Rend. Istit. Mat. Univ. Trieste
Vol. XXXIII, 281–297 (2001)

On Γ-convergence in Anisotropic Orlicz-Sobolev Spaces

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Summary. - *In this paper we consider* Γ*-convergence for a class of lower semicontinuous functionals defined on Orlicz-Sobolev spaces. Particularly we prove compactness results for these type of functionals. Moreover, we compare* Γ*-convergence and convergence of minima.*

Introduction

In the study of variational problems in applied mathematics the concept of variational convergence called Γ-convergence has come to be a very important tool. One reason is its compactness properties for general classes of functionals and topologies. In addition almost all other variational convergences follow as consequences of the Γ-convergence. For an introduction to the theory we refer to Dal Maso [9].

In this paper we study Γ-convergence for a class of lower semicontinuous functionals defined on the Orlicz-Sobolev class $W^1L_G(\Omega)$ defined below. There are many advantages of such a development. The analysis in Orlicz-Sobolev spaces uses properties like convexity and growth (Δ_2-property) in such a way that one can obtain variational solutions to larger classes of nonlinear problems than in usual Sobolev spaces, see e.g. [5].

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The paper is organized as follows: In Section 1 we give some preliminary results on Γ-convergence and on Orlicz-Sobolev spaces. The main results are presented in Section 2. In particular we prove a Γ-compactness result (Theorem 2.2) for functionals defined on $W^1L_G(\Omega)$. The framework uses the localization method as presented in [9]. We also compare Γ-convergence and convergence of minima (Theorem 2.3 and Theorem 2.4). Section 3 is devoted to the proof of Theorem 2.2 and contains in particular an Orlicz-space version of the fundamental estimate. In Section 4, finally, we give some concluding remarks.

1. Preliminary results

Let X be a topological space and let $\mathcal{N}(x)$ denote the set of all open neighborhoods of $x \in X$. Further, let $\{F_h\}$ be a sequence of functions from X into $\overline{R}$.

DEFINITION 1.1. *The Γ-lower and Γ-upper limits of the sequence $\{F_h\}$ are the functions from X into $\overline{R}$ defined by*

$$F'(x) = \Gamma - \liminf_{h\to\infty} F_h(x) = \sup_{\omega\in\mathcal{N}(x)} \liminf_{h\to\infty} \inf_{z\in\omega} F_h(z)$$

and

$$F''(x) = \Gamma - \limsup_{h\to\infty} F_h(x) = \sup_{\omega\in\mathcal{N}(x)} \limsup_{h\to\infty} \inf_{z\in\omega} F_h(z),$$

respectively. If these two limits coincide, i.e. if there exists a unique function $F : X \to \overline{R}$ such that

$$F = \Gamma - \liminf_{h\to\infty} F_h(x) = \Gamma - \limsup_{h\to\infty} F_h(x),$$

we say that the sequence $\{F_h\}$ Γ-converges to F.

REMARK 1.2. *By the definition its obvious that $\{F_h\}$ Γ-converges to F if and only if*

$$\Gamma - \limsup_{h\to\infty} F_h \leq F \leq \Gamma - \liminf_{h\to\infty} F_h.$$

This means that Γ-convergence and lower semicontinuity are closely related concepts. We have the following sequential characterization of Γ-convergence, see [9, Proposition 8.1]:

THEOREM 1.3. *Let X be a separable metric space and let $\{F_h\}$ be a sequence of functionals from X into $\overline{R}$. Then*

