathrm{R}$ *be a* C^2*-map,* $G = \operatorname{grad} g$. *Let* $h\,(t, x, x')$ *be bounded by a constant* H. *Assume that a (positive) constant* L(0) *there exists such that, for any* r, s $\in \mathrm{R}^m$,

(j) $$(G'\,(\mathrm{r}) \cdot \mathrm{s} \mid \mathrm{s}) \leq L \cdot |\, \mathrm{s}\,|^2.$$

Let $K > 0$ *and suppose*

(h_1) $\lim\limits_{|s| \to +\infty} \dfrac{(G\,(\mathrm{s}) \mid \mathrm{s})}{|\,\mathrm{s}\,|} = +\infty$, *either* ($h'_1$) $\lim\limits_{|s| \to +\infty} \dfrac{(G\,(\mathrm{s}) \mid \mathrm{s})}{|\,\mathrm{s}\,|} = -\infty$.

(h_2) $\liminf\limits_{|s| \to +\infty} \dfrac{(F\,(\mathrm{s}) \mid \mathrm{s})}{|\,\mathrm{s}\,|} \geq K$, *either* ($k'_2$) $\limsup\limits_{|s| \to +\infty} \dfrac{(F\,(\mathrm{s}) \mid \mathrm{s})}{|\,\mathrm{s}\,|} \leq -K$.

Then the Rayleigh system (1) *has a periodic solution provided that*

$$H < K.$$

Proof. — Let us fix α in (2) such that $\alpha \neq \omega^2 n^2$, $\alpha \leq L$. Moreover, let $\alpha > 0$ if (h_1) holds and let $\alpha < 0$ if (h'_1).

Using the Leray-Schauder continuation principle, we are going to prove that, for any $\lambda \in (0, 1]$, the T-periodic solutions of (2) are

(0) The constant L is *not* related in any way with the eigenvalues of the second order linear scalar equation $x'' + ax = 0$ with T-periodic boundary conditions.

bounded in C^1_T by some constant (independed of λ).

Let x be a solution of (2) for some $\lambda \in (0,1]$.

STEP 1: BOUNDING $|x'|_1$. — Let us take the L^2-scalar product of both sides of (2) by x'. So, $< x'' | x' >_2 = 0 = < x | x' >_2$ and, as $G = \text{grad } g$, we have $< G(x) | x' >_2 = 0$ too. Thus, we obtain:

$$\lambda < F(x') | x' >_2 = \lambda < h(\cdot, x, x') | x' >_2 . \tag{3}$$

Then, dividing both sides of (3) by $\lambda > 0$ and using the Hölder inequality for the scalar product $< h(....) | x' >_2$, we have:

$$- H | x' |_1 \leq < F(x') | x' >_2 \leq H | x' |_1 \tag{4}$$

Using now (k_2) or (k_2'), we have

either $$< F(x') | x' >_2 \geq (H + \varepsilon) | x' |_1 - C_\varepsilon \tag{5}$$

or $$< F(x') | x' >_2 \leq -(H + \varepsilon) | x' |_1 + C_\varepsilon \tag{5'}$$

with $0 < \varepsilon < K - H$ and C_ε a suitable positive constant. From (4) and (5) or (5'), we immediately obtain that $| x' |_1$ is bounded by some positive constant, say C:

$$| x' |_1 \leq C . \tag{6}$$

STEP 2: BOUNDING $| x'' |_2$. — Let us take now the L^2-scalar product of both sides of (2), by x''. Observe that $< x | x'' >_2 = - | x' |_2^2$, $< F(x') | x'' >_2 = 0$, $< G(x) | x'' >_2 = - < G'(x) x' | x' >_2 \geq - L \cdot | x' |_2^2$. Hence we easely have:

$$\begin{aligned} | x'' |_2^2 &\leq (1 - \lambda) \alpha | x' |_2^2 + \lambda L | x' |_2^2 + H \cdot T^{1/2} | x'' |_2 \leq \\ &\leq L | x' |_2^2 + H \cdot T^{1/2} | x'' |_2 . \end{aligned} \tag{7}$$

(Remember that $\alpha \leq L$).

From the Holder inequality: $| x' |_2^2 \leq | x' |_\infty | x' |_1$ and from $| x' |_\infty \leq T^{1/2} | x'' |_2 + | x' |_1 / T$, we get from (7), using (6):

$$| x'' |_2^2 \leq (LC + H) T^{1/2} | x'' |_2 + LC^2/T$$

and hence we have that $| x'' |_2$ is bounded.

