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# A Symmetrical Two-Phase Stefan Problem with Supercooling

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SUMMARY. - *We consider a two-phase Stefan problem in cylindrical symmetry with supercooling of the liquid phase, when the melting temperature is supposed to be a constant and zero flux conditions are imposed on the fixed boundaries. We perform an a priori analysis of the possibility of continuing the solution to arbitrarily large time intervals, and we relate the occurrence of each possible case with the value of an energy integral involving the initial data. Analogous results are achieved considering superheating of the solid phase instead of supercooling of the liquid one, or spherical — instead of cylindrical — symmetry.*

## 1. Introduction

The freezing of a supercooled liquid and related problems have been studied by many authors, *mostly in plane symmetry*. In doing so, they have dealt with certain *one-phase* free-boundary problems for

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the heat equation in one space dimension, releasing the sign restrictions on the data and the latent heat usually required in the Stefan problem (see the review [14] ; see also [2] - [7], [13], [15], [16], and the references quoted therein). In particular, they have analyzed the special behaviour usually known as *blow-up* (see [8] - [10] for a deeper insight). Furthermore, Fasano and Primicerio have found in [6] interesting relations between the initial data and the occurrence of each possible type of solution.

In [1] , Andreucci extended the ideas and techniques of [6] to the case of *cylindrical* (eventually *spherical*) symmetry.

In [12] , we adapted the ideas of [6] to the *two-phase case* with plane symmetry. Subsequently, Turner extended the results of [2] to the same case ([17]). The purpose of the present work is to accomplish the adaptation of [12] to the *two-phase case with cylindrical (or spherical) symmetry.*

Indeed, it is possible to analyze simultaneously the three usual types of symmetry, by using the unified expressions for the heat operator $L$ , its adjoint $L^*$ and Green's Theorem in a suitable domain $D_t$ , namely:

$$Lf = \Delta f - f_t = r^{-m}(r^m f_r)_r - f_t = f_{rr} + \frac{m}{r} f_r - f_t, \qquad (1)$$

$$L^* g = g_{rr} - \left(\frac{m}{r} g\right)_r + g_t = g_{rr} - \frac{m}{r} g_r + g_t + \frac{m}{r^2} g, \qquad (2)$$

$$\iint_{D_t} (g\, Lf - f\, L^* g)\; dr\, d\tau = \oint_{\partial D_t} \left[ f g\, dr + \left( f_r\, g - f g_r + \frac{m}{r} f g \right)\, dr \right], \qquad (3)$$

where $m = 0, 1, 2$ for planar, cylindrical and spherical symmetry, respectively.

Nevertheless, for the sake of brevity and definiteness here we will start with the cylindrical case. In particular we will study, in the cylindrical region of $\mathbb{R}^3$ defined by $r_0 \le r \le r_1$ , where $0 < r_0 < r_1$, a two-phase Stefan problem with supercooling of the liquid phase, assuming that the melting temperature is a constant (say 0) and the fixed boundaries are isolated.

To be specific, given the data

$$\varphi \in C\left[r_0, a\right], \quad \psi \in C\ \left[a,\ r_1\right], \quad \varphi \not\equiv 0, \quad \psi \not\equiv 0, \tag{4}$$

with $a \in (r_0, r_1)$, let us consider problem (P) which consists of finding $(T, s\left(t\right), u\left(r,t\right), v\left(r,t\right))$ such that:

a) $T > 0$;

b) $s \in C[0,T]$, $s \in C^1(0,T)$, $r_0 < s(t) < r_1$ for $0 < t < T$;

c) $u\left(r,t\right)$ is a bounded function in $r_0 \le r \le s(t)$, $0 \le t \le T$ , continuous on the same region except perhaps at the points $\left(s\left(0\right),0\right)$ and $\left(s\left(T\right),T\right)$; $u_r\left(r,t\right)$ is continuous for $r_0 \le r \le s\left(t\right)$, $0 < t < T$; $u_{rr}$, $u_t$ are continuous for $r_0 < r < s\left(t\right)$, $0 < t < T$;