(i) for every $x \in X$ and for every sequence $\{x_h\}$ converging to x,

$$F'(x) \leq \liminf_{h\to\infty} F_h(x_h);$$

(ii) for every $x \in X$ there exists a sequence $\{x_h\}$ converging to x such that

$$F'(x) = \liminf_{h\to\infty} F_h(x_h);$$

(iii) for every $x \in X$ and for every sequence $\{x_h\}$ converging to x,

$$F''(x) \leq \limsup_{h\to\infty} F_h(x_h);$$

(iv) for every $x \in X$ there exists a sequence $\{x_h\}$ converging to x such that

$$F''(x) = \limsup_{h\to\infty} F_h(x_h).$$

Consequently $\{F_h\}$ Γ-converges to a function $F \in X$ if and only if

(v) for every $x \in X$ and for every sequence $\{x_h\}$ converging to x,

$$F(x) \leq \liminf_{h\to\infty} F_h(x_h)$$

and

(vi) for every $x \in X$ there exists a sequence $\{x_h\}$ converging to x such that

$$F(x) = \lim_{h\to\infty} F_h(x_h)$$

Moreover, Γ-convergence enjoys the following compactness property, see [9], Theorem 8.5:

THEOREM 1.4. *Let X be a separable metric space. Then every sequence $\{F_h\}$ of functionals from X into $\overline{R}$ has a Γ-convergent subsequence.*

We recall that a Young function $A : [0,\infty) \to [0,\infty]$ is a function of the form

$$A(t) = \int_0^t a(x)dx$$

where the function $a : [0,\infty) \to [0,\infty]$ is increasing, left continuous and not identically zero and not identically infinity on the interval $(0,\infty)$.

The Orlicz space $L_A(\Omega)$ is the set of measurable functions f on Ω such that $\|f\|_{A,\Omega} < \infty$, where

$$\|f\|_{A,\Omega} = \inf\left\{\theta > 0 : \int_\Omega A\left(\frac{|f(x)|}{\theta}\right) dx \leq 1\right\}$$

(the Luxemburg norm on $L_A(\Omega)$)

A G-function $G : \mathbf{R}^m \to [0,\infty]$ is a function with the following properties:

(i) $G(0) = 0$;
(ii) $\lim_{|x|\to\infty} G(x) = \infty,\qquad \left[x \in R^m : |x| = \left(\sum_{i=1}^m x_i^2\right)^{1/2}\right]$;
(iii) G is convex
(iv) G is symmetric i.e. $G(-x) = G(x)$, $x \in \mathbf{R}^m$;
(v) the set $G^{-1}(\infty) = \{x \in R^m; G(x) = \infty\}$ is separated from 0;
(vi) G is lower semi-continuous.

Additionally we will assume that G is monotonically increasing in each variable separately, that G and G^* (the convex polar) satisfies Δ_2 condition (this will guarantee that the separability and reflexivity of function spaces defined below, see [6]). The vector valued Orlicz-space $L_G(\Omega)$ is defined as follows:

Let G be a G-function and let Ω be a domain in $\mathbf{R}^n$, let $u_1, u_2, \ldots,$ u_m be real valued measurable functions defined on Ω and let $u =$

$(u_1, u_2, ..., u_m)$ be a vector valued function. Then, u is said to belong to $L_G(\Omega)$ if there exists a $\lambda > 0$ such that

$$\int_\Omega G(\lambda u(x)) < \infty.$$

The space $L_G(\Omega)$ is equipped with a norm corresponding to the Luxemburg norm given by

$$\|u\|_{G,\Omega} = \inf\left\{\theta > 0 : \int_\Omega G\left(\frac{u}{\theta}\right) dx \le 1\right\}.$$

There should not be any ambiguity for the same notations $L_A(\Omega)$ and $L_G(\Omega)$ used for Young function and G-function,respectively.

For a G-function G, the complementary function G^*_+ is defined by

$$G^*_+(u) = \sup_{v_i \ge 0} (u \cdot v - G(v)),$$

where $u \cdot v = \sum_{i=1}^m u_i v_i$.

Let G be a G-function of $(n+1)$ variables. The anisotropic Orlicz-Sobolev space, denoted by $W^1 L_G(\Omega)$, is defined to be the space of weakly differentiable functions u for which

$$(u, Du) = (u, D_1 u, D_2 u, ..., D_n u)$$

belongs to $L_G(\Omega)$. A norm for the space $W^1 L_G(\Omega)$ is given by

$$\|u\| = \|(u, Du)\|_{G,\Omega}.$$

For further details regarding Orlicz-Sobolev spaces we refer to the monographs [1] and [6].