From the bounds for $| x' |_1$ and $| x'' |_2$, it immediately follows that $| x' |_\infty$ is bounded by a suitable constant, $M > 0$ ($| x' |_\infty \leq M$) and $| F(x') |_\infty$ is bounded too since F is continuous. Thus we get that, in the equation (2), the term $x'' + (1 - \lambda) \alpha x + \lambda G(x) = \lambda h(....) - \lambda F(x')$ is bounded in the C^0_T-norm by some positive constant, say W.

$$| x'' + (1 - \lambda) \alpha x + \lambda G(x) |_\infty \leq W. \tag{8}$$

STEP 3: BOUNDING $| x(t_o) |$. — Let us suppose that (h_1) holds and

$\alpha > 0$. Let t_o be such that $|x(t_o)| = \min_{[0,T]} |x(t)|$. Then, multiplying the equation (2), evaluated in $t = t_o$, by $x(t_o)$, and using (8) and the Schwarz inequality, we have $(x''(t_o) | x(t_o)) + (1 - \lambda)\alpha |x(t_o)|^2 + \lambda (G(x(t_o)) | x(t_o)) \leq W |x(t_o)|$ and, since $(x''(t_o) | x(t_o)) + |x'(t_o)|^2 \geq 0$, we obtain:

$$(1 - \lambda)\alpha |x(t_o)|^2 + \lambda (G(x(t_o)) | x(t_o)) \leq W |x(t_o)| + |x'(t_o)|^2 \leq W |x(t_o)| + |x'|_\infty^2 \leq W |x(t_o)| + M^2.$$

Now, either $|x(t_o)| \leq M$ and hence it is bounded, or

$$(1 - \lambda)\alpha |x(t_o)| + \lambda (G(x(t_o)) | x(t_o)) / |x(t_o)| \leq W + M \quad (9)$$

Since $\lim_{|s| \to +\infty} \alpha |s| = \lim_{|s| \to +\infty} \dfrac{(G(s) | s)}{|s|} = +\infty$, we have that the function $s \mapsto (1 - \lambda)\alpha |s| + \lambda (G(s) | s) / |s|$ tends to $+\infty$, as $|s|$ goes to $+\infty$, *uniformly* with respect to λ. Hence, from this fact and from (9), it follows that $|x(t_o)|$ is bounded (independently upon λ).

If (h'_1) holds, we choose $\alpha < 0$ and obtain the same thing for $|x(t_1)|$, where $|x(t_1)| = \max_{[0,T]} |x(t)|$.

At last, from the boundedness of $|x'|_\infty$ and $|x(t)|$, for some value of t in $[0, T]$, we have that $|x|_\infty$ is bounded and so we get the boundedness of the solution x in C^1_T which implies the existence of solutions for (1).

COROLLARY 1. *Let* $z: [0, +\infty) \to R$, $z(0) = 0$ *be continuous and let* $\lim_{t \to +\infty} z(t) = +\infty$ *(either* $-\infty$*). Let* $g: R^m \to R$ *be a* C^2*-function such that* $G = \text{grad } g$ *and*

$$L_1 |s|^2 \leq (G'(r) s | s) \leqslant L_2 |s|^2, \quad 0 < L_1 \leq L_2,$$

for every $r, s \in R^m$. *Then the vector Rayleigh system* $x'' + F(x') + G(x) = h(t, x, x')$, *where* $F(s) = z(|s|) \cdot s/|s|$, *if* $s \neq 0$ $(s \in R^m)$, $F(0) = 0$, *has a periodic solution for any bounded continuous forcing term, h, periodic in the first variable.*

Proof. — (h_1) is obviously satisfied since $L_1 > 0$, and (k_2) (either (k_2')) holds since $\lim_{|s| \to +\infty} (F(s) | s)/|s| = \lim_{|s| \to +\infty} z(|s|) = +\infty$ (either $-\infty$). Moreover, $s \mapsto \int_0^{|s|} z(t)\, dt$ is a potential function for the vector field F.

REMARK 1. — Corollary 1 improves the theorem by *Caristi-Invernizzi* in [2], which can be obtained from our result choosing $z(t) = \beta t^\rho$, $\beta \neq 0$, $\rho > 1$, $n^2 < L_1 \leq L_2 < (n + 1)^2$, $(n = 0, 1, 2, \ldots)$ and $h(\ldots)$ bounded continuous and 2π-periodic in t.

COROLLARY 2. *Let* A: $R^m \to R^m$ *be a positive definite symmetric matrix and let* $F = \text{grad}\, f$ (f *being a* C^1*-function*). *Then the system* $x'' + F(x') + Ax = h(t, x, x')$ *has a T-periodic solution, whenever*

$$\liminf_{|s| \to +\infty} \frac{(F(s) \mid s)}{|s|} > \sup |h(\ldots)|.$$

REMARK 2. — Corollary 2 extends to the systems the *Ascari* theorem [1] in the part concerning the existence of periodic solutions for the scalar equation $x'' + F(x') + x = p(t)$. (Cfr. *Sansone-Conti* [14, page 499]).

