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A Result about Selectors in non-Archimedean Spaces

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Summary. - *A sort of strong completeness property for subsets of a non-Archimedean space is defined. On the subsets which satisfy this property there exists a Vietoris continuous selector. The set of discrete closed subsets of* $\mathbb{R}$ *has a continuous selector when it is equipped with the Vietoris topology induced by the Michael line. Some properties of the tree of a non-Archimedean space are used.*

1. Preliminaries

Throughout this paper, all spaces are Hausdorff spaces. Let $\mathfrak{F}(X)$ denote the space of all non-empty closed subsets of X equipped with the Vietoris topology [7, 5]. A continuous selector (or simply selector) on a subspace $\mathfrak{A}$ of $\mathfrak{F}(X)$ is a continuous map $\tau : \mathfrak{A} \longrightarrow X$ such that $\tau(F) \in F$ for every $F \in \mathfrak{A}$. This means that for every neighborhood W of $\tau(F)$ there exist open sets $\Omega, V_1, \ldots, V_n$ of X, with $F \subseteq \Omega$ and $V_i \cap F \neq \emptyset \, \forall i$, such that for every $G \in \mathfrak{A}$ with $G \subseteq \Omega$ and $G \cap V_i \neq \emptyset \, \forall i$, we have $\tau(G) \in W$.

We recall a fundamental theorem about selectors in metric spaces which was obtained in two different times. A metric space is said

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to be strongly zero-dimensional provided that every open cover is refined by a partition of open sets.

Theorem 1.1. *Let X be a strongly zero-dimensional metric space. The following conditions are equivalent:*

1) there exists a selector on $\mathfrak{F}(X)$;

2) X is a completely metrizable space.

The implication 2⇒1 was found by different authors [4, 6]; the implication 1⇒2 appeared several years later [11].

One may wonder whether the above theorem can be extended to ω_μ-metric spaces. For any uncountable regular cardinal ω_μ, recall that a ω_μ-metric space is a uniform space which admits a base of uniform coverings $\mathfrak{B} = \{\mathcal{U}_\alpha\}_{\alpha<\omega_\mu}$ which is well ordered (by star-refinement) by ω_μ. These spaces are particular non-Archimedean spaces with uncountable punctual character at any non-isolated point.

A space X is said to be non-Archimedean provided that there exists a base $\mathfrak{B}$ for the open sets such that two members of $\mathfrak{B}$ are either disjoint or contained in each other. It is well-known that X is non-Archimedean iff there exists a base $\mathfrak{B}$ which is a tree with respect to reverse inclusion [8, 10, 9].

On ω_μ-metric spaces, the existence of a selector is fully solved by the following theorem [2].

Theorem 1.2. *Let X be a non-archimedean space with uncountable punctual character at every non-isolated point. Then the following conditions are equivalent:*

1) X is scattered (i.e. every subset has an isolated point);

2) there exists a continuous selector on $\mathfrak{F}(X)$.

In [1] we have investigated a strong completeness condition in ω_μ-metric spaces. This condition, called $\mathfrak{B}$-completeness, allowed us to prove the existence of continuous selections for lower semicontinuous multivalued functions. In the present paper we shall define $\mathfrak{B}$-completeness in non-Archimedean spaces and prove that on the space $\mathfrak{F}_{\mathfrak{B}}(X)$ consisting of all $\mathfrak{B}$-complete subsets there exists a continuous selector.

2. $\mathfrak{B}$-completeness

Let X be a non-Archimedean space and let $\mathfrak{B}$ be a base for the open sets which is a tree for the reverse inclusion. This means that for every $B \in \mathfrak{B}$ the set $\hat{B} = \{A \in \mathfrak{B} : A \supsetneq B\}$ is well-ordered, that is every subset of $\hat{B}$ has a maximum for the inclusion of sets. In this context, all elements of $\mathfrak{B}$ are called *balls*. A chain of balls is a linearly ordered subset of $\mathfrak{B}$; a branch is a maximal chain. For any point $x \in X$, denote by $\mathcal{P}(x)$ the branch consisting of the basic neighborhoods of x, that is $\mathcal{P}(x) = \{B \in \mathfrak{B} : x \in B\}$.