Given two functions A and B, the notation $A \prec\prec B$ means that for every $\lambda > 0$

$$\lim_{t\to\infty} \frac{A(t)}{B(\lambda t)} = 0.$$

Let us recall the following imbedding result (see [4]).

Theorem 1.5. *Let Ω be a bounded domain in $\mathbf{R}^n$ with the cone property, let f be a continuous non-negative function on $[0,\infty)$ and let G be a G-function of $(n+1)$ variables on $[0,\infty)$ such that*

$$G^*_+(0, f(s), f(s), ... f(s)) \le s.$$

Furthermore, let A be a Young function given by

$$A^{-1}(|t|) = \frac{1}{\eta} \int_0^{|t|} \frac{f^{-1}(s)}{s^{1/n}} ds$$

for some constant $\eta > 0$. If B is a Young function such that $B \prec\prec A$, then $W^1 L_G(\Omega)$ is compactly imbedded in $L_B(\Omega)$.

2. The main results

Let the function G be defined as above and let us define G_0 and B as

$$G_0(\xi_1, \xi_2, \ldots, \xi_n) = G(0, \xi_1, \xi_2, \ldots, \xi_n)$$

and

$$B(u) = G(u, u, \ldots, u),$$

respectively, where we assume that B satisfies all the hypotheses of Theorem 1.3 above. We have the following compactness result:

THEOREM 2.1. *Suppose that G satifies the Δ_2-condition. Then every sequence of functionals $F_h : L_B(\Omega) \to \overline{R}$ has a $\Gamma(L_B)$-convergent subsequence.*

Proof. Since G satifies the Δ_2-condition, $L_B(\Omega)$ is separable, see e.g. Kufner et. al. [6], and thus the result follows from the compactness Theorem 1.2 above. □

Let us now define the space $\mathcal{M} = \mathcal{M}(c, \beta)$ of Caratheodory functions $f : \Omega \times R^n \to [0, +\infty)$ satisfying the conditions:

(1) $f(x, \xi)$ is convex in ξ.

(2) $G_0(\xi_1, \ldots, \xi_n) \leq f(x, \xi) \leq c(1 + G_0(\xi_1, \ldots, \xi_n))$.

(3) G satisfies the Δ_2-condition with constant β.

Let us also define the class $\mathcal{F}(\mathcal{M})$ of functionals $F : L_B(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty)$ given by

$$F(u, A) = \int_A f(x, Du(x)) dx,$$

for $f \in \mathcal{M}$ and $A \in \mathcal{A}(\Omega)$, where $\mathcal{A}(\Omega)$ denotes the family of all open subsets of Ω. We extend in the usual way the functionals to $+\infty$ on $L_B(\Omega) \setminus W^1 L_G(\Omega)$.

The main objective is now to establish a result which says that the Γ-limit of a sequence

$$F_h(u, A) = \int_A f_h(x, Du(x))dx,$$

in $\mathcal{F}(\mathcal{M})$ has an integral representation

$$F_0(u, A) = \int_A \varphi(x, Du(x))dx, \tag{1}$$

where also $\varphi \in \mathcal{M}$.

The main result of this paper is the following compactness result:

THEOREM 2.2. *For every sequence $\{F_h\}$ in $\mathcal{F}(\mathcal{M})$ there exists a subsequence $\{F_{h_k}\}$ and a functional $F_0 \in \mathcal{F}(\mathcal{M})$ such that $F_{h_k}(\cdot, A)$ $\Gamma(L_B)$-converges to F_0 for every $A \in \mathcal{A}(\Omega)$.*

REMARK 2.3. *$F_0(u, \cdot)$ is the restriction of a Borel measure to $\mathcal{A}(\Omega)$ and moreover, the local property of the Γ-limit shows that in the integral representation (2.1) the function $\varphi \in \mathcal{M}$ is independent of A.*

REMARK 2.4. *By the definition of Γ-convergence it easily follows that*

(i) F_0 is lower semicontinuous.