d) $v\left(r,t\right)$ satisfies similar conditions in $s\left(t\right) \le r \le r_1$, $0 \le t \le T$;

e) $u$, $v$, $s$ satisfy

$$Lu = 0 \text{ in } D_T^u = \{(r,t) : r_0 < r < s(t),\ 0 < t < T\}, \quad \text{(liquid phase)}, \tag{5}$$

$$Lv = 0 \text{ in } D_T^v = \{(r,t) : s(t) < r < r_1,\ 0 < t < T\}, \quad \text{(solid phase)}, \tag{6}$$

$$s\left(0\right) = a, \tag{7}$$

$$u\left(r,0\right) = \varphi\left(r\right), \quad r_0 < r < a, \tag{8}$$

$$v\left(r,0\right) = \psi\left(r\right), \quad a < r < r_1, \tag{9}$$

$$u_r\left(r_0,t\right) = 0, \quad 0 < t < T, \tag{10}$$

$$v_r\left(r_1,t\right)=0,\quad 0<t<T, \tag{11}$$

$$u\left(s\left(t\right),t\right)=0,\quad 0<t<T, \tag{12}$$

$$v\left(s\left(t\right),t\right)=0,\quad 0<t<T, \tag{13}$$

$$v_r\left(s\left(t\right),t\right)-u_r\left(s\left(t\right),t\right)=\dot{s}\left(t\right),\quad 0<t<T. \tag{14}$$

REMARK 1.1. We exclude the situations $\varphi\equiv 0$ or $\psi\equiv 0$, because they correspond to the one-phase problem.

As usual, from Green's identity it is easily derived an useful integral formulation of Stefan condition (14). In our case, if $(T,s,u,v)$ is a solution of (P), we have the following energy balance formula:

$$s^2\left(t\right)-r_0^2+2\int_{r_0}^{s(t)} r\,u\left(r,t\right)\,dr+2\int_{s(t)}^{r_1} r\,v\left(r,t\right)\,dr=Q,\quad 0<t<T, \tag{15}$$

where $Q=a^2-r_0^2+2\int_{r_0}^{a} r\,\varphi(r)\,dr+2\int_{a}^{r_1} r\,\psi(r)\,dr$.

Since we are interested in the case of the supercooled liquid, we add to (P) the following assumptions:

$$\varphi\left(r\right)\leq 0,\qquad r_0\leq r\leq a, \tag{16}$$

$$\psi\left(r\right)\leq 0,\qquad a\leq r\leq r_1. \tag{17}$$

We assume also this condition:

$$\begin{gathered}\text{there is no constant } \delta\in(0,a-r_0)\\ \text{such that } \varphi\left(r\right)\leq -1,\quad a-\delta\leq r\leq a.\end{gathered} \tag{18}$$

Indeed, (18) is just a necessary condition (see similar ideas in [7] or [1]) for the existence of solutions of (P).

By working as in [15], [5] and [1], even under flux conditions more general than (10) and (11), it can be proved a trichotomy type result: if a solution of (P) exists, then one of the following three cases occurs:

(A) The problem has a solution with arbitrarily large $T$;

(B) There is a constant $T_0 > 0$ such that $\lim_{t \to T_0^-} s(t) = r_0$;

(C) There is a constant $T_1 > 0$ such that $\lim_{t \to T_1^-} s(t) > r_0$, $\liminf_{t \to T_1^-} \dot{s}(t) = -\infty$;

corresponding to global existence, finite time extinction of the liquid phase, and blow-up, respectively.

Moreover:

(i) (C) $\Rightarrow u$ is continuous up to $t = T_1$, $r_0 < r < \lim_{t \to T_1^-} s(t)$;

(ii) (C) $\Rightarrow v$ is continuous up to $t = T_1$ , $\lim_{t \to T_1^-} s(t) < r < r_1$;

(iii) (B) $\Rightarrow v$ is continuous up to $t = T_0$, $r_0 < r < r_1$.