Let Y be a subset of X and let $\mathcal{C}$ be a chain of balls. We say that $\mathcal{C}$ meets Y if $C \cap Y \neq \emptyset$ for every $C \in \mathcal{C}$.

In the following definition we extend to non-Archimedean spaces the notion of d-complete space introduced in [3] (see also [9], where the term "spherically complete" is used).

Definition 2.1. *A subset Y of X is said to be $\mathfrak{B}$-complete provided that, for every chain of balls $\mathcal{C}$ which meets Y, we have that $(\cap\mathcal{C}) \cap Y \neq \emptyset$.*

A path in $\mathfrak{B}$ is a chain $\mathcal{C}$ such that $\hat{B} \subseteq \mathcal{C}$ for every $B \in \mathcal{C}$. Of course every branch is a path and the intersection of two paths is a path.

Proposition 2.2. *Y is $\mathfrak{B}$-complete iff for every path which meets Y we have that its intersection meets Y.*

Since the balls which contain a point form a branch, we have immediately the following fact:

Proposition 2.3. *Every $\mathfrak{B}$-complete subset is closed.*

Examples of closed subsets which fail to be $\mathfrak{B}$-complete may be found in [1, Proposition 1.5]. In section 3 we study $\mathfrak{B}$-completeness in a classical non- Archimedean space.

Let $\mathfrak{F}_{\mathfrak{B}}(X)$ denote the space of non-empty $\mathfrak{B}$-complete subsets of X with the Vietoris topology.

Before proving the existence of a selector on $\mathfrak{F}_{\mathfrak{B}}(X)$, we must state some notations and properties about the tree $\mathfrak{B}$.

For every B belonging to $\mathfrak{B}$, the order type of $\hat{B}$ is called the height of B and denoted by $h(B)$.

If α is any ordinal, the α-th level of $\mathfrak{B}$ is the set $\mathfrak{B}_\alpha = \{B \in \mathfrak{B} : h(B) = \alpha\}$. The elements of $\mathfrak{B}_\alpha$ are mutually disjoint clopen sets and $\mathfrak{B}_0$ is a covering. If $\mathcal{C}$ is a path the height of $\mathcal{C}$ is $h(\mathcal{C}) = \min\{\alpha : \mathcal{C} \cap \mathfrak{B}_\alpha = \emptyset\}$. Obviously, the order type of $\mathcal{C}$ is $h(\mathcal{C})$. If A and B are elements of $\mathfrak{B}$, we define the equivalence relation $A \sim B$ if $\hat{A} = \hat{B}$. A *node* of $\mathfrak{B}$ is any equivalence class. Obviously, every node $\mathcal{N}$ is a subset of some level $\mathfrak{B}_\alpha$, and $\mathfrak{B}_0$ is the unique node of the 0-th level. If $\mathcal{C}$ is a path, then the node of $\mathcal{C}$ is $\mathcal{N}(\mathcal{C}) = \{B \in \mathfrak{B} : \hat{B} = \mathcal{C}\}$.

LEMMA 2.4. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be different paths and consider the node $\mathcal{N} = \mathcal{N}(\mathcal{P}_1 \cap \mathcal{P}_2)$. If $W_1 \in \mathcal{N} \cap \mathcal{P}_1$ and $W_2 \in \mathcal{N} \cap \mathcal{P}_2$, then $W_1 \cap W_2 = \emptyset$.*

THEOREM 2.5. *There exists a continuous selector on $\mathfrak{F}_{\mathfrak{B}}(X)$.*

Proof. For every node $\mathcal{N}$, we fix a well-ordering on $\mathcal{N}$. If $Y \in \mathfrak{F}_{\mathfrak{B}}(X)$, we construct by transfinite induction a path which meets Y in a single point.