(ii) If H is continuous, then

$$F_0 + H = \Gamma(L_B) - \lim_h F_h + H.$$

Theorem 2.2 will be proven in the next section. We end this section by giving examples of the relationship between Γ-convergence and convergence of minima. Let F_h and F belong to $\mathcal{F}(\mathcal{M})$ and let $H : L_B(\Omega) \to R$ be a continuous functional with the property that there exist some constants $c > 0$ and $b \in R$ such that

$$H(u) \geq c \int_{\Omega} B(u(x))dx - b \tag{2}$$

for all $u \in L_B(\Omega)$. Let us put

$$m_h = \inf_{u \in W^1 L_G(\Omega)} \{F_h(u) + H(u)\} \tag{3}$$

and

$$m = \inf_{u \in W^1 L_G(\Omega)} \{F(u) + H(u)\} \tag{4}$$

THEOREM 2.5. *If* $\{F_h\}$ Γ*-converges to* F *in* $L_B(\Omega)$ *then* m_h *converges to* m.

Proof. We recall that for any topological vector space X it holds that

$$\min_{x \in X} F(x) = \lim(\inf_{x \in X} F_h(x)) \tag{5}$$

whenever $\{F_h\}$ is a X-equi-coercive sequence of functionals which $\Gamma(X)$-converges to F (see e.g. [9] Theorem 7.8). The minima in (3) and (4) can be taken over $L_B(\Omega)$ instead of $W^1 L_G(\Omega)$. Moreover, by Remark 3, $\{F_h + H\}$ Γ-converges to $\{F + H\}$ in $L_B(\Omega)$. It holds that

$$F_h + H \geq k_1 \Psi - k_2$$

for some positive constants k_1 and k_2, where

$$\Psi(u) = \begin{cases} \int_{\Omega} G(u(x), Du(x))dx & \text{if } u \in W^1 L_G(\Omega) \\ +\infty & \text{otherwise} \end{cases}.$$

This follows from the fact that

$$\begin{aligned} G(u(x), Du(x)) &= G(\frac{1}{2}(2u(x)) + \frac{1}{2}0, \frac{1}{2}\overrightarrow{0} + \frac{1}{2}(2Du(x))) \\ &\overset{\text{convexity}}{\leq} \frac{1}{2}G(2(u(x), 0, ..., 0)) + \frac{1}{2}G(2(0, Du(x))) \\ &\leq \beta(G((u(x), 0, ..., 0)) + G((0, Du(x)))) \\ &\overset{\text{G increasing}}{\leq} c(G((u(x), ..., u(x))) + G_0(Du(x))) \end{aligned}$$

Moreover, we observe that $\Psi(u) \leq 1$ if $\|u\| \leq 1$ (by the definition of the Luxenburg norm) and that $\|u\| \leq \Psi(u)$ if $1 < \|u\|$ (use that by the definition of the Luxenburg norm and by convexity $1 < \Psi(\frac{u}{\theta}) \leq \theta^{-1}\Psi(u)$ for all $1 < \theta < \|u\|$). Thus, the set $\{u : \Psi(u) \leq t\}$ is bounded in $W^1L_G(\Omega)$ for all $t > 0$. Moreover, by the imbedding result Theorem 1.5, it holds that $\overline{\{u : \Psi(u) \leq t\}}$ is compact in $L_B(\Omega)$ which implies that the sequence $\{F_h + H\}$ is equi-coercive in $L_B(\Omega)$. Consequently we obtain that $m_h \to m$ by replacing X by $L_B(\Omega)$, F_h by $F_h + H$ and F by $F + H$ in (5). □

THEOREM 2.6. *Assume that all hypoteses are satisfied as in Theorem 2.5 except that 2 is replaced by the assumption that there exists a bounded set U in $W^1L_G(\Omega)$ such that*

$$\inf_{u \in W^1L_G(\Omega)} \{F_h(u) + H(u)\} = \inf_{u \in U} \{F_h(u) + H(u)\}$$

for all h. Then, if $\{F_h\}$ Γ-converges to F in $L_B(\Omega)$ it holds that m_h converges to m.

