REPRESENTATION OF POST L-ALGEBRAS BY RINGS OF SETS (*)

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SOMMARIO. - *Usando la topologia di Priestly, si associa a ciascuna Post L-algebra P uno spazio compatto P* con ordinamento totale sconnesso e si mostra che P è isomorfa al Post L-anello dei sottoinsiemi chiusi crescenti di P*. Si determinano poi i Post L-spazi e si mostra ch'essi sono in corrispondenza biunivoca con le Post L-algebre. Si mostra infine che se L è finito, una α-Post L-algebra P = (B, L) è isomorfa ad un α-Post L-anello (di sottoinsiemi di P*), modulo un α-Post L-ideale, se e solo se B è un α-rappresentabile algebra di Boole.*

SUMMARY. - *Using the Priestly topology, we assign to each Post L-algebra P a compact totally order disconnected space P* and show that P is isomorphic to the Post L-ring of clopen increasing subsets of P*. Post L-spaces are identified and are shown to be in one to one correspondence with Post L-algebras. It is also shown that if L is finite, then an α-Post L-algebra P = (B, L) is isomorphic to an α-Post L-ring (of subsets of P*) modulo an α-Post L-ideal if and only if B is an α-representable Boolean algebra.*

In 1974 Saloni [4] used the fact that a Post algebra $P = (B, C)$ is isomorphic to the coproduct $B * C$ to define the dual space of P as $X \times Y$, where X and Y are the Stone spaces of B and C. Since a Post L-algebra is isomorphic to the coproduct of a Boolean alge-

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bra and the fixed lattice of constants L, Saloni's definition can be extended to include the dual space of a Post L-algebra. However, in this note we find it more convenient to use the ordered spaces of Priestly [3], rather than the classical spaces of Stone, in defining the dual space of a Post L-algebra.

In Section 2 we assign to each Post L-algebra P a compact totally order disconnected space P^* and show that P is isomorphic to the Post L-ring of all the clopen increasing subsets of P^*. We also define Post L-spaces and show that these spaces are in one to one correspondence with Post L-algebras. The relationship between Saloni's dual space and the one considered here is pointed out. In Section 3 we show that for a finite distributive lattice L, an α-Post L-algebra $P = (B, L)$ is isomorphic to an α-Post L-ring (of subsets of P^*) modulo an α-Post L-ideal if and only if B is α-representable. This generalizes the corresponding result for Post algebras given by Traczyk in [8].

1. Post L-algebras.

All lattices considered in this note will be distributive lattices with 0 and 1; all lattice homomorphisims will preserve 0 and 1; and all sublattices will have the same 0 and 1 as the containing lattice. The least upper bound and greatest lower bound of x and y will be denoted by $x + y$ and xy respectively. A finite subset $\{a_i\}$ of nonzero elements of a distributive lattice is called a *partition* of 1 if $\Sigma a_i = 1$ and $a_i a_{i'} = 0$ for $i \neq i'$.

DEFINITION 1.1 - Let L be a fixed distributive lattice. A distributive lattice P will be called a *Post L-algebra* if P has a Boolean sublattice B and a sublattice $L' \cong L$ such that every element $x \in P$ can be expressed uniquely as $x = \sum_{i=1}^{n} a_i l_i$, where n depends on x, $\{a_i : 1 \leq i \leq n\} \subseteq B$ is a partition of 1, $\{l_i : 1 \leq i \leq n\} \subseteq L'$, and $l_i \neq l_j$ if $i \neq j$. The above representation of x will be called the *minimal representation* of x by elements of B and L'. B and L' will be called the *underlying Boolean algebra* and the *lattice of costants* of P respectively.

Post L-algebras were introduced by Speed in [7] and further investigated by the author in [9] and [10]. It is shown in [10] that the class $Post_L$ of Post L-algebras can be defined as a class of similar abstract algebras in which the elements of L are constants (i.e. nullary operations) and, moreover, that $Post_L$ is equationally definable if and only if L is finite.

