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Surgery description of orientation-preserving periodic maps on compact orientable 3-manifolds

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"Dedicated to the memory of Marco Reni"

Summary. - *We show that every orientation-preserving periodic diffeomorphism f on a closed orientable 3-manifold M has a "surgery description", that is, there is a framed link $\mathcal{L}$ in S^3 which is invariant by a standard rotation φ around a trivial knot, such that M is obtained by surgery on $\mathcal{L}$ and that f is conjugate to the periodic diffeomorphism induced by φ. We will illustrate this result, by visualizing isometries of the complements of 2-component hyperbolic links with ≤ 9 crossings which do not extend to periodic maps of S^3.*

1. Introduction

In [16], Montesinos has found a beautiful relationship between surgery on strongly invertible links and double branched coverings of S^3. Namely, he has proved that a closed orientable 3-manifold is a double branched covering of S^3 if and only if M is obtained by (rational) surgery on a strongly invertible link L in S^3. This enable us to "see" the covering involution of M, because it is conjugate to the involution of M induced by that of S^3 preserving L.

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In addition to this, Rolfsen shows in his book [20, Sections 6D and 10C] that every cyclic covering of S^3 branched over a knot is the result of (integral) surgery on a periodic link. This gives a "visualization" of the periodic diffeomorphism generating the covering transformation group.

Thus it is natural to ask if one can "visualize" an arbitrary orientation-preserving periodic diffeomorphism on a closed orientable 3-manifold. The main purpose of this paper is to give an affirmative answer to this question, by proving the following theorem (see Theorem 2.1 for the full statement):

THEOREM 1.1. *Let f be an orientation-preserving periodic diffeomorphism of period n on a closed orientable 3-manifold M. Then there is an integral framed link $\mathcal{L}$ in S^3 which is invariant by a standard $2\pi/n$-rotation φ_n around a trivial knot, such that M is obtained by surgery on $\mathcal{L}$ and that f is conjugate to the periodic diffeomorphism on M induced by φ_n.*

The theorem above gives a generalization of the results of Przytycki and Sokolov [19] for periodic maps of prime periods. We note that they also give nice applications of their results. (See also [6], [12], and [18] for related results and an interesting conjecture.) We hope our generalization also has some applications.

In the last section of this paper, we illustrate the theorem above by "visualizing" isometries of the complements of 2-component hyperbolic links with ≤ 9 crossings which do not extend to periodic maps of S^3. This part was motivated by the work of Henry and Weeks [9], which determines the isometry groups of the complements of the hyperbolic knots and links in the table of Rolfsen [20] by improving the computer program SnapPea [24]. For some of the links L in the table, there is a gap between the isometry group $\mathrm{Isom}(S^3 - L)$ of the complement and the symmetry group $\mathrm{Sym}(S^3, L)$, that is some isometry of the link complement does not extend to a homeomorphism of the pair (S^3, L) and hence we cannot "see" the isometry. We "visualize" the exotic isometry by giving its surgery description (cf. Corollary 2.4). As a matter of fact, it is this observation that motivated the author to prove Theorem 1.1.

The observations in the final section were found when the author was staying at the University of Toronto during the period Septem-

ber, 1991 - May, 1992, through discussion with Jeffrey Weeks by e-mail. He would like to express his hearty thanks to Kunio Murasugi for his kind hospitality and to Jeffrey Weeks for valuable discussion. His thanks also go to Kazuo Habiro for enlightening conversation and to Masako Kobayashi for informing him of the work of Przytycki and Sokolov [19]. Their paper has enabled the author to simplify his previous, rather complicated proof of a weaker version of the main result (see Section 3) and to prove the main theorem. Finally, he would also like to thank José M. Montesinos and Józef H. Przytycki for their informative comments on a preliminary version of the paper.