Proof. We recall that for any topological vector space X it holds that

$$\min_{x \in X} F(x) = \lim(\inf_{x \in X} F_h(x)) \tag{6}$$

whenever $\{F_h\}$ $\Gamma(X)$-converges to F and there exists a compact set K such that

$$\inf_{x \in X} \{F_h(u)\} = \inf_{x \in K} \{F_h(u)\}$$

for all h (see [9] Theorem 7.4.). Minimizing over $X = L_B(\Omega)$, and $K = \overline{U}$ and replacing F_h by $F_h + H$ and F by $= F + H$ in 6 we therefore obtain the desired result. □

3. Some results related to Theorem 2.2 and its proof

The proof of Theorem 2.2 will be divided into a number of lemmas. Inspired by the pedagogical presentation in Dal Maso [9] we will establish the result by using localization and by proving that functionals $F \in \mathcal{F}(\mathcal{M})$ satisfies the fundamental estimate in Orlicz-Sobolev spaces. A neccesary condition for the integral representation (2.1) is that $F_0(u,\cdot)$ is a measure. For this purpose we introduce increasing set functions:

DEFINITION 3.1. *A set function $\sigma : \mathcal{A}(\Omega) \to [0,+\infty]$ is called*

(i) an increasing set function if $\sigma(\emptyset) = 0$ and $\sigma(A_1) \leq \sigma(A_2)$ for $A_1 \subset A_2$.

(ii) subadditive if

$$\sigma(A_1 \cup A_2) \leq \sigma(A_1) + \sigma(A_2),$$

for all $A_1,\ A_2 \in \mathcal{A}(\Omega)$.

(iii) superadditive if

$$\sigma(A_1 \cup A_2) \geq \sigma(A_1) + \sigma(A_2),$$

for all $A_1,\ A_2 \in \mathcal{A}(\Omega)$ with $A_1 \cap A_2 = \emptyset$.

(iv) inner regular if

$$\sigma(A) = \sup\{\sigma(B) :\ B \in \mathcal{A}(\Omega),\ B \subset\subset A\},$$

for all $A \in \mathcal{A}(\Omega)$.

LEMMA 3.2. *Let $\sigma : \mathcal{A}(\Omega) \to [0,+\infty]$ be an increasing set function. The following statements are equivalent:*

(1) σ is is the restriction to $\mathcal{A}(\Omega)$ of a Borel measure on Ω;

(2) σ is subadditive, superadditive and inner regular;

(3) the set function

$$\nu(E) = \inf\{\sigma(A) :\ A \in \mathcal{A}(\Omega),\ E \subseteq A\}$$

is a Borel measure on Ω.

Proof. See e.g. [9, Theorem 14.23]. □

We will use the properties of increasing set functions to obtain the integral representation of the Γ-limit F_0. We begin with

LEMMA 3.3. *Let $\{F_h\}$ be a sequence of functionals in $\mathcal{F}(\mathcal{M})$. Suppose that for every $u \in W^1L_G(\Omega)$*

$$\sigma'(A) = \Gamma(L_B) - \liminf_h F_h(u, A)$$

and

$$\sigma''(A) = \Gamma(L_B) - \limsup_h F_h(u, A)$$

define inner regular increasing set functions. Then there exists a subsequence $\{F_{h_k}(u, A)\}$ which $\Gamma(L_B)$-converges for all $u \in W^1L_G(\Omega)$ and $A \in \mathcal{A}(\Omega)$.