In this paper we will discuss neither existence of solutions of (P) nor the trichotomy's theorem just mentioned. Here we will only look for relations between the initial data and the occurrence of the three cases, with the hope of providing *a simple test to predict the type of solution.*

This a priori analysis is made in Sec. 2. The case of an *superheated solid* (eventually in contact with an supercooled liquid) is considered in Sec.3. Finally, in Sec.4 we briefly comment the results for *spherical symmetry*.

## 2. A priori analysis of the solution's type

Recall that we are analyzing problem (P) in cylindrical symmetry (i.e. with $m = 1$).

PROPOSITION 2.1. *If $(T, s, u, v)$ solve (P), then*

*(i)* $Q < a^2 - r_0^2$;

*(ii)* $u < 0$ *in* $D_T^u$ *and* $v < 0$ *in* $D_T^v$;

*(iii)* $s$ *is strictly decreasing and* $r_0 < s(t) < a$, $0 < t < T$;

*(iv)* $s^2(t) > r_0^2 + Q$, $0 < t < T$.

*Proof.* Obviously $Q \leq a^2 - r_0^2$ , but $Q = a^2 - r_0^2 \Leftrightarrow \varphi \equiv \psi \equiv 0 \Leftrightarrow$ $(s \equiv a, u \equiv v \equiv 0)$ .

Part (ii) is a consequence of the (strong) maximum principle. Hence, by using (14) and the Vyborny - Friedman theorem ([11, p. 49]) , (iii) is obtained. From (15) and (ii), (iv) follows. □

PROPOSITION 2.2. *If (P) has a solution, then*

*(i) (B)* $\Rightarrow Q < 0$;

*(ii) (A)* $\Rightarrow 0 \leq Q < a^2 - r_0^2$ *and* $\lim_{t\to+\infty} s\,(t) = (r_0^2 + Q)^{1/2}$.

*Proof.* In case (B), we apply in (15) Lebesgue's bounded convergence theorem for $t \to T_0$ , and $Q \leq 0$ follows. But $Q = 0$ is immediately excluded because of $\psi \not\equiv 0$ and the maximum principle. In case (A), it can be shown (by proving a suitable comparison lemma as in [6] or [1]) that $u$ and $v$ tend uniformly to zero as $t \to +\infty$ . Then, from (15) we obtain $\lim_{t\to+\infty} s^2\,(t) = r_0^2 + Q$, whence $Q \geq 0$. □

We want to get some kind of converse for Prop. 2.2 (ii). In order to do so, after Prop. 2.2 (i), our aim is to exclude (C).

LEMMA 2.3. *Let* $(T, s, u, v)$ *a solution of (P) such that* $s(T^-) \equiv \lim_{t\to T^-} s\,(t) > r_0$ . *Let* $d > 0$, $\alpha \in (0, T)$, $z_1 \in (0, 1)$, $z_2 > 0$ *such that* $d \leq \min\{s\,(T^-) - r_0, r_1 - a\}$,

$$u\,(r,t) \geq -z_1 \text{ in the closure of } D_T^u \bigcap E_\alpha, \tag{19}$$

*and*

$$v\,(r,t) \geq -z_2 \text{ in the closure of } D_T^v \bigcap E_\alpha, \tag{20}$$

