FREE PRODUCTS OF COMMUTATIVE RINGS WITH AMALGAMATION (*)

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SOMMARIO. - *Si dimostra un teorema che dà condizioni sufficienti sopra una classe $\mathcal{K}$ di anelli commutativi affinchè esistano in $\mathcal{K}$ prodotti liberi con amalgamazione. Questo teorema viene poi usato per mostrare l'esistenza di prodotti liberi con amalgamazione nella classe di tutti gli anelli che soddisfano all'equazione $x^n = x$. Nel caso speciale $n = 2$ si ritrova un risultato noto per gli anelli di Boole.*

SUMMARY. - *We prove a theorem giving sufficient conditions on a class $\mathcal{K}$ of commutative rings in order that free products with amalgamation exist in $\mathcal{K}$. This theorem is then used to show that free products with amalgamation exist in the class of all rings satisfying the equation $x^n = x$. The special case where $n = 2$ gives a known result for Boolean rings.*

Let $\mathcal{K}$ be a class of commutative rings and let $\{A_t\}_{t \in T} \subseteq \mathcal{K}$. Let $B \in \mathcal{K}$ such that for every $t \in T$, there exists a monomorphism $f_t : B \to A_t$. The *free product of* $\{A_t\}_{t \in T}$ *in* $\mathcal{K}$ *with amalgamated subring* B is a pair $(A, \{g_t\}_{t \in T})$, where $A \in \mathcal{K}$ and for every $t \in T$, $g_t : A_t \to A$ is a monomorphism and the following conditions are satisfied:

(i) For every $t_1, t_2 \in T$, $g_{t_1} f_{t_1} = g_{t_2} f_{t_2}$.

(ii) A is generated by $\bigcup_{t \in T} g_t(A_t)$.

(iii) If $R \in \mathcal{K}$ and $\{h_t\}_{t \in T}$ is a set of homomorphisms such that $h_t : A_t \to R$ and for every $t_1, t_2 \in T$, $h_{t_1} f_{t_1} = h_{t_2} f_{t_2}$, then there exists a homomorphism $h : A \to R$ such that $h g_t = h_t$ for every $t \in T$.

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We say that *free products with amalgamation exist* in $\mathcal{K}$ if the free product of $\{A_t\}_{t \in T}$ in $\mathcal{K}$ with amalgamated subring B exists for every $\{A_t\}_{t \in T} \subseteq \mathcal{K}$ and $B \in \mathcal{K}$.

The existence of free products with amalgamation in the class of all Boolean rings (with unity) was proved in [1], and free products with amalgamation in classes of universal algebras are discussed in [5]. In this note we prove a theorem (Theorem 1) giving sufficient conditions on a class $\mathcal{K}$ of commutative rings in order that free products with amalgamation exist in $\mathcal{K}$. We then use this theorem to show (Theorem 2) that free products with amalgamation exist in the class of all rings satisfying the equation $x^n = x$ (for all x and a fixed integer $n > 1$). The case where $n = 2$ gives the result for Boolean rings which we referred to earlier in [1]. Finally, we consider the existence of free products with amalgamation in the class of all rings A with the property that for every $x \in A$ there exists an integer $n(x) > 1$ such that $x^{n(x)} = x$ (See Theorem 3).

Free products with amalgamation are closely related to the following amalgamation property. A class $\mathcal{K}$ of commutative rings has the *amalgamation property* if for every $A_1, A_2, B \in \mathcal{K}$ and for every monomorphisms $f_1 : B \to A_1$ and $f_2 : B \to A_2$, there exist $A \in \mathcal{K}$ and monomorphisms $g_1 : A_1 \to A$ and $g_2 : A_2 \to A$ such that $g_1 f_1 = g_2 f_2$.

The amalgamation property has been investigated for a number of algebraic systems and a detailed discussion of this property with references to the literature is given in [4]. Clearly, if free products with amalgamation exist in a class $\mathcal{K}$ of commutative rings, then $\mathcal{K}$ has the amalgamation property. The converse, however, does not hold: The class $\mathcal{F}$ of all fields has the amalgamation property [4] but free products with amalgamation do not exist in $\mathcal{F}$ (not even free products exist in $\mathcal{F}$ [5]). It is not difficult to show, however, that if $\mathcal{K}$ is a variety (i. e. an equationally defined class), then $\mathcal{K}$ has the amalgamation property if and only if free products with amalgamation exist in $\mathcal{K}$ (see Lemma 2).