To simplify the notation we shall identify L with L' in Definition 1.1 and thus consider L as a sublatice of every Post L-algebra

P. It is shown in [7] that the Post L-algebra P with underlying Boolean algebra B and lattice of constants L is isomorphic to the coproduct $B * L$ of B and L. Consequently we shall denote P by $P = (B, L)$. If L is a finite chain with n elements, $n \geq 2$, then every Post L-algebra $P = (B, L)$ is a Post algebra of order n.

Homomorphisms between members of $Post_L$ will by called *Post L-homomorphisms* and subalgebras of members of $Post_L$ will be called *Post L-subalgebras*. These concepts can also be characterized as follows (cf. [10]):

(1.2) *A lattice homomorphism h of a Post L-algebra $P = (B, L)$ into a Post L-algebra $P' = (B', L)$ is a Post L-homomorphism if and only if $h[B] \subseteq B'$ and $h(l) = l$ for every $l \in L$.*

(1.3) *A sublattice P' of a Post L-algebra $P = (B, L)$ is a Post L-subalgebra of P if and only if P' is generated by $B' \cup L$ for some Boolean sublattice B' of B.*

DEFINITION 1.4 - Let X be a set and L a distributive lattice. A ring R of subsets of X will be called a *Post L-ring of sets* if R has a subring $L' \cong L$ and a Boolean subring (i.e. a field) F such that every member of R has a unique minimal representation by members of F and L'. R will be denoted by $R = (F, L)$.

A Post L-algebra $P = (B, L)$ will be called *α-complete* if P is an α-complete lattice. A Post L-ring of sets $L = (F, L)$ will be called an *α-Post L-ring of sets* if R is an α-ring of sets.

2. Post L-spaces.

In this section we shall assign to each Post L-algebra $P = (B, L)$ a compact totally order disconnected space P^*. The space P^* will be the dual space of the distributive lattice P as was defined by Priestly in [3].

A subset S of a partially ordered set is called *increasing* if for every $x \in S$ and every $y \in Y, y \geq x$ implies $y \in S$. A *decreasing* set is defined similarly. A topological space $(X, \mathfrak{I}, \leq)$ is called an *ordered topological space* (briefly, an *ordered space*) if it has a partial ordering relation $\leq$. $(X, \mathfrak{I}, \leq)$ is called *totally order disconnected* if given $x, y \in X$ with $x \not\geq y$, then there exist a clopen increasing set U containing x and a clopen decreasing set V containing y such that $U \cap V = \phi$.

Let A be a distributive lattice and for every $a \in A$, let $X_a = \{0, 1\}$ be the two-element lattice endowed with the discrete topology. Let 2^A be the topological product of $\{X_a : a \in A\}$ and define $\leq$ on 2^A

by $f \leq g$ if and only if $f(a) \leq g(a)$ for all $a \in A$ and $f, g \in 2^A$. Then the subspace $X \subseteq 2^A$ consisting of all lattice homomorphisms of A onto $\{0,1\}$ will be called the (Priestly) *dual space* of A.

We shall denote the dual space of a distributive lattice A by A^*. It is shown in [3] that A^* is a compact totally order disconnected space and, moreover, A is isomorphic to the lattice of all clopen increasing subsets of A^*.

