ON BOUNDED POSITIVE SOLUTIONS OF SEMILINEAR SCHRÖDINGER EQUATIONS (*)

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Sommario. - *L'equazione semilineare di Schrödinger* $\Delta u + f(x,u) = 0$ *viene considerata in un dominio esterno di* R^n, $n \geq 3$. *Vengono date condizioni su* f *sufficienti affinchè l'equazione abbia soluzioni positive* $u(x)$ *con* $u(x) \to 0$ *quando* $|x| \to \infty$.

Summary. - *The semilinear Schrödinger equation* $\Delta u + f(x,u) = 0$ *is considered in an exterior domain of* R^n, $n \geq 3$. *Conditions on* f *are given which are sufficient for the equation to have positive solutions* $u(x)$ *with* $u(x) \to 0$ *as* $|x| \to \infty$.

1. Let us consider the semilinear Schrödinger equation

$$Lu = \Delta u + f(x,u) = 0, \qquad x \in G_A, \tag{1}$$

in an exterior domain $G_A = \{x \in R^n : \ |x| > A\}$ (here $A > 0$), $n \geq 3$, subject to the assumptions

(i) $f \in C^{\lambda}_{loc}(G_A \times R, R)$ for some $\lambda \in (0,1)$ (local Hölder continuous) with $f(x,u)$ odd in u, i.e. $f(x,-u) = -f(x,u)$;

(ii) $0 \leq f(x,t) \leq a(|x|)w(t)$ for all $x \in G_A$ and all $t \geq 0$ for some $a, w \in C(R_+, R_+)$ with w nondecreasing, $w(0) = 0$.

A solution of (1) in $G_B = \{x \in R^n : \ |x| > B\}$ for $B \geq A$ is a function $u \in C^2(G_B, R)$ such that $Lu(x) = 0$ for all $x \in G_B$.

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We will give conditions which guarantee that (1) has a positive solution in G_B (for some $B \geq A$) that satisfies $u(x) \to 0$ for $|x| \to \infty$.

2. Let us denote $S_B = \{x \in R^n : \ |x| = B\}$ for $B \geq A$.

In the sequel we will need the following

LEMMA 1. [1]. *Let L be the operator defined by* (1) *where f is nonnegative for $u \geq 0$ and satisfies assumption* (i) *in an exterior domain G_A. If there exists a positive solution v_1 and a nonnegative solution v_2 of $Lv_1 \leq 0$ and $Lv_2 \geq 0$, respectively, in G_B for some $B \geq A$ such that $v_2(x) \leq v_1(x)$ throughout $G_B \cup S_B$, then equation* (1) *has at least one solution $u(x)$ in G_B satisfying $u(x) = v_1(x)$ on S_B and $v_2(x) \leq u(x) \leq v_1(x)$ throughout G_B.*

Consider now the differential equation

$$y'' + F(t, y) = 0 \tag{2}$$

where $F(t, u)$ is continuous on $\{(t, u) : \ t \geq T, \ u \in R\}$ where $T > 0$. We have that

LEMMA 2. [2]. *Suppose that for each $p > 0$ there is a continuous function $\mu(s)$ such that*

$$|F(s, y)| \leq \mu(s), \qquad |y| \leq p, \quad s \geq T,$$

and

$$\int_T^\infty s\mu(s)ds < \infty.$$

Then every point $y_0 \in R$ is the limit of some solution of (2) *as $t \to \infty$.*

3. Let us now prove the following

THEOREM. *If for every* $p > 0$,

$$\int_A^\infty sa(s)\, w\left(\frac{p}{s^{n-2}}\right) ds < \infty \tag{3}$$

then there is a positive solution $u(x)$ *of* (1) *in* G_B *(for some* $B \geq A$*) with* $u(x) = O(|x|^{2-n})$ *as* $|x| \to \infty$.

Proof. We consider the differential equation

$$\frac{d}{dr}\left\{r^{n-1}\frac{dy}{dr}\right\} + r^{n-1}a(r)w(y) = 0, \quad r \geq A, \tag{4}$$

where we extended w to R by defining $w(-y) = -w(y)$ for $y > 0$.

The change of variables

$$r = \beta(s) = \left\{\frac{1}{n-2}s\right\}^{\frac{1}{n-2}}, \qquad h(s) = sy(\beta(s),$$

transforms (4) into

$$h''(s) + \frac{\beta'(s)\beta(s)}{n-2}\, a(\beta(s))\, w\left(\frac{h(s)}{s}\right) = 0, \quad s \geq \beta^{-1}(A). \tag{5}$$

In view of Lemma 2, condition (3) guarantees that (5) has a bounded solution $h(s)$ which is positive in some interval $[b, \infty)$ with $b \geq \beta^{-1}(A)$. Returning to (4), this yields a solution $y(r)$ of (4) which is positive in $[B, \infty)$ where $B = \beta(b) \geq A$.

Let us define $v_1(x) = y(r),\ |x| = r \geq B$. Observe that

$$\begin{aligned} r^{n-1}\, Lv_1(x) &= \frac{d}{dr}\{r^{n-1}\frac{dy}{dr}\} + r^{n-1}f(x, v_1(x)) \\ &\leq \frac{d}{dr}\left\{r^{n-1}\frac{dy(r)}{dr}\right\} + r^{n-1}a(r)w(y(r)) \end{aligned}$$

and hence $Lv_1(x) \leq 0$ for all $x \in G_B$. If we define $v_2(x) = 0$ for $|x| \geq B$ we clearly have that $Lv_2(x) \geq 0$ in G_B and so an application of Lemma 1 yields a solution $u(x)$ of (1) in G_B with $0 \leq u(x) \leq y(r)$ for $|x| = r \geq B$ and $u(x) = y(B)$ for $|x| = B$.

Since $u(x) \geq 0$ in $G_B \cup S_B$ satisfies $\Delta u \leq 0$ in $G_B \cup S_B$, we have that (see [3, page 917])

$$u(x) \geq \left\{ \frac{B}{|x|} \right\}^{n-2} \inf_{|x|=B} \{u(x)\} = L_0 |x|^{2-n} > 0, \qquad |x| \geq B,$$

for some constant $L_0 > 0$. On the other hand, we have that $u(x) \leq y(r)$ for $|x| = r \geq B$ and since $y(r) = \frac{h(\beta^{-1}(r))}{\beta^{-1}(r)} \leq L r^{2-n}$ for some constant $L > 0$ (the solution $h(s)$ of (5) is bounded), we have that the solution $u(x)$ of (1) satisfies

$$L_0 |x|^{2-n} \leq u(x) \leq L |x|^{2-n}, \ |x| \geq B.$$

$\diamondsuit$

To show the applicability of our theorem and the relation to the results of Swanson [4], let us have a look at the following

EXAMPLE. Consider (1) with

$$f(x,u) = \begin{cases} a(|x|) \sqrt{|u|} \ sgn(u), & 0 \leq |u| \leq 1, \\ a(|x|) \ u|u|, & |u| \geq 1, \end{cases}$$

where $a \in C(R_+, R_+)$.

The results from [4] are not applicable (it is not possible to majorize $\frac{f(x,u)}{u}$ above by a nonnegative function $g(|x|, u)$ which is monotone in u for $u > 0$) but, by our Theorem, we know that if

$$\int_A^\infty s^{2-\frac{n}{2}} a(s) ds < \infty$$

then the corresponding equation (1) has a positive solution $u(x)$ in G_B (for some $B \geq A$) with $u(x) = O(|x|^{2-n})$ as $|x| \to \infty$. $\diamondsuit$

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