Throughout the following, $\mathcal{K}$ will denote a variety of commutative rings. Moreover, for every $\{A_t\}_{t \in T} \subseteq \mathcal{K}$ and $B \in \mathcal{K}$ such that for every $t \in T$, there exists a monomorphism $f_t : B \to A_t$, we define the ideal $I(\{A_t\}_{t \in T}, B)$ as follows. Let $(E, \{i_t\}_{t \in T})$ be the free product of $\{A_t\}_{t \in T}$ in $\mathcal{K}$ ([5], p. 103), and for simplicity of notation identify each A_t with $i_t(A_t)$. Then $I(\{A_t\}_{t \in T}, B)$ is the ideal of E generated by $\{f_{t_1}(x) - f_{t_2}(x) \mid t_1, t_2 \in T, x \in B\}$.

The following two lemmas follow easily from the preceding definitions.

LEMMA 1. *The free product of* $\}A_t\{_{t\in T}$ *in* $\mathcal{K}$ *with amalgamated subring* B *exists if and only if* $I(\}A_t\{_{t\in T}, B) \cap A_t = (0)$ *for every* $t \in T$.

PROOF. Let $(I\}A_t\{_{t\in T}, B) = I$. To show the necessity of the condition, let $(A, \}g_t\{_{t\in T})$ be the free product of $\}A_t\{_{t\in T}$ in $\mathcal{K}$ with amalgamated subring B. Since E is the free product of $\}A_t\{_{t\in T}$, there exists a homomorphism $g: E \to A$ such that for every $t \in T$, $g \mid A_t = g_t$, where $g \mid A_t$ denotes the restriction of g to A_t. Let J be the kernel of g. Then for every $x \in B$, $g(f_{t_1}(x) - f_{t_2}(x)) = 0$, ant it follows from the definition of I that $I \subseteq J$. But for every $t \in T$, $J \cap A_t = (0)$, hence $I \cap A_t = (0)$.

Conversely, suppose that $I \cap A_t = (0)$ for every $t \in T$, and let g_t be the restriction to A_t of the natural homomorphism of E onto E/I. Then it can be shown, in exactly the same way as in ([1], p. 228), that $(E/I, \}g_t\{_{t\in T})$ is the free product of $\}A_t\{_{t\in T}$ in $\mathcal{K}$ with amalgamated subring B.

LEMMA 2. *Let* $\mathcal{K}$ *be a variety of commutative rings. Then free products with amalgamation exist in* $\mathcal{K}$ *if and only if* $\mathcal{K}$ *has the amalgamation property.*

PROOF. Suppose first that the amalgamation property holds in $\mathcal{K}$. Let $\}A_t\{_{t\in T} \subseteq \mathcal{K}$ and $B \in \mathcal{K}$ such that for every $t \in T$, there exists a monomorphism $f_t: B \to A_t$. Let $I = I(\}A_t\{_{t\in T}, B)$. We shall show that $I \cap A_t = (0)$ for every $t \in T$. Suppose $I \cap A_{t_0} \neq (0)$ for some $t_0 \in T$, and let $a \in I \cap A_{t_0}$, $a \neq 0$. Clearly I is also generated by $\}f_{t_0}(x) - f_t(x) \mid t \in T,\ x \in B\{$. Hence

$$a = \sum_{i=1}^{n} r_i(f_{t_0}(x_i) - f_{t_i}(x_i)) + n_i(f_{t_0}(x_i) - f_{t_i}(x_i)), \dots (*)$$

where $r_i \in E$, $x_i \in B$, and n_i is an integer. Since the amalgamation property holds in $\mathcal{K}$, there exist $C \in \mathcal{K}$ and monomorphisms $g_i: A_{t_i} \to C$, such that $g_{t_i} f_{t_i} = g_{t_j} f_{t_j}$ for all i, j, $0 \leq i, j \leq n$. For every $t \in T$ such that $t \neq t_i$, $0 \leq i \leq n$, let $g_t: A_t \to C$ be the zero homomorphism. Since E is the free product of $\}A_t\{_{t\in T}$, there exists a homomorphism $g: E \to C$ such that $g \mid A_t = g_t$ for every $t \in T$. Then from equation (*),

$$g(a)=\sum_{i=1}^{n} g(r_i)\,(gf_{t_0}(x_i)-gf_{t_i}(x_i))+n_i\,(gf_{t_0}(x_i)-gf_{t_i}(x_i))$$

$$=\sum_{i=1}^{n} g(r_i)\,(g_{t_0}f_{t_0}(x_i)-g_{t_i}f_{t_i}(x_i))+n_i\,(g_{t_0}f_{t_0}(x_i)-g_{t_i}f_{t_i}(x_i))$$

$$=0,\ \text{since}\ g_{t_i}f_{t_i}=g_{t_j}f_{t_j}\,.$$

On the other hand, since $a \in A_{t_0}$ and g_{t_0} is a monomorpism, $g(a)=$ $=g_{t_0}(a)\neq 0$. This contradiction shows that $I \cap A_t=(0)$ for all $t \in T$. Hence, by Lemma 1, free products with amalgamation exist in $\mathcal{K}$. The converse is obvious.