Proof. Consider the countable family $\mathcal{R}$ of all finite unions of open rectangles of Ω with rational vertices. For every fixed sequence $\{F_h\}$ we can use a diagonal procedure and Theorem 2.1 to extract a subsequence $\{F_{h_k}(u, R)\}$ which $\Gamma(L_B)$-converges for all $R \in \mathcal{R}$ and $u \in W^1L_G(\Omega)$. Now, let $A \in \mathcal{A}(\Omega)$ and $u \in W^1L_G(\Omega)$. By hypothesis $\sigma'(A)$ and $\sigma''(A)$ define inner regular increasing set functions. This gives

$$\begin{aligned}
&\Gamma(L_B) - \liminf_h F_{h_k}(u, A) = \sigma'(A) = \\
&= \sup\{\sigma'(B) : \ B \in \mathcal{A}(\Omega), \ B \subset\subset A\} \\
&= \sup\{\sigma'(R) : \ R \in \mathcal{R}(\Omega), \ R \subset\subset A\} \\
&= \sup\{\sigma''(R) : \ R \in \mathcal{R}(\Omega), \ R \subset\subset A\} \\
&= \sup\{\sigma''(B) : \ B \in \mathcal{A}(\Omega), \ B \subset\subset A\} \\
&= \sigma''(A) = \Gamma(L_B) - \limsup\nolimits_h F_{h_k}(u, A).
\end{aligned}$$

We proceed by proving a fundamental estimate in L_B which will guarantee that the Γ-limits define inner regular increasing set functions. □

DEFINITION 3.4. *We say that F satisfies the L_B-fundamental estimate if for every A, A' and B in $\mathcal{A}(\Omega)$ with $A' \subset\subset A$ and $\alpha > 0$ there exists $M_\alpha > 0$ such that for all u, $v \in W^1L_G(\Omega)$ there exists a cut-off function ψ between A' and A such that*

$$F(\psi u+(1-\psi)v, A'\cup B) \le (1+\alpha)(F(u,A)+F(v,B)) + + M_\alpha \int_{(A\cap B)\setminus A'} B(u-v)\,dx+\alpha.$$

Moreover, we say that the class $\mathcal{F}(\mathcal{M})$ satisfies the L_B-fundamental estimate uniformly if every functional $F \in \mathcal{F}(\mathcal{M})$ satisfies the fundamental estimate and the constant $M_\alpha > 0$ can be chosen uniformly on $\mathcal{F}(\mathcal{M})$.

REMARK 3.5. *Let A, $A' \in \mathcal{A}(\Omega)$ with $A' \subset\subset A$. We say that ψ is a cut-off function between A' and A if ψ is smooth with compact support in A, $0 \le \psi \le 1$ and $\psi \equiv 1$ on A'.*

LEMMA 3.6. *The class $\mathcal{F}(\mathcal{M})$ satisfies the L_B-fundamental estimate uniformly.*

Proof. Let $F \in \mathcal{F}(\mathcal{M})$ and let A, A' and B in $\mathcal{A}(\Omega)$ with $A' \subset\subset A$. Define

$$\delta = \operatorname{dist}(A', \partial A)$$

and take $0 < \eta < \delta$ and $0 < r < \delta - \eta$. Let ψ be a cut-off function between

$$\{x \in A:\ \operatorname{dist}(x, A') < r\} \text{ and } \{x \in A:\ \operatorname{dist}(x, A') < r+\eta\},$$

with $|D\psi| \le 2/\eta$. Define the sets

$$B_r^\eta = \{x \in B:\ r < \operatorname{dist}(x, A') < r+\eta\},$$

$$I_1 = \{x \in B : \operatorname{dist}(x, A') \ge r+\eta\}$$

and

$$I_2 = \{x \in A' \cup B : \operatorname{dist}(x, A') \le r\}.$$

For u, $v \in W^1L_G(\Omega)$ a repeated use of the convexity and the Δ_2-

property of G yield

$$\begin{aligned}
& F(\psi u+(1-\psi)v, A'\cup B) \\
= & \int_{A'\cup B} f(x, \psi Du+(1-\psi)Dv+(u-v)D\psi)dx \\
= & \int_{I_1} f(x, Dv)dx+\int_{I_2} f(x, Du)dx+\int_{B_r^\eta} f(x, \psi Du+ \\
& (1-\psi)Dv+(u-v)D\psi)dx \\
\leq & F(u, A)+F(v, B)+ \\
& c\int_{B_r^\eta}(1+G_0(\psi Du+(1-\psi)Dv+(u-v)D\psi))dx
\end{aligned}$$