2. Statement of the results

Let M be a smooth orientable 3-manifold and K a knot in M, i.e., a smoothly embedded circle in M. By a *slope* of K, we mean an isotopy class ν of an essential simple loop on $\partial N(K)$. (Throughout this paper, the symbol $N(\cdot)$ denotes a regular neighborhood.) The symbol $[\nu]$ denotes the (integral) homology class in $H_1(\partial N(K))$, or its image in the first homology group of a subspace of $M-K$ containing $\partial N(K)$, supported by ν with an arbitrary orientation. Similarly, if X is a subspace of M containing K, then the homology class in $H_1(X)$ supported by K with an arbitrary orientation is denoted by the symbol $[K]$. The *meridian* μ of K is the slope of K determined by the boundary of the meridian disk of $N(K)$. By a *longitude* λ, we mean a slope of K which is homotopic to the core K of $N(K)$. When M is a homology sphere, the *preferred longitude* is the longitude λ, such that $[\lambda] = 0 \in H_1(M - K)$.

By a *rational framed knot* (a framed knot, in brief) in M, we mean a pair (K, ν), where K is a knot in M and ν is a slope of K. We call ν the *framing* or a *slope* of the framed knot. A framed knot (K, ν) is *integral* if ν is a longitude of K in $N(K)$. A *rational framed link* (a framed link, in brief) $\mathcal{L}$ in M is a disjoint finite collection of framed knots $\{(K_1, \nu_1), \cdots, (K_m, \nu_m)\}$. Each (K_i, ν_i) is called a *component* of $\mathcal{L}$. It is *integral* if each component of $\mathcal{L}$ is integral. We often denote $\mathcal{L}$ by $(L, \boldsymbol{\nu})$, where $L = K_1 \cup \cdots \cup K_m$ and $\boldsymbol{\nu} = (\nu_1, \cdots, \nu_m)$. Then $\chi(\mathcal{L})$ denotes the manifold obtained by surgery on $\mathcal{L}$, that is, the smooth manifold (see [10, Section 8.2]) obtained from M by removing the interior of the regular neighborhood $N(L) = N(K_1) \cup \cdots \cup N(K_m)$ of L and sewing back so that the meridian of $N(K_i)$ is identified with ν_i. The core of the sewed back solid torus $N(K_i)$ forms a knot in $\chi(\mathcal{L})$, which we denote by K_i^*, and the meridian of $K_i \subset M$ is regarded as a framing of K_i^*, which we denote by ν_i^*. We call the framed link $\mathcal{L}^* = \{(K_1^*, \nu_1^*), \cdots, (K_m^*, \nu_m^*)\}$ in $\chi(\mathcal{L})$, the *dual framed link* to $\mathcal{L}$. Then $\mathcal{L}^*$ is integral if and only if $\mathcal{L}$ is integral.

A slope of a knot K in S^3 is expressed by an element $r \in \boldsymbol{Q} \cup \{\infty\}$ in the usual way (see [20]), and it is integral if and only if $r \in \boldsymbol{Z} \cup \{\infty\}$. So, a framed link in S^3 is expressed by a pair $(L, \boldsymbol{r})$ with $\boldsymbol{r} = (r_1, \cdots, r_m)$, where m is the number of components of L and $r_i \in \boldsymbol{Q} \cup \{\infty\}$ $(1 \leq i \leq m)$.

By a *periodic rational framed link of period* n, (a periodic framed link, in brief) we mean a pair $(\mathcal{L}, \varphi_n)$ of a framed link $\mathcal{L} = (L, \boldsymbol{r})$ in S^3 and a periodic diffeomorphism φ_n on S^3 which satisfies the following conditions:

1. The map φ_n is conjugate, by a diffeomorphism of S^3 to itself, to the standard $2\pi/n$-rotation of S^3 around a trivial knot.

2. The map φ_n preserves L, and each component of L is disjoint from $\mathrm{Fix}(\varphi_n)$ or identical with $\mathrm{Fix}(\varphi_n)$.

3. For any pair of components (K_i, r_i) and (K_j, r_j) of $\mathcal{L}$, we have $r_i = r_j$ whenever $\varphi_n(K_i) = K_j$.