We now prove the main theorem.

THEOREM 1. *Let $\mathcal{K}$ be a variety of commutative rings satisfying the following two conditions :*

(1) *For every $A \in \mathcal{K}$, A is semisimple (i.e. the Jacobson radical of A is (0)).*

(2) *For every $A \in \mathcal{K}$ and every subring B of A, a proper ideal M of B is maximal if and only if $M=B \cap M'$ for some maximal ideal M' of A.*

Then free products with amalgamation exist in $\mathcal{K}$.

PROOF. By Lemma 2, it suffices to show that the amalgamation property holds in $\mathcal{K}$. Thus let A_1, A_2, $B \in \mathcal{K}$ and suppose that there are monomorphisms $f_1 : B \longrightarrow A_1$ and $f_2 : B \longrightarrow A_2$. Let $(E, \{i_1, i_2\})$ be the free product of A_1 and A_2 in $\mathcal{K}$ and for simplicity of notation identify A_i with $i_i(A_i)$, $i=1, 2$. Let I be the ideal of E generated by $\{f_1(x)-f_2(x) \mid x \in B\}$. We shall show that $I \cap A_i=(0)$, $i=1, 2$. Suppose that $I \cap A_1 \neq (0)$, and let $a \in I \cap A_1$, $a \neq 0$. Since A_1 is semisimple, there exists a maximal ideal M_1 of A_1 such that $a \notin M_1$. Let $N_1=M_1 \cap f_1(B)$. Then by condition (2), $N_1=f_1(B)$ or N_1 is a maximal ideal of $f_1(B)$. Suppose that $N_1=f_1(B)$. Let $h_1 : A_1 \longrightarrow A_1/M_1$ be the natural homomorphism, and let $h_2 : A_2 \longrightarrow A_1/M_1$ be the zero homomorphism. Since E is the free product of A_1 and A_2, there is a homomorphism $h : E \longrightarrow A_1/M_1$ such that $h \mid A_i=h_i$, $i=1, 2$. Now since $a \in I$,

$$a=\sum_{j=1}^{n} r_j\,(f_1(x_j)-f_2(x_j))+n_j\,(f_1(x_j)-f_2(x_j))\,, \dots\,, (*)$$

where $r_j \in E$, $x_j \in B$, and n_j is an integer. Thus

$$h(a) = \sum_{j=1}^{n} h(r_j)(h_1 f_1(x_j) - h_2 f_2(x_j)) + n_j (h_1 f_1(x_j) - h_2 f_2(x_j)) = 0,$$

since $h_i f_i(x) = 0$ for all $x \in B$, $i = 1, 2$. On the other hand, since $a \notin M_1$, $h(a) = h_1(a) \neq 0$. This contradiction shows that $N_1 \neq f_1(B)$. Thus N_1 is a maximal ideal of $f_1(B)$. Hence the ideal $N_2 = f_2 f_1^{-1}(N_1)$ is maximal in $f_2(B)$. Hence by condition (2), there is a maximal ideal M_2 of A_2 such that $M_2 \cap A_2 = N_2$. Let $h_i' : A_i \longrightarrow A_i/M_i$, $i = 1, 2$, be the natural homomorphism. Then it follows from condition (1) that for every $i = 1, 2$, A_i/M_i is a field and $f_i(B)/N_i$ is a subfield of A_i/M_i. Since the amalgamation property holds in the class of all fields [4], there exists a field F and monomorphisms $h_i'' : A_i/M_i \longrightarrow F$, $i = 1, 2$, such that $h_1'' h_1' f_1 = h_2'' h_2' f_2$. Moreover F can be chosen such that $F \in \mathcal{K}$. Since E is the free product of A_1 and A_2, there is a homomorphism $h : E \longrightarrow F$ such that $h \mid A_i = h_i'' h_i'$, $i = 1,2$. Now from (*),

$$h(a) = \sum_{j=1}^{n} h(r_i)(h_1'' h_1' f_1(x_j) - h_2'' h_2' f_2(x_j)) + n_j (h_1'' h_1' f_1(x_j) - h_2'' h_2' f_2(x_j)) = 0.$$

On the other hand, since $a \notin M$, $h(a) = h_1(a) \neq 0$. This contradiction shows that $I \cap A_1 = (0)$. Similarly $I \cap A_2 = (0)$.