*where* $E_\alpha \equiv \{(r,t) : \alpha < t < T, s(t) - d < r < s(t) + d\}$. *Then there exists a constant* $K > 0$ *such that*

$$\dot{s}\,(t) \geq -K, \quad \alpha < t < T.$$

*Proof.* By decreasing $d$ if necessary, we can also assume with no loss of generality that

$$d \cdot \max\{|u_r(r,\alpha)| : r \in [r_0, s(\alpha)]\} \leq z_1, \tag{21}$$

and

$$d \cdot \max\{|v_r(r,\alpha)| : r \in [s(\alpha), r_1]\} \leq z_2. \tag{22}$$

For any $\epsilon \in (0, T-\alpha)$ we define $\sigma_\epsilon \equiv \inf\{\dot{s}\,(t) : t \in (\alpha, T-\epsilon)\}$,

$$A_\alpha^\epsilon \equiv \{(r,t) : \alpha < t < T-\epsilon, s(t) - d < r < s(t)\},$$

and

$$B_\alpha^\epsilon \equiv \{(r,t) : \alpha < t < T-\epsilon, s(t) < r < s(t) + d\}.$$

In the closure of the domain $A_\alpha^\epsilon$ we compare $u$ with the function $w_1(r,t) = -z_1(1-e^{-bd})^{-1}(1-\exp[b(r-s(t))])$ , where $b$ is a positive constant to be determined.

It is easy to prove that $w_1\,(r,t) \leq u\,(r,t)$ in the parabolic boundary of $A_\alpha^\epsilon$, because of (12), (19), (21) and convexity arguments (note that $w_{1,rr} > 0$).

Now, we choose $b = -\sigma_\epsilon$ and obtain

$$Lw_1 = z_1 \left(1 - e^{-bd}\right)^{-1} e^{b(r-s(t))} \left[b\,(b + \dot{s}\,(t)) + \frac{b}{r}\right] > 0 \text{ in } A_\alpha^\epsilon.$$

At this point, the maximum principle yields $w_1\,(r,t) < u\,(r,t)$ in $A_\alpha^\epsilon$, and

$$\begin{aligned} u_r\,(s\,(t)\,,t) \leq w_{1,r}\,(s\,(t)\,,t) &= \frac{z_1 b}{1 - e^{-bd}} \\ &= -z_1\sigma_\epsilon\,[1 - \exp\,(\sigma_\epsilon d)]^{-1}. \end{aligned}$$

Now, we compare $v$ with the function

$$w_2\,(r,t) = \frac{z_2}{d^2}\,(r - s\,(t))\,(r - s\,(t) - 2d)$$

in the closure of $B_\alpha^\epsilon$. We have

$$Lw_2 = \frac{2z_2}{rd^2}\,[(r - s(t)) + (r-d) + r\,(r - s\,(t) - d)\,\dot{s}(t)] > 0$$

in $B_\alpha^\epsilon$; working as in the liquid phase we obtain $v_r(s(t),t) \geq -\frac{2z_2}{d}$, whence $\dot{s}\,(t) \geq -\frac{2z_2}{d} + z_1\sigma_\varepsilon[1-\exp(\sigma_\epsilon d)]^{-1}$.

Then $\sigma_\epsilon \geq -\frac{2z_2}{d} + z_1\sigma_\varepsilon[1-\exp(\sigma_\epsilon d)]^{-1}$, and the conclusion easily follows by taking $K = -\beta$, where $\beta$ is the (unique) zero of the strictly increasing function $f(x) = x - z_1 x(1-e^{dx})^{-1} + \frac{2z_2}{d}$. □

Remark 2.4. We have used $\dot{s}(t) < 0$ to prove $Lw_2 > 0$, but not when verifying that $Lw_1 > 0$. Note that condition (16) (and then $\dot{s} < 0$) has not been essential in obtaining similar results for the one-phase problem. In fact, in [6] and [1] they have used $\inf\{s(t) : t \in (0,T)\}$ instead of $s(T^-)$, and then reasoned as follows: if $\sigma_\epsilon \geq 0$ for all $\epsilon \in (0, T-\alpha)$, the conclusion is obviously true; on the other hand, it suffices to make a convenient comparison in $A_\alpha^\epsilon$.