If $(X, \mathfrak{J}, \leq)$ is an ordered space, then the lattice of all clopen increasing subsets of X will be called the *dual lattice* of X. We shall denote the dual lattice of X by X^* also. The duality between distributive lattices and compact totally order disconnected spaces is given by the following theorem (cf. [3]:

(2.1) *Let* $(X, \mathfrak{J}, \leq)$ *be a compact totally order disconnected space,* $A = X^*$, *and* $(X', \mathfrak{J}', \leq') = A^*$. *Then* $(X, \mathfrak{J}, \leq)$ *and* $(X', \mathfrak{J}', \leq')$ *are homeomorphic as topological spaces and isomorphic as partically ordered sets.*

If B is a Boolean algebra, then in the dual space $B^* = (X, \mathfrak{J}, \leq)$ of B, the relation $\leq$ is the trivial order (i.e. $x \leq y$ if and only if $x = y$) and in this case B^* reduces to the classical Stone space of B. More generally, the following is proved in [3]:

(2.2) *A distributive lattice* A *is a Boolean algebra if and only if* A^* *has the trivial order.*

LEMMA 2.3: *Let* A *and* L *be distributative lattices and let* $A * L$ *be their coproduct. Then the spaces* $(A * L)^*$ *and* $A^* \times L^*$ *are homeomorphic as topological spaces and isomorphic as partially ordered sets.*

Proof: Since $A^* \times L^*$ is a compact totally order disconnected space, it suffices by (2.1) to show that the lattice D of all the clopen increasing subsets of $A^* \times L^*$ is isomorphic to $A * L$. If U is a clopen increasing subset of A^* and V is a clopen increasing subset of L^*, then $U \times V \in D$. Hence D contains the sublattice D' generated by all sets $U \times V$, where U and V are clopen increasing subsets of A^* and L^* respectively. To show the converse containment $D \subseteq D'$, let $E \in D$. Since E is compact, it is the union of a finite number of members of the base for the product topology of $A^* \times L^*$; that is, $E = \bigcup_{i=1}^{n} (A_i \times L_i)$, where each A_i and L_i are nonempty clopen subsets of A^* and L^* respectively. Moreover, we may assume that $L_i \cap L_j = \phi$ for all $i \neq j$. Then using the disjointness of the sets L_i and the hypothesis that E is increasing, it is not difficult to show that each A_i, $1 \leq i \leq n$, is increasing. Now the sets L_i need not be increasing. However, if we let $L'_i = \cup \{L_j : A_i \times L_j \subseteq E, 1 \leq j \leq n\}$, $1 \leq i \leq n$, then $E = \bigcup_{i=1}^{n} (A_i \times L'_i)$ and we shall show that each L'_i is increasing. Suppose that for some $i = i_o$, L'_{i_o} is not increasing. Then there is $b \in L'_{i_o}$ and $y \in L^*$ such that $y \geq b$ and $y \notin L'_{i_o}$, and it follows from the definition of L'_{i_o} that there is $a \in A_{i_o}$ such that $(a, y) \notin E$. On the other hand, $(a, y) \in E$ since $(a, y) \geq (a, b) \in E$ and E is increasing. This contradiction shows that L'_{i_o} is increasing. Thus we have shown that $E = \bigcup_{i=1}^{n} (A_i \times L'_i)$, where each $A_i \times L'_i \in D'$. Hence $E \in D'$ and it follows that $D = D'$. Now since A and L are isomorphic to the lattices of clopen increasing subsets of A^* and L^* respectively, it follows from the representation theorem for coproducts (cf. [5]) that $D' \cong A^* L$. Hence $D \cong A^* L$ and the proof is complete.

We define the (Priestly) *dual space* of a Post L-algebra $P = (B, L)$ to be the space $B^* \times L^*$, where B^* and L^* are dual spaces of B and L respectively. Thus a Post L-algebra $P = (B, L)$ has two dual spaces: the space $B^* \times L^*$ and the dual space P^* of P when P is considered as a distributive lattice. That these two dual spaces are homeomorphic as topological spaces and isomorphic as partially ordered sets is given by Lemma 2.3. Consequently we shall use the notation $P^* = B^* \times L^*$ to denote the dual space of $P = (B, L)$.