Now let $C = E/I$, $g : E \longrightarrow E/I$ be the natural homomorphism, and $g_i = g \mid A_i$, $i = 1, 2$. Since $I \cap A_i = (0)$, each g_i is a monomorphism. Moreover, since $f_1(x) - f_2(x) \in I$ for every $x \in B$, $g(f_1(x) - f_2(x)) = 0$. Hence $g_1 f_1 = g_2 f_2$. This shows that the amalgamation property holds in $\mathcal{K}$ and completes the proof of the theorem.

We now apply Theorem 1 to the equationally defined class $\mathcal{L}$ which is defined as follows. Let $n > 1$ be a fixed integer, and let $\mathcal{L}$ be the class of all rings A satisfying the equation $x^n = x$ for all $x \in A$. It is known [3] that for every $A \in \mathcal{L}$, A is commutative and semisimple. Moreover, it is shown in [2] that for every $A \in \mathcal{L}$, A has the *congruence extension property*; that is, for every subring B of A, if I is an ideal of B, then $I = B \cap I^*$ for some ideal I^* of A.

THEOREM 2. *Free products with amalgamation exist in the class $\mathcal{L}$.*

PROOF. We show that conditions (1) and (2) of Theorem 1 hold in $\mathcal{L}$. As we already noted, condition (1) holds. To show that condition (2) holds, we first observe that if J is an ideal of $R \in \mathcal{L}$, then the intersection of all the maximal ideals of R/J is (0). Hence every proper ideal of R is the intersection of all the maximal ideals of R containing it. Now, let B be a subring of $A \in \mathcal{L}$, and suppose first that M is a maximal ideal of B. Since B has the congruence extension property, there exists an ideal M^* of A such that $M^* \cap B = M$. Moreover, M^* is proper. Hence M^* is the intersection of all the maximal ideals of A containing M^*. Thus we can find a maximal ideal M' of A such that $M' \supseteq M^*$ and $M' \cap B$ is proper in B. By the maximality of M, $M' \cap B = M$.

Conversely, let M' be a maximal ideal of A. Since $A/M' \in \mathcal{L}$, A/M' is a field. Let $x \in B/M' \cap B$, $x \neq 0$. Then $x^n = x$ Hence $x^{n-1} = 1$, and the multiplicative inverse of x is in $B/M' \cap B$. Hence $B/M' \cap B$ is a field and $M' \cap B$ is a maximal ideal of B. Thus condition (ii) holds and the proof is complete.

The following two corollaries follow immediately from Theorem 2. A ring A is called a *p-ring*, where p is a fixed prime, if for all $x \in A$, $x^p = x$ and $px = 0$.

COROLLARY 1. *The class $\mathcal{L}$ has the amalgamation property.*

COROLLARY 2. *Free products with amalgamation exist in the class of all p-rings.*

We now consider the class $\mathcal{L}^*$ consisting of all rings A with the property that for every $x \in A$, there exists an integer $n(x) > 1$ such that $x^{n(x)} = x$. Members of $\mathcal{L}^*$ have the congruence extension property [2], and for every $A \in \mathcal{L}^*$, A is commutative and semisimple [3]. However, we cannot apply Theorem 1 to $\mathcal{L}^*$ since it is not a variety. On the other hand, the proof of Theorem 1 can be used to show that $\mathcal{L}^*$ has the amalgamation property (although the free product of an arbitrary subset of $\mathcal{L}^*$ need not exist in $\mathcal{L}^*$, the free product of a finite number of members of $\mathcal{L}^*$ does exist in $\mathcal{L}^*$). Moreover, the argument used in the proof of Lemma 2 can be also used to show that if $\mathcal{K}'$ has the amalgamation property and $\mathcal{K}'$ is a subclass of the variety $\mathcal{K}$, then the free product of $\{A_t\}_{t \in T}$ in $\mathcal{K}$ with amalgamated subring B exists for every $\{A_t\}_{t \in T} \subseteq \mathcal{K}'$ and $B \in \mathcal{K}'$. Thus we have the following

THEOREM 3. *Free products with amalgamation need not exist in $\mathcal{L}^*$. However, if $\mathcal{R}$ is the class of all commutative rings, $\{A_t\}_{t \in T} \subseteq$*

$\subseteq \mathcal{L}^*$, $B \in \mathcal{L}^*$, *and for every* $t \in T$, *there exists a monomorphism* $f_t : B \longrightarrow A_t$, *then the free product of* $\{A_t\}_{t \in T}$ *in* $\mathcal{R}$ *with amalgamated subring* B *exists.*

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