ON THE GENERAL HYPERPLANE SECTION OF A CURVE IN CHAR. P (*)

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SOMMARIO. - *Sia C una curva di grado d in $\mathbf{P}^n$, $n \geq 4$. In caratteristica positiva la generica sezione iperpiana, $C \cap H$, di C può avere varie patologie (e.g. il suo gruppo di monodromia G può non essere il gruppo simmetrico S_d). Assumiamo che il lemma delle trisecanti valga per C. Allora si dimostra qui che $d = 2^k$ con $k \geq n-1$ e G è isomorfo al gruppo affine $AGL(k,2)$ su F_2 e l'isomorfismo rispetta l'azione di G su $C \cap H$ e l'azione di $AGL(k,2)$ su F_2^k.*

SUMMARY. - *Let $C \subset \mathbf{P}^n$, $n \geq 4$, be a curve of degree d; in characteristic $p > 0$ the general hyperplane section of C may have monodromy group, G, different from the full symmetric group. Assume that the trisecant lemma holds for C and that $d > 22$. Here we prove that $d = 2^k$ for some integer $k \geq n-1$ and $G \equiv AGL(k,2)$ (the affine group over F_2): this isomorphism respects the action of G on the general hyperplane section and the action of $AGL(k,2)$ on F_2^k. Furthermore if $n \geq 5$, then $p = 2$.*

This note is an addendum to the characteristic p part of the very nice thesis of J. Rathmann (see [R]).

Fix an algebraically closed field $\mathbf{F}$. All the schemes considered in this note will be algebraic over $\mathbf{F}$. Fix an integral, non degenerate curve $C \subset \mathbf{P}^n$. Let $\mathbf{P}^{n^*}$ be the set of hyperplanes of $\mathbf{P}^n$ and $\Gamma \subset C \times \mathbf{P}^{n^*}$ be the incidence correspondence. The projection $p_2 \colon \Gamma \to \mathbf{P}^{n^*}$ is finite and separable (Bertini's theorem). Let G be the Galois group of the normal extension of the function field $\mathbf{F}(\mathbf{P}^{n^*})$ of $\mathbf{P}^{n^8}$ generated by $\mathbf{F}(\Gamma)$. G is called the monodromy group of the general hyperplane section of C, because it acts as permutation group of the general fiber of p_2. If $\mathrm{char}(\mathbf{F}) = 0$,

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then G is always the full symmetric group on $\deg(C)$ elements ([H]). This implies that for a general hyperplane H, any subset S of $H \cap C$ with $\text{card}(S) \leq n$, spans a linear space of dimension card $(S) - 1$; we will say that the "linear general position" holds for C (or for $C \cap H$) if this last property is satisfied (in [SV] it is called "general position"). However the linear general position property may fail if chart $(\mathbf{F}) = p > 0$ (and hence the Galois group can be smaller); as remarked in [R], to have the linear general position property (and much more!) it is sufficient to know that the monodromy group contains the alternating group. From now on we assume always $\text{char}(\mathbf{F}) = p > 0$. As in [B] we will say that the curve C is very strange if the linear general position property fails for C. In [R], th 2.4, J. Rathmann showed that if $n \geq 4$ and C is not very strange then either G is the full symmetric group or the alternating group or one of the Mathieu groups M_d, $d = 11, 12, 23$ or 24, in their standard representation (hence $d = \deg(C)$). Rathmann's theorem used the classification of 4-transitive finite groups. Indeed in [R], th. 2.4, it is given also the list of 3-transitive groups (with the obvious slip of the 3-transitive group M_{22}); this list obviously gives the possible monodromy groups for not very strange C even for $n = 3$. Simultaneously Rathmann proved that if $d > 24$ and C is very strange either a general secant line to C intersects C at more than 2 points (i.e. the "trisecant lemma" fails) or every plane spanned by points of C contains at least 4 points of C. The aim of this note is the proof that if $n \geq 5$ and $p \neq 2$ the trisecant lemma fails for all very strange curves; if $p = 2$ this may not be true (see the examples in 2.2). More precisely we prove the following result.