We shall exhibit a second way of working in $B_\alpha^\epsilon$ to prove Lemma 2.3. This applies under the additional hypothesis $z_1 + z_2 < 1$ . If $\sigma_\epsilon \geq -\frac{1}{s(T^-)}$ for all $\epsilon \in (0, T-\alpha)$, there is nothing to prove. Then we can suppose that there is a small $\epsilon_0$ such that

$$\sigma_\epsilon < -\frac{1}{s(T^-)} \text{ for all } \epsilon \in (0, \epsilon_0). \tag{23}$$

For any $\epsilon < \epsilon_0$ we compare $v$ in the closure of the domain $B_\alpha^\epsilon$ with the function $w_2 = -z_2\left(1 - e^{cd}\right)^{-1}(1 - \exp[c(r - s(t))])$, where $c$ is a suitable negative constant, say $c = \sigma_\epsilon$. Thus, $Lw_2 = z_2\left(1 - e^{cd}\right)^{-1} \cdot e^{c(r-s(t))}c[(c + 1/r) + \dot{s}(t)] > 0$ in $B_\alpha^\epsilon$ because of (23). As before, we have used $\dot{s}(t) < 0$ for proving $Lw_2 > 0$ . Now, we have $v_r(s(t), t) \geq z_2\sigma_\epsilon[1 - \exp(\sigma_\epsilon d)]^{-1}$, then $\dot{s}(t) \geq (z_1 + z_2)\sigma_\varepsilon[1 - \exp(\sigma_\epsilon d)]^{-1}$, from which we easily obtain $\sigma_\epsilon \geq \frac{1}{d}\log(1 - z_1 - z_2)$.

Note that $z_1 + z_2 < 1$ is neither an illogical nor an excessive assumption. Indeed, taking into account the continuity of $v$ in an adequate compact set (the closure of $D_T^v \bigcap E_\alpha$) or even the maximum principle, we can reduce $z_2$ by redefining $d$.

Remark 2.5. If (C) occurs, the isotherm $u = -1$ exists and reaches the free boundary at $t = T_1$.

Remark 2.6. If $\varphi \geq -1$ we have $u > -1$ (because (18) excludes $\varphi \equiv -1$ ). Thus, (C) cannot occur.

Now then, we look for conditions more general than those of Remark 2.6, ensuring we are in position of using Lemma 2.3. As in [6] (and later in [1] and [12]), here we add the following useful hypothesis:

(H) The equation $\varphi(r) = -1$ has at most one root in $[r_0, a]$;

or the somewhat weaker:

(H') There is no triple $(r_1^*, r\prime, r_2^*)$ with $r_0 < r_1^* < r\prime < r_2^* < a$, such that

$$\begin{array}{ll} \varphi(r\prime) < -1 & [\text{respectively } \varphi(r\prime) > -1], \\ \varphi(r_i^*) \geq -1 & [\leq -1], \quad i = 1, 2, \\ \varphi(r_1^*)\,\varphi(r_2^*) < 1 & [> 1]. \end{array}$$

LEMMA 2.7. *Let $(T, s, u, v)$ solve (P). If $Q \geq 0$ and $\varphi$ satisfies (H), then: either there are no points in $D_T^u$ where $u(r,t) = -1$, or the isotherm $u = -1$ is separated by a positive distance from $r = s(t)$ for all $t \in [0, T]$ such that $s(t) > r_0$.*

*Proof.* Since $Q \geq 0$ excludes (B), it is worth noting that the inequality $s(t) > r_0$ is satisfied even for $t = T$. After Remark 2.6, if $\varphi \geq -1$ there is nothing to prove. In the other case, let us note however that the average temperature in the liquid fase is always greater than $-1$, because

$$Q \geq 0 \Rightarrow P \equiv a^2 - r_0^2 + 2\int_{r_0}^{a} r\,\varphi(r)dr > 0$$

and then

$$\frac{1}{\pi\left(a^2 - r_0^2\right)}\int_{r_0}^{a} 2\pi r\,\varphi(r)\,dr > -1;$$

for $t > 0$ we have

$$s^2(t) - r_0^2 + 2\int_{r_0}^{s(t)} r\,u(r,t)\,dr > 0,$$

whence

$$\frac{1}{\pi\left(s^2(t) - r_0^2\right)}\int_{r_0}^{s(t)} 2\pi r\,u(r,t)\,dr > -1. \tag{24}$$