Notes and comments

First of all we note that the Theorem 1 seems to be more «interesting» when (h_1) holds. In fact, in the case (h'_1), as proved by *Mawhin* [7, Corollary 6.3], it sufficies only to suppose that $F(x')$ has the same or the opposite direction of x' (as, for instance, in our Corollary 1) and the existence of periodic solutions for (1) is ensured without any other assumption on F and G.

Moreover, we note that, when in the Theorem 1, $L < \omega^2$ holds, then, as was proved by *Invernizzi* in [5], the existence of periodic solutions for (1) is ensured without assuming (k_2) and (k'_2) for the function F. Hence, for $L < \omega^2$, our result is a consequence of the theorem in [5], while, for $L \geq \omega^2$, some condition upon the function F is necessary in order to avoid the resonance, as we do.

At last we point out that, by a more careful proof of the Theorem 1, it could be proved that our result is still true if we replace the growth condition (h_1) (either (h'_1)) on G with the two typotheses $\lim_{|s| \to +\infty} |G(s)| = +\infty$ *and* $\liminf_{|s| \to +\infty} (G(s) \mid Us) / |G(s)| \cdot |s| > -1$ (for a suitable unitary symmetric matrix, U), which have been introduced in [5] by *Invernizzi*.

The scalar case

At the end of this paper we want to show how a light modification of the proof of the Theorem 1 permits us to obtain, for the scalar case, in a simple way, a theorem which improves some known results for the scalar Rayleigh equation.

THEOREM 2. *Let* $m = 1$, $|h(\ldots)| \leq H$ *and assume that*

(k_1) $\liminf_{|s| \to +\infty} G(s) \operatorname{sign} s > H + |F(0)|$ *holds.*

Then, equation (1) *has a periodic solution if one of the following hypotheses holds:*

(k_2) $\liminf_{|s| \to +\infty} F(s) \operatorname{sign} s > H$

(k_2) $\limsup_{|s| \to +\infty} F(s) \operatorname{sign} s < -H.$

Proof. — Let us fix α in (2), $0 < \alpha < \omega^2$ and let x be a solution of (2) for some $\lambda \in (0, 1]$.

Assume that $|x(t_o)| = \min_{[0,T]} |x(t)|$. Then, $|x(t_o)| = 0$ or $|x(t_o)| > 0$ and hence $x'(t_o) = 0$, $x''(t_o)\, x(t_o) \geq 0$, $F(x'(t_o))\, x(t_o) = = F(0)\, x(t_o)$. Now, working as in the Step 3 of the Theorem 1, we have: $x''(t_o)\, x(t_o) + (1 - \lambda)\, \alpha\, x(t_o)^2 + \lambda\, G(x(t_o))\, x(t_o) = = \lambda\, h(t_o, x(t_o), x'(t_o))\, x(t_o) - \lambda\, F(0)\, x(t_o) \leq \lambda\, (H + |F(0)|)\, |x(t_o)|$. So, we immediately obtain

$$G(x(t_o))\, x(t_o) \leq (H + |F(0)|)\, |x(t_o)| \tag{10}$$

From (10) and from (k_1) we have that $|x(t_o)|$ is bounded by some constant B (for every λ).

$$|x(t_o)| \leq B \tag{11}$$

Moreover, from (k_2) (either (k'_2)) we deduce that $|x'|_1$ is bounded (that is inequality (6)) as in the Step 1 of Theorem 1. Then, (6) and (11) give the bound for $|x|_\infty$ and so we can state that $|G(x)|_\infty$ is bounded too: $|G(x)|_\infty \leq D$.

Now, let us take the L^2-scalar product of both sides of (2) by x'' and obtain:

$$|x''|_2^2 \leq (1 - \lambda)\, \alpha\, |x'|_2^2 + (D + H)\, T^{1/2}\, |x''|_2 \leq \alpha\, |x'|_2^2 + (D + H)\, T^{1/2}\, |x''|_2,$$

and using the Wirtinger inequality $\omega\, |x'|_2 \leq |x''|_2$, we have

$$(\omega^2 - \alpha)\, |x''|_2^2 \leq \text{const}\, |x''|_2. \tag{12}$$

From (12) we get the bounding for $|x''|_2$, since $\alpha < \omega^2$ and so we have proved that x is bounded in C^1_T.

REMARK 3. — It is not difficult to show that the Theorem 2 remains true if we change the conditions (k_2) and (k'_2) with the following one:

(j) *Assume that G is continuously differentiable and* $G'(s) \leq L < \omega^2$ *for every* $s \in \mathbb{R}$.