Let B_0 be the minimum element of the node $\mathfrak{B}_0$ such that $B_0 \cap Y \neq \emptyset$. If B_0 is not a singleton, let B_1 be the minimum element of $\mathcal{N}(\{B_0\}) = \{B \in \mathfrak{B}_1 : B \subseteq B_0\}$ such that $B_1 \cap Y \neq \emptyset$. Suppose that we have defined a path $\mathcal{C} = \{B_\gamma\}_{\gamma<\alpha}$ such that:

$\boxed{1}$ $B_\gamma \cap Y \neq \emptyset$ for every $\gamma < \alpha$;

$\boxed{2}$ B_γ is the minimum element of the node $\mathcal{N}(\hat{B}_\gamma)$ for which $B_\gamma \cap Y \neq \emptyset$.

The $\mathfrak{B}$-completeness of Y ensures that $(\cap\mathcal{C}) \cap Y \neq \emptyset$. If this intersection is not a singleton, there exists a ball B of the node $\mathcal{N}(\mathcal{C})$ such that $B \cap Y \neq \emptyset$: denote by B_α the minimum of such balls. Since we can go on up to the height of the tree, we construct a branch $\mathcal{C}(Y)$ with properties $\boxed{1}$ and $\boxed{2}$ and moreover $\cap\mathcal{C}(Y)$ is a single point $q = \tau(Y)$ belonging to Y. Observe that $\mathcal{C}(Y) = \mathcal{P}(q)$.

It remains to prove that τ is Vietoris continuous. Let V be a ball containing $q = \tau(Y)$. We are going to construct an open subset Ω of X containing Y such that for every $\mathfrak{B}$-complete subset Z, if $Z \subseteq \Omega$ and $Z \cap V \neq \emptyset$, then $\tau(Z) \in V$.

Let α be the height of V. For every $x \in Y \backslash V$, the path $\mathcal{P}(x) \cap \hat{V}$ has height less than or equal to α.

Consider the node $\mathcal{N} = \mathcal{N}(\hat{V} \cap \mathcal{P}(x))$. Let W_x be the unique element of $\mathcal{N}$ belonging to $\mathcal{P}(x)$ and let V_x be the unique element of $\mathcal{N}$ belonging to $\hat{V}$. By Lemma 2.4, we have $W_x \cap V_x = \emptyset$, hence $W_x \cap V = \emptyset$. Since V_x is the minimum element of $\mathcal{N}$ which meets Y, then V_x is less then W_x in the order of $\mathcal{N}$.

Let $\Omega = V \cup \bigcup_{x \in Y \setminus V} W_x$. Suppose that a $\mathfrak{B}$-complete set Z is contained in Ω and meets V. Let $r = \tau(Z)$ and suppose by contradiction that $r \notin V$. Then there exists $x \in Y \setminus V$ such that $r \in W_x$, hence $W_x \in \mathcal{P}(r) \cap \mathcal{P}(x)$.

Since $W_x \notin \hat{V}$, then $\hat{V} \cap \mathcal{P}(r) = \hat{V} \cap \mathcal{P}(x)$. In the node $\mathcal{N}(\hat{V} \cap \mathcal{P}(r))$, both V_x and W_x meet Z and Y. Since V_x is less than W_x, then by $\boxed{2}$ the ball W_x cannot belong to $\mathcal{P}(r)$, a contradiction. ☐

Since every finite subset is $\mathfrak{B}$-complete, we have:

COROLLARY 2.6. *Let X be a non-Archimedean space. Then there exist a selector on the finite subsets of X.*

3. The Michael line

The Michael line is the topological space M obtained by adding to the usual topology of $\mathbb{R}$ all irrational numbers as isolated points [5, 10]. The closed sets of M have the form $C \setminus S$, where C is a closed set of $\mathbb{R}$ and S is a subset of $\mathbb{R} \setminus \mathbb{Q}$.

M is a hereditarily paracompact non-metrizable space with character ω_0. Furthermore, M is a non-Archimedean space with a base $\mathfrak{B}$ which is a tree of height $\omega_0 + 1$. Such a base may be constructed in the following way.

Consider the dense subset of $\mathbb{R}$ given by $D = \{\pi + \frac{k}{2^n} : \ k \in \mathbb{Z},\ n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, the n-level of $\mathfrak{B}$ is:

$$\mathfrak{B}_n = \{\ ]\pi + \frac{k}{2^n}, \pi + \frac{k+1}{2^n}[\ : \ k \in \mathbb{Z}\} \cup \{\ \{\pi + \frac{k}{2^n}\} : \ k \in \mathbb{Z}\}$$

while $\mathfrak{B}_{\omega_0} = \{\ \{s\} : \ s \notin \mathbb{Q} \cup D\}$.
Since $\bigcap_n]\pi, \pi + \frac{1}{2^n}[= \emptyset$, the space M is not $\mathfrak{B}$-complete. With a similar argument, one sees that no set containing a usual interval is $\mathfrak{B}$-complete.