Let L be a distributive lattice and let L^* be the dual space of L. A compact totally order disconnected space $(X', \mathfrak{I}', \leq')$ will be called a *Post L-space* if there exists a compact totally order disconnected space $(X, \mathfrak{I}, \leq)$ with the trivial order such that $(X', \mathfrak{I}', \leq')$ and $X \times L^*$ are homeomorphic as topological spaces and isomorphic as partially ordered sets. (Note that the space $(X, \mathfrak{I}, \leq)$ is the Stone space of a Boolean algebra; namely, the Boolean algebra of its clopen subsets).

LEMMA 2.4: *Let $X \times L^*$ be a Post L-space and let F be the field of clopen subsets of X. Then the lattice D of clopen increasing subsets of $X \times L^*$ is isomorphic to the Post L-ring of sets (F, L).*

Proof: By (2.1) X can be considered as the dual space F^* of F. Hence it follows from the proof of Lemma 2.3 that $D \cong F * L$. Hence $D \cong (F, L)$.

We define the *dual algebra* of a Post L-space $X \times L^*$ to be the Post L-algebra of all the clopen increasing subsets of $X \times L^*$. The duality between Post L-algebras and Post L-spaces is given by the following theorem which follows immediately from (2.1) and Lemmas 2.3 and 2.4.

THEOREM 2.5: (i) *Let $P = (B, L)$ be a Post L-algebra and let $P^* = B^* \times L^*$ be the dual space of P. Then P^* is a Post L-space and the ring of all clopen increasing subsets of P^* is a Post L-ring of sets isomorphic to P.*

(ii) *Let $Y = X \times L^*$ be a Post L-space, $P = (B, L)$ the dual algebra of Y, and $P^* = B^* \times L^*$ the dual space of P. Then Y and P^* are homeomorphic as topological spaces and isomorphic as partially ordered sets.*

REMARK: If A is distributive lattice with (Priestly) dual space A^*, then A is also isomorphic to the lattice of clopen decreasing subsets of A^*. Moreover, for any ordered space $(X, \mathfrak{J}, \leq)$, the family $\mathfrak{L}$ of all open decreasing subsets of X is a base for some topology $\mathfrak{J}'$ which is called the *lower topology* for X. If $P_n = (B, C)$ is a Post algebra of order $n \geq 2$, then the (Priestly) dual space of P_n is $P_n^* = B^* \times C^*$. On the other hand, the dual space of P_n which Saloni defined in [4] is the space $P'_n = B^* \times C'$, where C' is the space C^* with the lower topology. The two spaces P_n^* and P'_n are not homeomorphic (the former is a T_2 space while the latter is not even T_1); however, the Post algebra of sets which is isomorphic to P_n is the same in both cases, provided we define the dual space of a Post L-space to be the lattice of clopen decreasing (rather than increasing) sets.

3. α-Representable Post L-algebras.

An ideal I of a Post L-algebra $P = (B, L)$ is called a *Post L-ideal* if there exists an ideal I_o of B such that $I = \{x \in P : x \leq b$ for some $b \in I_o\}$. It is easy to verify that $I_o = I \cap B$.

If I is an ideal of a distributive lattice A, then we shall denote the congruence relation determined by I by $\Theta(I)$; that is, $\Theta(I) = \{(x, y) \in A^2 : x + u = y + u$ for some $u \in I\}$. We shall write A/I instead of $A/\Theta(I)$ and denote the elements of A/I by $[x]_I, x \in A$.

LEMMA 3.1: *Let I be a proper Post L-ideal of $P = (B, L)$ and let $I_o = I \cap B$. Then P/I is a Post L-algebra isomorphic to $(B/I_o, L)$.*