We say that $(\mathcal{L}, \varphi_n)$ is *augmented* or *non-augmented* according to whether $\mathrm{Fix}(\varphi_n) \subset L$ or not. Note that we can choose $N(L)$ in the construction of $\chi(\mathcal{L})$ to be ϕ_n-invariant and that the restriction of ϕ_n to $\partial N(L)$ extends to a periodic diffeomorphism of the sewed back $N(L)$. Thus φ_n induces an orientation-preserving periodic diffeomorphism of period n on $\chi(\mathcal{L})$. Furthermore, by using the uniqueness of invariant tubular neighborhood (see [5, Section VI.2 and Remark in p.314]), we can see that such periodic diffeomorphisms are unique modulo conjugacy by diffeomorphisms smoothly isotopic to the identity. We denote the periodic diffeomorphism on $\chi(\mathcal{L})$ by $\phi(\mathcal{L}, \varphi_n)$.

Let f be an orientation-preserving periodic diffeomorphism of period n on a compact orientable 3-manifold M. Then f induces a smooth action of the cyclic group $\boldsymbol{Z}_n = \langle t \,|\, t^n = 1\rangle$, with a specified generator t, on M by $t \cdot x = f(x)$ $(x \in M)$. For a subset X of M, $\mathrm{Stab}_f(X) = \{g \in \boldsymbol{Z}_n \,|\, g(X) = X\}$ denotes the stabilizer of X, and $\Sigma(f) = \{x \in M \,|\, \mathrm{Stab}_f(x) \neq \{1\}\}$ denotes the singular set. Then $\Sigma(f)$ is a (possibly empty) 1-dimensional submanifold in M (see [5, Corollary VI.2.4]). A periodic framed link $(\mathcal{L}, \varphi_n)$ is called a *rational surgery description* (surgery description, in brief) of f, if f is conjugate to $\phi(\mathcal{L}, \varphi_n)$, i.e., there is a diffeomorphism $h : M \to \chi(\mathcal{L})$ such that $f = h^{-1} \circ \phi(\mathcal{L}, \varphi_n) \circ h$. A surgery description $(\mathcal{L}, \varphi_n)$ is said to be *integral* if $\mathcal{L}$ is integral. We prove the following theorem, which generalizes Theorems 1.1 and 1.2 of Przytycki and Sokolov [19].

THEOREM 2.1. *Let f be an orientation-preserving periodic diffeomorphism of period n on a closed orientable 3-manifold M. Then the following hold.*

(1) The periodic diffeomorphism f has an integral surgery description.

(2) The periodic diffeomorphism f has an integral surgery description by a non-augmented periodic framed link, if and only if there is a component C of $\Sigma(f)$, such that the action of f on an f-invariant regular neighborhood $N(C)$ is a $2\pi/n$-rotation around the core C of $N(C)$.

COROLLARY 2.2. *Let f be an orientation-preserving periodic diffeomorphism of period n on a closed orientable 3-manifold M. Then there is a compact orientable 4-manifold W admitting an orientation-preserving periodic diffeomorphism $\bar{f}$, such that $\partial W = M$ and the restriction of $\bar{f}$ to M is equal to f.*

REMARK 2.3. In Theorem 2.1 and Corollary 2.2, the assumption that the periodic map f is smooth can be replaced with the weaker assumption that f is *locally linear*, that is, every point in M has an f-invariant neighborhood V_x such that $V_x \cong \boldsymbol{R}^3$ and that the restriction of f on V_x is equivalent to an orthogonal linear map. Because, it has been proved by Kwasik and Lee [13, Theorem 2.1] that any locally linear topological action of a finite group on a closed 3-manifold is smoothable. Furthermore, the uniqueness of equivariant smoothings is established by Kwasik and Schultz [14, Theorem 4.1].

For periodic diffeomorphisms on 3-manifolds with boundary, we have the following.