$$\begin{aligned}
\leq & F(u, A)+F(v, B)+ \\
& c\int_{B_r^\eta}(1+G_0(2(\frac{1}{2}(\psi Du+(1-\psi)Dv)+\frac{1}{2}(u-v)D\psi)))dx \\
\leq & F(u, A)+F(v, B)+ \\
& c\int_{B_r^\eta}(1+\beta G_0(\frac{1}{2}(\psi Du+(1-\psi)Dv))+\frac{1}{2}(u-v)D\psi))dx \\
\leq & F(u, A)+F(v, B)+ \\
& c\int_{B_r^\eta}(1+\frac{\beta\psi}{2}G_0(Du)+\frac{\beta(1-\psi)}{2}G_0(Dv))dx+ \\
& \int_{B_r^\eta}\frac{\beta^\kappa}{2}G_0((u-v)D\psi/|D\psi|))dx \\
\leq & F(u, A)+F(v, B)+\frac{c\beta}{2}\int_{B_r^\eta}(1+G_0(Du)+G_0(Dv))dx \\
& +\frac{c\beta^\kappa}{2}\int_{(A\cap B)\setminus A'} G_0((u-v)D\psi/|D\psi|)dx \\
\leq & F(u, A)+F(v, B)+\frac{c\beta}{2}\int_{B_r^\eta}(1+G_0(Du)+G_0(Dv))dx \\
& \frac{c\beta^\kappa}{2}\int_{(A\cap B)\setminus A'} B(u-v)dx
\end{aligned} \tag{7}$$

where $\kappa = 1-\frac{\log\eta}{\log 2}$. Now define

$$\mu(U)=\frac{c\beta}{2}\int_U(1+G_0(Du)+G_0(Dv))dx.$$

By the structure conditions

$$\mu(A\cap B)\le \frac{c\beta}{2}(m(A\cap B)+F(u,A)+F(v,B)).$$

Moreover, for every $N=1,\ 2,\ldots,$

$$\mu(A\cap B)\ge \sum_{k=1}^{N}\mu(\{x\in B:\delta\frac{k-1}{N}<\operatorname{dist}(x,A')<\delta\frac{k}{N}\}).$$

Consequently, for every $N=1,\ 2,\ldots$ there exists $k\in\{1,\ldots,\ N\}$ such that

$$\mu(\{x\in B:\delta\frac{k-1}{N}<\operatorname{dist}(x,A')<\delta\frac{k}{N}\})\le$$

$$\le\frac{c\beta}{2N}(m(A\cap B)+F(u,A)+F(v,B)).$$

Hence, for fixed $\alpha>0$, by chosing

$$N\ge\frac{1}{\alpha}\max\{\frac{c\beta}{2}m(A\cap B),\frac{c\beta}{2}\},\quad \eta=\frac{\delta}{N}\ \text{ and }\ r=\frac{k-1}{N}\delta,$$

we obtain

$$M_\alpha=\frac{c\beta^\kappa}{2},$$

which depends only on A, A', B, c and β and can thus be chosen uniformly in the class $\mathcal{F}(\mathcal{M})$. □

In the next two lemmas we apply the fundamental estimate to show that the Γ-limits satisfies the measure properties subadditivity and inner regularity.

LEMMA 3.7. *Let $\{F_h\}$ be a sequence in $\mathcal{F}(\mathcal{M})$ which satisfies the L_B-fundamental estimate as $h\to\infty$. Then*

$$F'(u,A'\cup B)\le F'(u,A)+F''(u,B)$$

and

$$F''(u,A'\cup B)\le F''(u,A)+F''(u,B),$$

for all $u\in W^1L_G(\Omega)$ and A, A' and B in $\mathcal{A}(\Omega)$ with $A'\subset\subset A$.