Now, we consider the isotherm $u(r,t) = -1$ originating from $t = 0$, which is unique because of (H) and the maximum principle. By using

(H) again, we see that if $u(\bar{r},\bar{t}) = -1$, then $u(r,\bar{t}) < -1$ for $r < \bar{r}$ and $u(r,\bar{t}) > -1$ for $r > \bar{r}$. Then, from (24) we obtain the thesis for all $t < T$. To rule out $u(s(T^-),T) = -1$, it suffices to take the limit for $t \to T^-$ in (24) and recall once more the maximum principle. □

REMARK 2.8. Lemma 2.7 remains valid, with minor changes in the proof, if we suppose (H') instead of (H).

PROPOSITION 2.9. *If (H') is satisfied, then (C)* $\Rightarrow Q < 0$.

*Proof.* If $Q \geq 0$, then the thesis of Lemma 2.7 is true. Thus, it is possible to apply Lemma 2.3, excluding (C). □

Therefore, we have completely proved the following

THEOREM 2.10. *If* $\varphi$ *satisfies (H'), then*

$$\text{(A)} \quad \Leftrightarrow \quad 0 \leq Q < a^2 - r_0^2.$$

Thus, we have $Q < 0$ iff either (B) or (C).

Next we will see that it is impossible to discriminate between (B) and (C) only by the value of $Q$; they also depend on the initial configuration $(\varphi, \psi)$. Roughly speaking, (B) and (C) "share" each value of $Q$ belonging to the interval $(-\infty, 0)$. This fact *contrasts the one-phase problem* and shows that the presence of a solid phase at negative temperature (instead of $v \equiv 0$) can modify substantially the behaviour of the free boundary.

LEMMA 2.11. *Assume that* $\psi(r) \geq -M$ *in* $[a, r_1]$ *for some* $M > 0$. *If* $Q \leq -M\left(r_1^2 - r_0^2\right)$, *then (C) occurs.*

*Proof.* The fact that $Q < 0$ excludes (A). In case (B) we proceed as in the proof of the Prop. 2.2 (i) to get a contradiction:

$$Q = 2\int_{r_0}^{r_1} rv(r, T_0^-)dr > -M(r_1^2 - r_0^2).$$

□

By choosing $K = \frac{M}{r_0}$ and noting that $-M \geq -Kr$ in $[r_0, r_1]$, we can modify the preceding proof to conclude that if $Q \leq -\frac{2}{3}K\left(r_1^3 - r_0^3\right)$, then (C) occurs.

THEOREM 2.12. *For each $Q < 0$ there exists two pairs $(\varphi_1, \psi_1)$ and $(\varphi_2, \psi_2)$, both corresponding to $Q$, such that*

*(i) $(\varphi_1, \psi_1)$ produces a solution of type (B);*

*(ii) For $(\varphi_2, \psi_2)$, (C) occurs.*

*Proof.* Given $Q < 0$ , we can choose $(\varphi_1, \psi_1)$ satisfying (4), (16)-(17), such that $\varphi_1 (r) > -1$ in $(r_0, a)$ and

$$Q = a^2 - r_0^2 + 2 \int_{r_0}^{a} r\varphi_1 (r)\, dr + 2 \int_{a}^{r_1} r\, \psi_1 (r)\, dr.$$

For instance $\varphi_1(r) = \frac{r-a}{a-r_0}$, $\psi_1(r) = -\alpha(r-a)$, with

$$\alpha = \frac{(a-r_0)(2a+r_0) - 3Q}{(r_1-a)^2(2r_1+a)}.$$

Then (B) occurs, and thus (i) is proved. The other part is an elementary application of Lemma 2.11.