COROLLARY 2.4. *Let f be an orientation-preserving periodic diffeomorphism of period n on a compact orientable 3-manifold M which satisfies the following conditions:*

1. $\Sigma(f) \cap \partial M = \emptyset$.
2. *For each component F of ∂M, $\mathrm{Stab}_f(F)$ extends to a smooth free action on a handlebody bounded by F.*

Then there is a periodic framed link $(\mathcal{L}, \varphi_n)$ and an φ_n-invariant graph Γ in $S^3 - (L \cup \mathrm{Fix}(\varphi_n))$, such that f is conjugate to the restriction of $\phi(\mathcal{L}, \varphi_n)$ to $\chi(\mathcal{L}) - \mathrm{int}N(\Gamma)$.

We call the triple $(\mathcal{L}, \Gamma, \varphi_n)$ above an *integral surgery description* of f.

3. Construction of a rational surgery description of periodic diffeomorphisms

In this section, we prove a weaker version of Theorem 2.1. Namely, we show the following:

1. *Every orientation-preserving periodic diffeomorphism f on a closed orientable 3-manifold has a rational surgery description.*
2. *f has a rational surgery description by a non-augmented periodic link if and only if the condition in Theorem 2.1 (2) holds.*

Let M be a closed orientable 3-manifold and f an orientation-preserving periodic diffeomorphism of period n on M. As in the previous section, f determines an action of $\boldsymbol{Z}_n = \langle t \,|\, t^n = 1 \rangle$ on M. Let $\mathcal{O} = M/\boldsymbol{Z}_n$ be the quotient (smooth) orbifold (see [1, Section 2.1]), $\pi : M \to \mathcal{O}$ the projection, and $\Sigma = \pi(\Sigma(f))$ the singular set of $\mathcal{O}$. Then the underlying space of $\mathcal{O}$ is a closed 3-manifold and has a natural smooth structure, such that the projection π is smooth and Σ is a smooth 1-submanifold (see the argument of [11, p.64] using the invariant tubular neighborhood theorem [5, Theorem IV.2.2]). We abuse notation by using $\mathcal{O}$ to denote the smooth 3-manifold above. Note that $\pi : M \to \mathcal{O}$ is an n-fold cyclic covering branched over Σ. The branched covering is uniquely determined by the unbranched covering $\pi : M - \Sigma(f) \to \mathcal{O} - \Sigma$, which is determined by an epimorphism $\psi : H_1(\mathcal{O} - \Sigma) \to \boldsymbol{Z}_n$. Thus the periodic diffeomorphism f is uniquely recovered from the triple $(\mathcal{O}, \Sigma, \psi)$ (see Lemma 3.2 below). The idea of the proof of Theorem 2.1 is to convert the triple $(\mathcal{O}, \Sigma, \psi)$ by a sequence of "surgeries" to the triple corresponding to the standard $2\pi/n$-rotation φ_n of S^3. To this end, we introduce a few definitions and lemmas.

Definition 3.1. (1) By a *$\boldsymbol{Z}_n$-orbifold*, we mean a triple $(\mathcal{O}, \Sigma, \psi)$, where $\mathcal{O}$ is a closed orientable smooth 3-manifold, Σ a (possibly empty link) in $\mathcal{O}$, and ψ an epimorphism $\psi : H_1(\mathcal{O} - \Sigma) \to \boldsymbol{Z}_n$, such that the image of the meridian of each component of Σ is not equal to 1.

(2) For a $\boldsymbol{Z}_n$-orbifold $(\mathcal{O}, \Sigma, \psi)$, $\tilde{\chi}(\mathcal{O}, \Sigma, \psi)$ denotes the smooth manifold obtained as the branched covering of $\mathcal{O}$ branched over Σ determined by ψ, and $f(\mathcal{O}, \Sigma, \psi)$ denotes the covering transformation of $\tilde{\chi}(\mathcal{O}, \Sigma, \psi)$ corresponding to the generator $t \in \boldsymbol{Z}_n$. We call $\tilde{\chi}(\mathcal{O}, \Sigma, \psi)$ the *covering manifold* of $(\mathcal{O}, \Sigma, \psi)$ and $f(\mathcal{O}, \Sigma, \psi)$ the *periodic diffeomorphism determined by* $(\mathcal{O}, \Sigma, \psi)$.