Proof: Let $L' = \{[l]_I : l \in L\}$. Then L' is a sublattice of P/I and we shall show that the mapping $h: L \to L'$ defined by $h(l) = [l]_I$ is an isomorphism. Clearly h is a homomorphism of L onto L'. To show that h is one to one, let $[l_1] = [l_2]$. Then $l_1 + u = l_2 + u$ for some $u \in I$. Since I is a Post L-ideal, there is $u_o \in I_o$ such that $u \leq u_o$, and it follows that $l_1 + u_o = l_2 + u_o$. Hence $1 \cdot l_1 \leq u_o + l_2$, and it follows from the criterion for coproducts (cf. Theorem VII. 1 of [1]) that $1 = u_0$ or $l_1 \leq l_2$. Since I is proper, $l_1 \leq l_2$. Similarly it follows that $l_2 \leq l_1$, so $l_1 = l_2$ and h is one to one. Next let $B' = \{[b]_I : b \in B\}$ and let $g : B/I_o \to B'$ be defined by $g([b]_{I_o}) = [b]_I$. Then g is a homomorphism of B/I_o onto P'. Moreover if $[b_1]_I = [b_2]_I$, then it follows that $b_1 + u_o = b_2 + u_o$ for some $u_o \in I_o$. Hence $[b_1]_{I_o} = [b_2]_{I_o}$, so g is one to one and $B' \cong B/I_o$. Thus to show that $P/I \cong (B', L')$ it suffices to show that every element $[x]_I \in P/I$ has a unique minimal representation by elements of B' and L'. Let $[x]_I \in P/I$ and let $x = \sum_{i=1}^{n} b_i l_i$ be the minimal representation of x by elements of B and L. Then $[x]_I = \sum_{i=1}^{n} [b_i]_I [l_i]_I$ is a minimal representation of $[x]_I$. Suppose also that $[x]_I = \sum_{i=1}^{k} [a_i]_I [m_i]_I$ is another minimal representation of $[x]_I$ by elements of B' and L'. Then it follows that $(\sum_{i=1}^{n} b_i l_i) + u_o = (\sum_{i=1}^{k} a_i m_i) + u_o$ for some $u_o \in I_o$. Let $\bar{u}_0$ denote the complement of u_o; then $\sum_{i=1}^{n} (b_i \bar{u}_o) l_i = \sum_{i=1}^{k} (a_i \bar{u}_o) m_i$. Thus if we let

$$b = (\sum_{i=1}^{n} b_i \bar{u}_o)^- \text{ and } a = (\sum_{i=1}^{k} a_i \bar{u}_o)^-,$$

$$\text{then } b \cdot 0 + \sum_{i=1}^{n} (b_i \bar{u}_o) l_i = a \cdot 0 + \sum_{i=1}^{k} (a_i \bar{u}_o) m_i$$

and the expression on the two sides of the last equation are both minimal representations of the same element in P. Thus by the uniqueness of the minimal representation in P, it follows that $a = b$, $n = k, a_i = b_i$, and $l_i = m_i$ for $1 \leq i \leq n$. This completes the proof of the lemma.

A Post L-algebra $P = (B, L)$ is called *α-representable* if $P \cong R/I$, where R is an α-Post L-ring of sets and I is an α-Post L-ideal of R. It follows from this that if $P = (B, L)$ is α-representable, then P is α-complete. Moreover, it is shown in [7] that if L is finite, then $P = (B, L)$ is α-complete if and only if B is an α-complete Boolean algebra. On the other hand, if L is an infinite α-complete lattice, then it can be shown that $P = (B, L)$ is α-complete if and only if B s finite. Thus it is natural to examine the α-representability of $P = (B, L)$ only when L is finite.

For the remainder of this section we shall use the following notation. If $P^* = B^* \times L^*$ is the dual space of a Post L-algebra $P = (B, L)$, then $F_o(B)$ will denote the field of clopen subsets of $B^*, F_\alpha(B)$ the α-field (of subsets of B^*) which is generated by $F_o(B)$, and N_α the α-ideal (of $F_\alpha(B)$) consisting of all sets of the α-categorary belonging to $F_\alpha(B)$ (cf. [6]). It is known [6] that B is α-representable if and only if $N_\alpha \cap F_o(B) = (\phi)$.