□

REMARK 2.13. Note that we have proved the Theorem 2.12 without using (H').

When $Q < 0$, one may try to look for information through the liquid phase energy, as in the one-phase problem. We consider

$$Q_1 (t) = s^2 (t) - r_0^2 + 2 \int_{r_0}^{s(t)} r\, u\, (r, t)\, dr, \quad 0 \leq t < T$$

and recall $P = a^2 - r_0^2 + 2 \int_{r_0}^{a} r\, \varphi\, (r)\, dr = Q - 2 \int_{a}^{r_1} r\, \psi\, (r)\, dr$; then $Q_1 (0) = P$.

The global heat balance equation (15) indicates that the energy $Q$ is a constant with respect to time. Is the same true for $Q_1$ ? By introducing a slight modification in the proof of (15), we obtain the heat balance equation in the supercooled region:

$$Q_1(t) = P + 2 \int_{0}^{t} s(\tau) v_r(s(\tau), \tau) d\tau, \quad 0 < t < T. \tag{25}$$

The Vyborny-Friedman theorem implies that $Q_1(t) < Q_1(0) = P$, $0 < t < T$. Moreover:

(i) $\int_0^t s(\tau)v_r(s(\tau),\tau)d\tau$ is a decreasing function in $(0,T)$.

(ii) $Q_1(t_2) = Q_1(t_1) + 2\int_{t_1}^{t_2} s(\tau)\, v_r(s(\tau),\tau)\, d\tau < Q_1(t_1)$ if $0 < t_1 < t_2 < T$.

Therefore, the next two propositions are trivial.

PROPOSITION 2.14. *If (B) occurs, then:*

*(i)* $P > 0$,

*(ii)* $Q_1(t) = -2\int_t^{T_0} s(\tau)\, v_r(s(\tau),\tau)\, d\tau > 0$ *for all* $t \in [0, T_0)$,

*(iii)* $Q_1(T_0^-) = 0$,

*(iv) the average* $\frac{1}{\pi(s^2(t) - r_0^2)} \int_{r_0}^{s(t)} 2\pi r u(r,t)\, dr > -1$.

PROPOSITION 2.15. *If* $P \leq 0$, *then (C) occurs. Moreover,*

*(i)* $Q < 0$,

*(ii)* $Q_1(t) < 0$, $0 < t < T_1$,

*(iii) the average* $\frac{1}{\pi(s^2(t) - r_0^2)} \int_{r_0}^{s(t)} 2\pi r u(r,t)\, dr < -1$.

REMARK 2.16. (i) Considering the influence of the negative temperature in the solid phase, it is natural that the one-phase result $P < 0 \Rightarrow$ (C) remains valid for the two-phase problem. (ii) Owing to the same influence, now $P = 0 \Rightarrow$ (C), despite $P = 0, \psi \equiv 0$, (H) $\Rightarrow$ (B). (iii) The case $P > 0$ has been connected to (A) when

$\psi \equiv 0$. But in the two-phase problem the type of solution depends in the first place on the value $Q - P = 2 \int_a^{r_1} r\psi(r)\, dr$ to check whether $Q < 0$, and then on the whole profile $(\varphi, \psi)$.

We end this section by mentioning that the rôle played by the level curves $u = -1$ has been the same both in the one-phase problem as in the two-phase one.

## 3. The case of an superheated solid

In this section, we initially remove the conditions (16) and (17) and assume

$$\varphi(r) \geq 0, \quad r_0 \leq r \leq a, \tag{26}$$

$$\psi(r) \geq 0, \quad a \leq r \leq r_1. \tag{27}$$

Since this case is entirely similar to $\varphi \leq 0$, $\psi \leq 0$, with imaginable conclusions, we shall confine ourselves to comment that the proofs of Sec.2 remain essentially valid and the isotherm $v = 1$ plays in the superheated solid the same "critical" rôle as the isotherm $u = -1$ did in the case of the supercooled liquid.