The following lemma is proved by the argument of [11, Lemma 1.1] using the uniqueness of invariant tubular neighborhood:

Lemma 3.2. *Let $(\mathcal{O}, \Sigma, \psi)$ and $(\mathcal{O}', \Sigma', \psi')$ be $\boldsymbol{Z}_n$-orbifolds and let $h : \mathcal{O} \to \mathcal{O}'$ be a diffeomorphism such that $h(\Sigma) = \Sigma'$ and $\psi = \psi' \circ h_*$, where $h_* : H_1(\mathcal{O} - \Sigma) \to H_1(\mathcal{O}' - \Sigma')$ is the isomorphism induced by the restriction of h to $\mathcal{O} - \Sigma$. Then there is a smooth isotopy of M, fixed on Σ, carrying h to a diffeomorphism h' which lifts to a diffeomorphism $\tilde{h}' : \tilde{\chi}(\mathcal{O}, \Sigma, \psi) \to \tilde{\chi}(\mathcal{O}', \Sigma', \psi')$, such that $\phi(\mathcal{O}', \Sigma', \psi') = \tilde{h}' \circ \phi(\mathcal{O}, \Sigma, \psi) \circ \tilde{h}'^{-1}$.*

Two $\boldsymbol{Z}_n$-orbifolds are said to be *equivalent* if there is a diffeomorphism $h : \mathcal{O} \to \mathcal{O}'$ satisfying the condition in the lemma above. By virtue of the lemma, we may identify $\boldsymbol{Z}_n$-orbifolds which are equivalent. We note that, by [14, Theorem 4.1], two $\boldsymbol{Z}_n$-orbifolds are equivalent if there is a homeomorphism h satisfying the condition in the lemma.

Definition 3.3. By a *framed link* for a $\boldsymbol{Z}_n$-orbifold $(\mathcal{O}, \Sigma, \psi)$, we mean a framed link $\mathcal{L} = (L, \boldsymbol{\nu})$ in $\mathcal{O}$, such that each component of L is disjoint from Σ or identical with a component of Σ. We use the symbol $\mathcal{O}(\mathcal{L})$ (instead of $\chi(\mathcal{L})$) to denote the 3-manifold obtained from $\mathcal{O}$ by surgery on $\mathcal{L}$.

Lemma 3.4. *Let $\mathcal{L} = (L, \boldsymbol{\nu}) = \{(K_i, \nu_i)\}$ be a framed link for a $\boldsymbol{Z}_n$-orbifold $(\mathcal{O}, \Sigma, \psi)$. Let L_ψ be the sublink of L consisting of those components K_i, such that $\psi(\nu_i) \neq 1$. Put $\Sigma(\mathcal{L}) = (\Sigma - L) \cup L_\psi^* \subset \mathcal{O}(\mathcal{L})$. (To be precise, $\Sigma(\mathcal{L})$ is the union of the image of $\Sigma - L$ in $\mathcal{O}(\mathcal{L})$ and the dual link L_ψ^* to L_ψ.) Then there is an epimorphism $\psi(\mathcal{L}) : H_1(\mathcal{O}(\mathcal{L}) - \Sigma(\mathcal{L})) \to \boldsymbol{Z}_n$, such that $\psi \circ j = \psi(\mathcal{L}) \circ j^*$, where j and j^* denote the homomorphisms among the first homology groups induced by the inclusion maps from $\mathcal{O} - (\Sigma \cup L) = \mathcal{O}(\mathcal{L}) - ((\Sigma - L) \cup L^*)$ to $\mathcal{O} - \Sigma$ and $\mathcal{O}(\mathcal{L}) - \Sigma(\mathcal{L})$, respectively. Moreover, $\psi(\mathcal{L})$ sends the meridian of each component of $\Sigma(\mathcal{L})$ to a non-trivial element of $\boldsymbol{Z}_n$. Thus $(\mathcal{O}(\mathcal{L}), \Sigma(\mathcal{L}), \psi(\mathcal{L}))$ is a $\boldsymbol{Z}_n$-orbifold.*