In fact, we can prove that $|x''|_2$ is bounded in (2) using the same argument as in [4] (that is, using the Wirtinger inequality in the inequality (7) of the Step 2 of our Theorem 1) and then we use (11) (which depends by (k_1)) and deduce that x is bounded in the C^1_T-norm.

In the variant (j), Theorem 2 gives an improvement of a result in [4, Corollary 1] (which generalizes a result by *Reissig* [11]) because we obtain a simple explicitly computable «level» $(H + |F(0)|)$ which must be exceeded by G in order to get the solutions of (1).

Moreover, the Theorem 2 (with (k_2), either (k'_2)) gives an improvement of various theorems on the existence of periodic solutions for the scalar Rayleigh equation, as those by *Ascari* [1], *Reuter* [12], *Mawhin* [6, Théoreme 5.2 $d_{2p+1} > 0$]. For various generalizations of the Ascari and Reuter theorems on the periodic solutions, see also the book of *Reissig - Sansone - Conti* [9]. For other recent results, under sign conditions for F and G, see also Dancer [3].

At last we remark that, when $h = h(t)$, by more careful proof of the Theorem 2, one can see that hypothesis (k_1) can be weakened as

$$G(s) < \min\{h(t): t \in [0, T]\} - F(0), \text{ for } s \leq s_1 \leq 0,$$

$$G(s) > \max\{h(t): t \in [0, T]\} - F(0), \text{ for } s \geq s_2 \geq 0;$$

while (k_2) and (k'_2) can be weakened as

$$\text{either } \liminf_{|s| \to +\infty} F(s) \operatorname{sign} s > \frac{1}{2} (\max\{h(t)\} - \min\{h(t)\})$$

$$\text{or } \limsup_{|s| \to +\infty} F(s) \operatorname{sign} s < \frac{1}{2} (\min\{h(t)\} - \max\{h(t)\}),$$

respectively.

Finally, we note that, using a standard perturbation like $G(x) + \varepsilon x$ of the function G, letting $\varepsilon \to 0^+$, via the Ascoli-Arzelà theorem, we get the existence of periodic solutions in the Theorem 2 also when we replace the strict inquality $>$ in (k_1) which the weaker one $\geq$.

REFERENCES

[1] A. ASCARI: *Studio asintotico di un'equazione relativa alla dinamica del punto,* Rend. Ist. Lomb. Sc. Lett., (3), 16 (1952), 278-288.

[2] G. CARISTI - S. INVERNIZZI: *Forced Oscillations of Systems with Nonlinear Damping,* Boll. Un. Mat. Ital. (5), 17 - B, (1980), to appear.

[3] E. N. DANCER: *On the Ranges of Certain Damped Nonlinear Differential Equations,* Ann. Mat. Pura Appl. 119, (1979), 281-295.

[4] S. INVERNIZZI - F. ZANOLIN: *Periodic Solutions of a Differential Delay Equation of Rayleigh Type,* Rend. Sem. Mat. Univ. Padova, 61 (1979), 115-124.

[5] S. INVERNIZZI: *Periodic solutions of nonlinear Rayleigh Systems,* Boll. Un. Mat. Ital. Supplemento (1980), to appear.

[6] J. MAWHIN: *Degré topologique et solutions périodiques des systémes différentielles non linéaires,* Bull. Soc. Roy. Sci. Liège, 38 (1969), 308-398.

[7] J. MAWHIN: *Boundary Value Problems for Nonlinear Second - Order Vector Differential Equations,* Journal of Diff. Eq., 16 (1974), 259-269.

[8] C. MIRANDA: *Problemi di esistenza in analisi funzionale,* Scuola Normale Superiore, Pisa 1975 (reprint).

[9] R. REISSIG - G. SANSONE - R. CONTI: *Qualitative theorie nichtlinearer differentialgleichungen,* Ed. Cremonese, Roma 1963.

[10] R. REISSIG: *Contractive mappings and periodically perturbed non-conservative systems,* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., 58 (1975), 696-702.

[11] R. REISSIG: *Extension of some results concerning the generalized Liénard equation,* Ann. Mat. Pura Appl. 104 (1975), 269-281.

[12] G. E. H. REUTER: *Boundedness theorems for non linear differential equations of the second order,* Jour. London Math. Soc., 27 (1952), 48-58.

[13] N. ROUCHE - J. MAWHIN: *Equations Differentielles Ordinaires, (2 volumes),* Masson, Paris 1973.

[14] G. SANSONE - R. CONTI: *Equazioni differenziali non lineari,* Ed. Cremonese, Roma 1956.