We now put an superheated solid in contact with an supercooled liquid, that is we retain (27) and suppose (16) instead of (26). Assume also (18) and this analogous condition:

$$\begin{array}{l}\text{there is no constant } \delta \in (0, r_1 - a)\\ \text{such that } \psi(r) \geq 1,\ a \leq r \leq a + \delta.\end{array} \tag{28}$$

We define $T^* \equiv \sup\{\, T > 0 : \text{there exists } (T, s, u, v) \text{ solving (P)}\}$.

In our setup it is not easy to analyze the problem, because

- $s(t)$ can be a non monotone function,
- both phases can have a "critical" isotherm,
- $Q$ may assume any real value.

Moreover, there are more than three possible kinds of behaviour, namely

(A) Global existence, i.e. $T^* = +\infty$.

($B_{left}$) Finite time extinction of the liquid phase, i.e. $T^* < +\infty$ and $\liminf_{t\to T^*-} s(t) = r_0$.

($B_{right}$) Finite time extinction of the solid phase, i.e. $T^* < +\infty$ and $\limsup_{t\to T^*-} s(t) = r_1$.

(C) Blow-up, i.e. $T^* < +\infty$, $\liminf_{t\to T^*-} s(t) > r_0$, $\limsup_{t\to T^*-} s(t) < r_1$, and $\limsup_{t\to T^*-} |\dot{s}(t)| = +\infty$.

Actually, we should examine a variety of possibilities in case (C), but we will avoid to do so for the sake of brevity.

For the above reasons, we merely obtain partial conclusions, essentially necessary conditions for some kinds of behaviour of the solution (for more details, see analogous case in [12]), for instance:

(i) (A) $\Rightarrow \quad 0 < Q < r_1^2 - r_0^2$.

(ii) ($B_{left}$) $\Rightarrow \quad Q = 2\int\limits_{r_0}^{r_1} rv(r, T^{*-})dr > 0$.

(iii) ($B_{right}$) $\Rightarrow \quad Q = r_1^2 - r_0^2 + 2\int\limits_{r_0}^{r_1} ru(r, T^{*-})dr < r_1^2 - r_0^2$.

(iv) $Q \leq 0 \quad \Rightarrow$ We are either in case (C) or ($B_{right}$).

(v) $Q \geq r_1^2 - r_0^2 \quad \Rightarrow$ We are either in case (C) or ($B_{left}$).

## 4. The problem in spherical symmetry

As far as the spherical symmetry is concerned, we have to use (1)-(3) with $m = 2$. Then, instead of (15) we obtain

$$s^3(t) - r_0^3 + 3\int_{r_0}^{s(t)} r^2 u(r,t)\,dr + 3\int_{s(t)}^{r_1} r^2 v(r,t)\,dr = R \tag{29}$$

where $R \equiv a^3 - r_0^3 + 3\int\limits_{r_0}^{a} r^2\varphi(r)\,dr + 3\int\limits_{a}^{r_1} r^2\psi(r)\,dr$, that is, the energy balance equation within a factor of $\frac{4}{3}\pi$.

By using (29) appropriately, all the results of Sec.2 and 3 can be adapted in a very elementary way to the spherical symmetry case. We only list a few properties corresponding to the assumptions $\varphi \leq 0$, $\psi \leq 0$, $\varphi \not\equiv 0$, $\psi \not\equiv 0$, namely:

(i) $R < a^3 - r_0^3$.

(ii) (A) $\Rightarrow \quad \lim_{t \to +\infty} s(t) = (r_0^3 + R)^{1/3}$.

(iii) (B) $\Rightarrow \quad R < 0$.

(iv) If (H') is verified, then (C) $\Rightarrow R < 0$.

(v) If $\varphi$ satisfies (H'), then (A) $\Leftrightarrow \quad 0 \leq R < a^3 - r_0^3$.

(vi) Theorem 2.12 of Sec.2 remains valid with $R$ instead of $Q$.

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