Proof. This follows from the isomorphism

$$H_1(\mathcal{O}(\mathcal{L}) - \Sigma(\mathcal{L})) \cong H_1(\mathcal{O} - (\Sigma \cup L))/\langle [\nu_i] \,|\, K_i \subset L - L_\psi \rangle$$

and the fact that $\psi(\nu_i) = 1$ if and only if $K_i \subset L - L_\psi$. □

Definition 3.5. (1) Let $\mathcal{L}$, $(\mathcal{O}, \Sigma, \psi)$, and $(\mathcal{O}(\mathcal{L}), \Sigma(\mathcal{L}), \psi(\mathcal{L}))$ be as in Lemma 3.4. Then the $\boldsymbol{Z}_n$-orbifold $(\mathcal{O}(\mathcal{L}), \Sigma(\mathcal{L}), \psi(\mathcal{L}))$ is said to be *obtained from $(\mathcal{O}, \Sigma, \psi)$ by surgery on $\mathcal{L}$.*

(2) By a *covering framed link* of a framed link $\mathcal{L} = (L, \boldsymbol{\nu})$ for $(\mathcal{O}, \Sigma, \psi)$, we mean a framed link $\tilde{\mathcal{L}} = (\tilde{L}, \tilde{\boldsymbol{\nu}})$ in the covering manifold $\tilde{\chi}(\mathcal{O}, \Sigma, \psi)$ obtained as follows:

- $\tilde{L}$ is the inverse image of L.
- Let $\tilde{K}_i$ be a component of $\tilde{\mathcal{L}}$ which is a lift (a connected component of the inverse image) of a component K_i of L. Then its framing is given by a lift of ν_i in $\partial N(\tilde{K}_i)$.

The following lemma is obvious.

Lemma 3.6. *Let $\mathcal{L} = (L, \boldsymbol{\nu})$ be a framed link for a $\boldsymbol{Z}_n$-orbifold $(\mathcal{O}, \Sigma, \psi)$. Then the following hold:*

(1) The covering manifold of $(\mathcal{O}(\mathcal{L}), \Sigma(\mathcal{L}), \psi(\mathcal{L}))$ *is obtained from the covering manifold of* $(\mathcal{O}, \Sigma, \psi)$ *by surgery on the covering framed link* $\tilde{\mathcal{L}}$ *of* $\mathcal{L}$.

(2) Suppose that $(\mathcal{O}, \Sigma, \psi) = (S^3, O, \psi_O)$, *where* O *is a trivial knot in* S^3 *and* ψ_O *is the epimorphism* $H_1(S^3 - O) \to \boldsymbol{Z}_n$ *sending the meridian of* O *to the generator* t *of* $\boldsymbol{Z}_n$. *Let* φ_n *be the periodic diffeomorphism determined by* (S^3, O, ψ_O). *Then* $(\tilde{\mathcal{L}}, \varphi_n)$ *is a periodic framed link and is a surgery description of the periodic diffeomorphism determined by* $(\mathcal{O}(\mathcal{L}), \Sigma(\mathcal{L}), \psi(\mathcal{L}))$.

We now return to the proof of the weaker version of Theorem 2.1, and convert the $\boldsymbol{Z}_n$-orbifold $(\mathcal{O}, \Sigma, \psi)$ into (S^3, O, ψ_O) by a sequence of surgeries on framed links through the following Steps 1 - 4.

Step 1. *We show that there is a framed link* $\mathcal{L}_0$ *for* $(\mathcal{O}, \Sigma, \psi)$, *such that the* $\boldsymbol{Z}_n$-*orbifold* $(\mathcal{O}_1, \Sigma_1, \psi_1)$ *obtained from* $(\mathcal{O}, \Sigma, \psi)$ *by surgery on* $\mathcal{L}_0 = (L_0, \boldsymbol{\nu}_0)$ *satisfies the condition* $\Sigma_1 = \emptyset$.