THEOREM 0.1. *Fix a very strange curve C spanning $\mathbf{P}^n$, $n \geq 4$; set $d := \deg(C)$; let G be the monodromy group of the general hyperplane section of C. Assume that the trisecant lemma holds for C and that $d > 22$. Then $d = 2^k$ for some integer $k \geq n-1$ and $G \simeq AGL(k,2)$ (the affine group over F_2; this isomorphism respects the action of G on the general hyperplane section and the action of $AGL(k,2)$ on F_2^k. Furthermore if $n \geq 5$, then $p = 2$.*

The proof of this statement uses results and methods from [R] and [BH]. The proof of 0.1 will be given in the first part of §2. Then in 2.1

we will show that 0.1 gives some restrictions on the degree of very strange curves. In 2.2 and 2.3 we will consider in more detail the case $p = 2$; in 2.2 we will give the promised examples, while in 2.3 we will analyze the structure of their general linear sections, and show that if $k = n - 1$ (the minimal degree case) up to a projective transformation the example given is the only rational very strange curve for which the general secant line is not multisecant. In 2.4 we will discuss why the results in [B] and 0.1 allows one to avoid an unpleasant restriction (characteristic 0 or the assumption that the linear general position principle holds for the general linear section) in [SV], th. 2.

This paper is dedicated to the memory of Giorgio.

§1. Fix a generically etale dominant morphism $a : X \to Y$ with Y integral and a general point $x \in Y_{\text{reg}}$ over which a is finite and etale. Set $n := \deg(a)$. We want to define the monodromy group $M(X/Y)$ (or $M(Y)$ if there is no danger of misunderstanding) of a. Let $t : T \to Y$ be a morphism with T integral and $x \in t(T)$; assume that the pull-back (by T) $b : X_T \to T$ has n disjoint section s_i, $1 \leq i \leq n$, i.e. X_T splits into n components mapped isomorphically by b onto T. Fix points $e, f \in t^{-1}(x)$. Use t to identify $b^{-1}(e)$ and $b^{-1}(f)$ with $a^{-1}(x)$; then $s_i(e) \to s_i(f)$ induces a permutation of $a^{-1}(x)$. The group of permutations of $a^{-1}(x)$ obtained in this way for general x is called the monodromy group $M(X/Y)$ of a ([BH]). If X is integral, let L be the normalization of the degree n extension of function field $\mathbf{F}(X)/\mathbf{F}(Y)$ and let G be the Galois group of $L/\mathbf{F}(Y)$. By [BH], prop. 1, G contains $M(X/Y)$ (by 1.2 below $G = M(X/Y)$, but we do not need this result). In this section, in view of future applications, we give 2 addenda (1.1 and 1.2) to [BH]; to prove 0.1 we do not need 1.2 and also 1.1 will not be used in a very essential way.

LEMMA 1.1. *If T is irreducible in Y_{reg} and $X \to Y$ is generically unramified at the points of T, there is an inclusion of $M(X_T/T)$ into $M(X/Y)$.*

Proof. By assumption there is an open subset W of Y such that $W' := W \cap T \neq \emptyset$ and $X \to Y$ is etale over W. Of course, if we have a section of

$X_{W'} \to W'$ this section will not lift in general to a section of $X_W \to W$, even on some Zariski open dense subset of W. Assume, after an etale base change $T' \to T$, that $X_{T'} \to T'$ has n sections s_i defining an element g of $M(X_T/T)$. We may find etale morphisms $Z \to A \to W$ with $A \to W$ extending $T \to T$ and such that X_Z/Z has d sections w_i which, on a suitable base point, agree with the value of the pull-back of the sections s_i. By the irreducibility of T and T', these sections w_i will agree, when both are defined, with the pull-back of the sections s_i. This means that $g \in M(X/Y)$. ◇

REMARK 1.2. If X is integral, then the monodromy group $M := M(X/Y)$ is the Galois group G of the normalization L of $\mathbf{F}(X)/\mathbf{F}(Y)$. Indeed take (passing to a Zariski open subvariety) a covering $t : Y' \to Y$ with $\mathbf{F}(Y') = L$. By the definition of the splitting field of a polynomial, the fiber product $j : X' \to Y'$ of $X \to Y$ by t splits into n components mapped isomorphically over the base, i.e. j has n sections s_i. Take a general $x \in X$. Choosing suitable $a, b \in t^{-1}(x)$, we may assume that the permutation of the fiber over x given by $s(a)$ and $s(b)$ is any fixed element of G, buy the definition of the action of the Galois group as permutation group. ◇

§2. *Proof of* 0.1: Fix an integral non-degenerate curve $C \subset \mathbf{P}^n$, $n \geq 4$ such that a general secant line to C intersects C exactly at two points, but such that a general plane spanned by 3 points of C contains exactly a points of C with $a \geq 4$. Set $d := \deg(C)$. Let G be the monodromy (or, if you prefer, the Galois) group associated to C. The assumption on the secant lines to C means that G is 3-transitive, while the assumption on the secant planes means that G does not act 4-transitively (e.g. use semicontinuity and 1.1).