THEOREM 3.2. *Let L be a finite distributive lattice. Then a Post L-algebra $P = (B, L)$ is α-representable if and only if B is an α-complete, α-representable Boolean algebra. Moreover if P is α-representable, then $P \cong (F_\alpha(B), L)/N$, where N is the α-Post L-ideal generated by N_α.*

Proof: Suppose first that B is an α-complete, α-representable Boolean algebra. Let $P^* = B^* \times L^*$ be the dual space of P and identify L with the lattice of clopen increasing subsets of L^*. Let R be the ring (of subsets of P^*) generated by $\{A \times U : A \in F_\alpha(B), U \in L\}$. Then by the representation theorem for coproducts of distributive lattices (cf. [5], $R \cong F_\alpha(B) * L \cong (F_\alpha(B), L)$. Moreover, since $F_\alpha(B)$ is an α-field, R is an α-Post L-ring of sets. We shall show that $P \cong R/N$, where N is the α-Post L-ideal of R generated by N_α; that is, $N = \{E \in R : E \subseteq U$ for some $U \in N_\alpha\}$. By Lemma 3.1, $R/N \cong (F_\alpha(B)/N_\alpha, L)$. But since B is α-representable, $F_\alpha(B)/N_\alpha \cong B$ (cf. [6]). Hence $(B, L) \cong R/N$, so (B, L) is α-representable.

Conversely, suppose that $P = (B, L)$ is α-representable and let $P \cong (F, L)/I$, where F is an α-field of sets and I is an α-Post L-ideal of (F, L). Then by Lemma 3.1, $(F, L)/I \cong (F/I_o, L)$, where $I_o = F \cap I$. Thus $P = (B, L) \cong (F/I_o, L)$ and it follows that $B = F/I_o$. Since F is an α-field, B would be α-representable once we show that I_o is an α-ideal of B. Now since P is α-complete, so is B (cf. [7]). Moreover, using the representation theorem for the coproducts of distributive lattices, it is not difficult to show that for every $S \subseteq B$ with $|S| \leq \alpha$, the least upper bound of S in B coincides with the least upper bound of S in P. From tthis and the fact that I is an α-ideal of P it follows that I_o is an α-ideal of B.

As a corollary to the last theorem we obtain the following result which is proved in [8] and [2].

COROLLARY 3.3: *A Post algebra (B, C) is α-representable if and only if B is an α-representable Boolean algebra.*

Since every σ-complete Boolean algebra is σ-representable (cf. [6]), Theorem 3.2 yields the following.

COROLLARY 3.4: *If L is a finite distributive lattice, then every σ-complete Post L-algebra is σ-representable.*

REFERENCES

[1] BALBES, R. and DWINGER, PH., *Distributive Lattices,* University of Missouri Press, Columbia, 1974.

[2] DWINGER, PH., *Notes on Post algebras I, II,* Indag. Math. 28 (1976), 462-478.

[3] PRIESTLY, H., *Representation of distributive lattices by means of ordered Stone spaces,* Bull. London Math. Soc 2 (1970), 186-190.

[4] SALONI, Z., *A topological representation of Post algebras and free Post algebras,* Colloq. Math. 31 (1974), 1-9.

[5] SIKORSKI, R., *Products of abstract algebras,* Fund. Math. 39 (1952), 211-228.

[6] SIKORSKI, R., *Boolean algebras,* Springer-Verlag (third ed.) New York, Berlin, Göttingen, Heidelberg, 1969.

[7] SPEED, T. P., *A note on Post algebras,* Colloq. Math 24 (1971), 37-44.

[8] TRACZYK, T., *A generalization of the Loomis-Sikorski Theorem,* Colloq. Math. 12 (1964), 156-161.

[9] YAQUB, F. M., *Free Post L-algebras,* Algebra Universalis 9 (1979), 45-53.

[10] YAQUB, F. M., *Equationally definable Post L-algebras, Algebra Universalis* (to appear).