One may wonder whether there exists a selector on the hyperspace of all non- empty closed subsets of M. Since $\mathbb{Q}$ is closed in M, the answer is negative by [6, Theorem 6.1]. However the selector does exist on the hyperspace of $\mathfrak{B}$-complete subsets, which are characterized in the next proposition.

Proposition 3.1. *The following conditions are equivalent:*
i) $E \in \mathfrak{F}_{\mathfrak{B}}(M)$.
ii) In the usual topology, E *is a closed subset without limit points in* D.

Proof. If E is a subset of $\mathbb{R}$, we denote by E' the set of limit points of E in the usual topology.
i)⇒ii) Take any point of D, say $d = \pi + \frac{j}{2^n}$. Since both paths

$$\mathcal{P}^+ = \{\ ]d, d + \frac{1}{2^m}[\ :\ m \geq 0\}, \qquad \mathcal{P}^- = \{\ ]d - \frac{1}{2^m}, d[\ :\ m \geq 0\}$$

have empty intersection, then $E' \cap D = \emptyset$.
If we take $x \in E'$, then x cannot belong to D. Therefore for each n there exists an interval $J_n \in \mathfrak{B}_n$ containing x. Since $J_n \cap E \neq \emptyset$ for each n, then $x \in E$ because $\{x\} = \bigcap_n J_n$.

ii)⇒i) It is enough to prove that if an infinite path $\mathcal{P}$ meets E then $\cap\mathcal{P}$ is a point of E. Since the path $\mathcal{P}$ is infinite, it consists of intervals $J_n =]a_n, b_n[$, where J_{n+1} is strictly contained in J_n and a_n and b_n are points of D. Since E has no limit points in D, both sequences a_n and b_n are not eventually constant. Thus the point $x = \sup a_n = \inf b_n$ belongs to every J_n and consequently $x \in E' \subseteq E$. □

Remark 3.2. Let $\mathfrak{G}(\mathbb{R})$ be the subspace of $\mathfrak{F}(\mathbb{R})$ consisting of the closed sets without limit points in D. Notice that $\mathfrak{G}(\mathbb{R})$ and $\mathfrak{F}_{\mathfrak{B}}(M)$ are different topological spaces with the same underlying set. The proof of [6, Proposition 5.1] shows that there is no selector on $\mathfrak{G}(\mathbb{R})$, while Proposition 3.1 proves that the selector does exist on $\mathfrak{F}_{\mathfrak{B}}(M)$. Consequently, on the hyperspace of discrete closed sets of $\mathbb{R}$ there exists a selector which is continuous for the Vietoris topology of M.

While the union of a finite number of $\mathfrak{B}$-complete sets is $\mathfrak{B}$-complete, in general the intersection of two $\mathfrak{B}$-complete sets is not necessarily $\mathfrak{B}$-complete.

Example 3.3. Denote by T the set of all ordinals less than or equal to ω_1 equipped with the topology induced by the base which has the following α-levels, for each $\alpha < \omega_1$:

$$\mathfrak{B}_\alpha = \{\{0\},\ \{1\},\ \{2\},\ \ldots,\ \{\alpha\},\ [\alpha+1, \omega_1]\}$$

The subset $\mathbb{N}$ of T is not $\mathfrak{B}$-complete, although $\mathbb{N} \cup \{\beta\}$ is $\mathfrak{B}$-complete, for every $\beta \notin \mathbb{N}$.

In contrast with Example 3.3, Proposition 3.1 shows that the family of $\mathfrak{B}$-complete subsets of the Michael line is closed under arbitrary intersections (so they form a topology, which fails to be Hausdorff). One can easily see that this situation occurs in every non-Archimedean space with a base for which every node of the ω_0-level is a singleton.

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