Step 1: Fix 3 general points P_i, $i = 1,2,3$ of C; let V be the plane spanned by these points. Consider the family T of hyperplanes containing V. By 1.1 the family T shows that the stabilizer in G of 3 elements has orbits of cardinality at least $d-a$. Thus card $(G) \geq d(d-1)(d-2)/(d-a)$. Now we want to check that $d-2 \geq (a-1)(a-2)$. Taking a general linear projection we reduce to the case $n = 4$. Take a general hyperplane

H and fix a point $P \in C \cap H$ and let π be the projection of H into $\mathbf{P}^2$ from P. By assumption $S := \pi((C \cap H)\backslash P)$ is formed by $d-1$ points such that each line containing 2 of this points contains $a-1$ points of S; projecting S from one of its points, we get the inequality.

Step 2: since the monodromy group is 3-transitive but not 4-transitive and $d > 22$, by [R], th. 2.4 (plus the case of M_{22}), we have to check only two cases for G.

(a) First assume that there is a prime r and an integer $k > 0$ such that $d = 1 + q$, where $q = r^k$, and G is a group containing $PSL(2,q)$ and contained in Aut($PSL(2,q)$) $=: P\Gamma L(2,q)$ (see below) such that the representation of G is isomorphic to the restriction of the following action of $P\Gamma L(2,q)$ on the projective line with $q+1$ elements: and element g of $P\Gamma L(2,q)$ is given by an automorphism t of F_q as field and by $a, b, c, d \in F_q$ with $ad - bc \neq 0$, so that $g(x) = (at(x) + b)/(ct(x) + d)$ if $x \in F_q \subset \mathbf{P}(F_q^2)$; thus card $(G) \leq$ card$(P\Gamma L(2,q)) = (q+1)q(q-1)k$, contradicting the lower bound for card (G) and the inequality for a found in step 1.

(b) Now assume $d = 2^k$ with $k > 0$ and $G \simeq AGL(k,2)$ (the affine linear group of F_2^k). Note that any element of G maps coplanar points into coplanar points (for the monodromy group use semicontinuity), But consider the action of $AGL(k,2)$ on F_2^k; $AGL(k,2)$ acts 4-transitively on the 4-ple of non-coplanar (in F_2^k) points. Thus $a = 4$. Now assume $n \geq 5$. For the same reason a general t-dimensional linear subspace V_t of $\mathbf{P}^n$ (with $1 \leq t \leq n-2$) spanned by points of C contains exactly 2^t points of C. We will see now that for $t = 3$ this implies that $p = 2$. Projecting V_3 into $\mathbf{P}^2$ from one of its points we find a configuration T of 7 points $[P_i]$, $1 \leq i \leq 7$, such that each line containing at least 2 points of T contains exactly 3 points of T; up to a change of indices, we choose homogeneous coordinates such that $P_1 = (1,0,0)$, $P_2 = (0,1,0)$, $P_3 = (0,0,1)$, $P_4 = (1,1,1)$ and then $P_5 = (1,1,0)$, $P_6 = (1,0,1)$, $P_7 = (0,1,1)$; $p = 2$ is equivalent to the fact that P_5, P_6 and P_7 are collinear. This gives easily also the inequality $k \geq n-1$ (or see remark 2.1 just below). $\diamond$

REMARK 2.1. We will show now that 0.1 imposes some restrictions (depending on p) on the possible (d, n, p) such that there is a non-degenerate very strange curve C in $\mathbf{P}^n$; in particular we will show that

$d \geq 2p^{n-2}$ if $p > 2$. Indeed fix such a curve C. Assume $n > 3$. By [R], th. 2.4, C is strange. Let $D \subset \mathbf{P}^{n-1}$ be the image of C under the projection from the strange point of C; D spans $\mathbf{P}^{n-1}$ and $\deg(D) \leq \deg(C)/p$. First assume $p \geq 2$. Since C is very strange, by 0.1 a general secant line to C is multisecant. Thus the same happens to D. Thus D is very strange; if $n-1 > 3$, we may continue, until we arrive at a very strange curve $A \subset \mathbf{P}^3$; we find $\deg(C) \geq p^{n-3}\deg(A)$; projecting A into $\mathbf{P}^2$ we see that $\deg(A) \geq 2p$. But if we know more on C or A, we have other bounds on $\deg(A)$ and $\deg(C)$; let $v(1)$ (resp. $v'(1)$) be the number of points at which a general secant line intersects C (resp. A); note that by construction $v'(1) \geq v(1)$; note that A is not contained in a surface of degree less that $v'(1)$; thus projecting A from one of its points we get a plane curve of degree at least $v'(1)$; thus we find $\deg(A) \geq 1 + v'(1)(v'(1)-1)$, hence $\deg(C) \geq p^{n-3}(1 + v(1)(v(1)-1))$. Now assume $p = 2$. The same proof works until we are in $\mathbf{P}^4$ and then show exactly when it fails (see also 2.3). ◇