Proof. By Theorem 1.1 there exists two sequences $\{u_h\}$ and $\{v_h\}$ converging to u strongly in $L_B(\Omega)$ such that

$$F'(u,A) = \liminf_h F_h(u_h, A)$$

and

$$F''(u,B) = \limsup_h F_h(v_h, B).$$

If we now apply the L_B-fundamental estimate as $h \to \infty$ to the functions u_h and v_h with fixed $\alpha > 0$, there exist M_α and h_α such that for all $h > h_\alpha$ there exists a sequence of functions

$$w_h = \psi_h u_h + (1 - \psi_h) v_h,$$

where ψ_h are cut-off functions between A' and A such that

$$\begin{aligned} F_h(w_h, A' \cup B) \quad \leq \quad & (1+\alpha)(F(u_h, A) + F(v_h, B)) \\ & + M_\alpha \int_{(A\cap B)\setminus A'} B(u_h - v_h)\, dx + \alpha, \end{aligned}$$

Now $w_h \to u$ in $L_B(\Omega)$. Moreover, since convergence in $L_B(\Omega)$ implies B-mean convergence, see e.g. Kufner et. al. [6], p. 157, it follows that

$$\int_{(A\cap B)\setminus A'} B(u_h - v_h)\, dx \to 0.$$

Consequently,

$$\begin{aligned} F'(u, A' \cup B) \quad & \leq \quad \liminf_h F_h(w_h, A' \cup B) \\ & \leq \quad (1+\alpha)(\liminf_h F_h(u_h, A) + \liminf_h F_h(v_h, B)) + \alpha \\ & = \quad (1+\alpha)(F'(u,A) + F''(v,B)) + \alpha. \end{aligned}$$

Since α can be chosen arbitrarily the first inequality follows. The second inequality is proved the same way. □

The last lemma concerns inner regularity of the Γ-limits.

LEMMA 3.8. *Let $\{F_h\}$, F' and F'' be defined as in Lemma 3.4. Let $u \in W^1L_G(\Omega)$. If $F'(u,\cdot)$ and $F''(u,\cdot)$ are increasing set functions and if*

$$F''(u,A) \leq \tilde{C} \int_A (1 + G_0(Du))dx,$$

for all $A \in \mathcal{A}(\Omega)$, then $F'(u,\cdot)$ and $F''(u,\cdot)$ are inner regular and moreover $F''(u,\cdot)$ is subadditive.

Proof. Since $\{F_h\}$ satisfies the L_B-fundamental estimate the proof follows along the line of Proposition 11.6 in [2], by taking Lemma 3.4 into account. □

Proof of Theorem 2.2. We extend as above the functionals to $+\infty$ on $L_B(\Omega)\backslash W^1L_G(\Omega)$. By Lemma 3.3 $\{F_h\}$ satisfies the L_B-fundamental estimate. Therefore, by Lemma 3.5, the Γ-lower and Γ-upper limits define inner regular increasing set functions. Compactness thus follows from Lemma 3.2 and the measure properties again follows from Lemma 3.5 if we take Lemma 3.1 into account. □

4. Some final comments and concluding remarks

Theorem 2.2 opens the possibility to find representations of the Γ-limit for large classes of interesting problems. In particular in the periodic case, i.e. when f_h is of the form

$$f_h(x,\xi) = f(hx,\xi),$$

it is possible, with the obvious modifications, to apply classical homogenization methods analogous to those presented in for instance Dal Maso [9]. Moreover, for the case when f_h is of the form

$$f_h(x,\xi) = f(x,hx,..,h^m x,\xi),$$

one can mimic the reiterated homogenization techniques presented in [3] and obtain homogenization results. Similar compactness and homogenization results are clearly also obtainable for corresponding nonlinear parabolic operators by combining the compactness result in this paper with the G-convergence and multi-scale convergence methods described in e.g. [11, 10, 7, 8, 12, 13]. These interesting questions will be discussed in a forthcoming paper.

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Received July 4, 2001.