REMARK 2.2. Here we assume $p = 2$. For every $n \geq 4$ and every $k \geq n-1$ we give an example (very well-known) of a rational non degenerate curve $C \subset \mathbf{P}^n$ with $\deg(C) = 2^k$ and $AGL(k,2)$ as Galois group. Fix n powers $a(1) < \ldots < a(n)$ of 2 with $a(1) = 1$ and $a(2) = 2$. Take as C the rational curve with affine parametrization $(t^{a(1)}, \ldots, t^{a(n)})$. Let V_t be a general linear space spanned by $C \cap V_t$ and with $\dim(V_t) = t$. It is easy to check that in V_t $C \cap V_t$ has exactly the linear structure of F_2^k. When $k \geq 4$ from these examples we get easily non-trivial families of rational examples: e.g. if $2a(n-1) < a(n)$, in the parametrization take $t^{a(n)} + c_1 t^{2a(n-1)} + \ldots$ instead of $t^{a(n)}$. ◇

PROPOSITION 2.3. *Fix $n \geq 4$ and any C as in the exceptional case of* 0.1 *(i.e. with monodromy group $AGL(k,n)$ with $k \geq n-1$). For $1 \leq t \leq n-2$ write V_t for a general t-dimensional linear space spanned by $C \cap V_t$. Then:*

(a) Card$(C \cap V_t) = 2^t$ for every t.

(b) For every t the set $C \cap V_t$ does not depend (up to a projective transformation) upon the choice of C.

(c) If $k = n-1$ the following holds: the normalization N of C has

genus $g \leq 1$ and if $g = 0$ then C is uniquely determined (up to a projective transformation) and hence it has a parametrization as in 2.2 with $a(i) = 2^{i-1}$.

Proof. Part (a) was proved in the last part of the proof of 0.1.

Proof of part (b): This is essentially an elementary assertion about points which in the terminology of 2.4 below are in linear semi-uniform position and are "very few". Fix a subset S of a linear space V, $\dim(V) = m$ and $S \subset V$, card $(V) = 2^{m-1}$, S spanning V, such that for every t and every t-dimensional linear subspace V_t spanned by $S \cap V_t$, we have card$(S \cap V_t) = 2^{t-1}$. Fix any point $P \in S$. Consider the projection h_P of $V\backslash\{P\}$ from P onto a projective subspace, W, with $r := \dim(W) = m - 1$. Set $T := (S\backslash\{P\})$. We have card $(T) = 2^{r-1}$; furthermore for every $t \geq 0$ and every t-linear subspace W_t of W with $\dim(W_t) = t$ and W_t spanned by $T \cap W_t$, we have card $(T \cap W_t) = 2^{t-1}$. First we show (by induction on r) the uniqueness of T up to projective transformations, i.,e. that T is uniquely determined if we give $T' \subset T$ with card $(T') = r + 2$ and T' in linear general position. The case $r = 2$ was the last part of the proof of 0.1. Now assume $r \geq 2$ and the result for $r - 1$; fix (W', T') and project from one of the points of T' we get exactly the case $r - 1$, for cardinality reasons (i.e. the fact that for such W_1, W_{r-1}, we have $W_1 \cap W_{r-1} \cap T \neq \emptyset$ even when W_{r-1} does not contain W_1); essentially we know $r + 1$ projections of T and this is more than enough. Now in the same way we pass from $(h_P(V\backslash\{P\}, h_P\{S\backslash\{P\}))$ (for all possible choices of $P \in S$) to (V, S).

Proof of part (c): Now we assume $k = n - 1$; we want to prove that $g \leq 1$ and the uniqueness of C when $g = 0$. We use induction on n; the starting point of the induction ($n = 4$), will be done at the end; for the moment assume $n \geq 4$. Now the same trick as in the proof of part (b) works even if $t = n - 1$. Fix C and C' with the same properties as in 0.1, C being given by the parametrization of 2.2 with $a(j) = 2^{j-1}$ for all j. We may choose a general hyperplane H and then move C' by an element of Aut$(\mathbf{P}^n)$ in such a way that $H \cap C = H \cap C'$ and that the strange point P, of C is the strange point of C'. Projecting C and C' from P we find very strange curves D and D' in $\mathbf{P}^{n-1}$. For degree reasons and the proof of 2.1, D and D' must be very strange curves of minimal degree 2^{n-2} in $\mathbf{P}^{n-1}$ (in particular $P \notin C'$ and the projection $C' \to D'$ is purely inseparable of degree 2). Since the projection is purely inseparable, C' and D' are birational; thus by induction $g \leq 1$. Assume $g = 0$. By induction D and D' are projectively equivalent; we may assume $D = D'$. Hence if $(u, u^2, \ldots)$ is the parametrization of $D' = D$, setting $u = t^2$ and $P = (0, 1; 0; \ldots; 0)$, we get a parametrization $(A(t), t^2, t^4 \ldots)$ of C'; changing coordinates we may even assume that $A(t)$ has no power of 2 as addenda with non-zero coefficient. Writing down the condition that the plane spanned by 3 general points of C contains a fourth point x of C' (knowing even that $D = D'$, i.e. knowing the projection of x in H from P we get the linearity of A. Now assume $n = 4$, hence $d = 16$ and use the same notations. Now $\deg(D') \leq 4$ for degree reasons and the inseparability of the projection (hence $p = 2$ even if $n = 4$). Thus D' (hence C') has geometric genus $g \leq 1$. Assume $g = 0$. C' is a projection of the rational normal curve $Y \subset \mathbf{P}^8$ from a 3-dimensional linear space Q. Take a parametrization $X_i = u_0^i u_1^{8-i}$, $0 \leq i \leq 8$ of Y; we see that every tangent line to Y intersects the linear space $Q' := \{X_0 = X_2 = X_4 = X_6 = X_8 = 0\}$; since C' is strange, Q has the same property; since Y spans $\mathbf{P}^8$, we have $Q = Q'$; this means exactly that $C = C'$ (up to a projective transformation). ◊

We stress that we do not have examples of curves C as in 2.3 for the case $g = 1$.

REMARK 2.4. Here we will show that 0.1 and 2.1 allows one to avoid the assumption of "general position" ("linear general position" in our terminology) from [SV], th. 2 i.e. to avoid the distinction between characteristic 0 and characteristic p in that statement (not in the proofs!). We do not claim that 0.1 and 2.1 are necessary for this purpose; the results and methods of [B] would be sufficient, but 0.1 and 2.1 allows one some simplifications. Recall from [B] that a finite set $S \subset \mathbf{P}^m$ is called in linear semi-uniform position if for every t with $0 < t < m$, there is an integer $v(t)$ such that every linear space $V \subset \mathbf{P}^m$ with $\dim(V) = t$ and V spanned by $S \cap V$, we have card $(S \cap V) = v(t)$; in particular S is in linear general position if and only if it is in linear semi-uniform position with $v(t) = t + 1$ for every t. The general hyperplane section of every integral non degenerate curve is in linear semi-uniform postition. By the structure of the proofs in [SV] (inductive proofs of th.1 and th. 2, the Main Lemma in §4) it is obvious that to reach our goal it is sufficient to fix an irreducible variety V of degree d and assume that a general 0-dimensional linear section S of V is formed by d points in linear semi-uniform position (but not in linear uniform position) in $\mathbf{P}^m$, and prove [SV], th. 1, for S. In this situation, part (i) of th. 1 in [SV] is [B], th. 0.1 (or see the proof of part (ii)). We leave to the reader to check (using also 2.1 and [R], §2) the cases with $d \leq 22$. Assume $d > 22$. By 0.1 we may assume that $v(1) > 2$ (i.e. $s = 1$ in the terminology of [B]). By Castenuovo-Mumford's lemma it is sufficient to check that $h^1(\mathbf{P}^m, I_S(t)) = 0$ for all t such that $(t+1) \geq \{d/m\}$ with "$\{x\}$" meaning "minimal integer $\geq x$"}. Look at [B], lemma 2.5; in our situation we have $s = 1$, $e := v(1) - 1 > 1$, $x := e - 1$. Fix the integer t with $(t+1) \geq \{d/m\}$. By [B], lemma 2.5 it is sufficient to check the following inequality:

$$1 + e((t - x)(m - 1) + 1) \geq d \qquad (\$)$$

If $e = 2$ (hence $x = 1$), ($) is easy. By the second part of 2.1, we have $t \geq 2x + 2$; hence we conclude if $e \geq 3$. ◇

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