To this end, we use the idea of Hartley [8, Theorem 1.1] (cf. [17, Proof of Theorem 3], [19, Proof of Theorem 1.2]). Let $K_1, \cdots, K_m$ be the components of Σ. Then we have the following.

Lemma 3.7. *For each component* K_i *of* Σ, *there is a slope* ν_i *of* K_i, *such that* $\psi([\nu_i]) = 1$.

Proof. Let j be the homomorphism $H_1(\partial N(K_i)) \to H_1(\mathcal{O} - \Sigma)$ induced by the inclusion. Then, since the image of the homomorphism $\psi \circ j : H_1(\partial N(K_i)) \to \boldsymbol{Z}_n$ is cyclic, its kernel contains a primitive element of $H_1(\partial N(K_i))$, and hence represented by a slope ν_i. □

Put $\boldsymbol{\nu}_0 = (\nu_1, \cdots, \nu_m)$, where ν_i is as in Lemma 3.7, and put $\mathcal{L}_0 = (\Sigma, \boldsymbol{\nu}_0)$. Then it is a framed link for $(\mathcal{O}, \Sigma, \psi)$ and satisfies the desired condition.

Step 2. *We show that there is an integral framed knot* $\mathcal{L}_1 = (L_1, \nu_1)$ *for* $(\mathcal{O}_1, \Sigma_1, \psi_1)$ *which satisfies the following conditions:*

- L_1 *is disjoint from the dual link* L_0^* *in* $\mathcal{O}_1$.
- *Let* $(\mathcal{O}_2, \Sigma_2, \psi_2)$ *be the* $\boldsymbol{Z}_n$-*orbifold obtained from* $(\mathcal{O}_1, \Sigma_1, \psi_1)$ *by surgery on* $\mathcal{L}_1$. *Then* $\Sigma_2 = L_1^*$ *and* ψ_2 *sends the meridian of* Σ_2 *to* t.

To this end, choose a knot L_1 in $\mathcal{O}_1$ disjoint from L_0^*, such that $\psi_1([L_1]) = t$. Let ν_1 be an arbitrary integral slope of L_1. Then we have $\psi_1([\nu_1]) = \psi_1([L_1]) = t$, and hence the framed knot $\mathcal{L}_1 = (L_1, \nu_1)$ for $(\mathcal{O}_1, \Sigma_1, \psi_1)$ satisfies the desired conditions.

Step 3. *We show that there is an integral framed link* $\mathcal{L}_2 = (L_2, \boldsymbol{\nu}_2)$ *for* $(\mathcal{O}_2, \Sigma_2, \psi_2)$ *which satisfies the following conditions:*

- L_2 *is disjoint from the dual link* $L_0^* \cup L_1^* = L_0^* \cup \Sigma_2$*.*
- *Let* $(\mathcal{O}_3, \Sigma_3, \psi_3)$ *be the* $\boldsymbol{Z}_n$*-orbifold obtained from* $(\mathcal{O}_2, \Sigma_2, \psi_2)$ *by surgery on* $\mathcal{L}_2$*. Then* $\mathcal{O}_3 \cong S^3$*,* Σ_3 *is equal to the image of the knot* Σ_2 *in* $\mathcal{O}_3$*, and* ψ_3 *sends the meridian of* Σ_3 *to* t*.*

By the celebrated theorem of Wallace [23] and Lickorish [15], there is an integral framed link $\mathcal{L}_2 = (L_2, \boldsymbol{\nu}_2)$ in M_2, such that $\chi(\mathcal{L}_2) = S^3$. We may assume $\mathcal{L}_2$ is disjoint from $L_0^* \cup L_1^*$, in particular, $L_2 \subset \mathcal{O}_2 - \Sigma_2$. Then the desired result is obtained by the following lemma, which we prove by using the idea of [19, Proof of Theorem 1.1].

LEMMA 3.8. *We can choose* $\mathcal{L}_2$ *so that it satisfies the following conditions:*

1. L_2 *is disjoint from* $L_0^* \cup L_1^*$*.*
2. *For each component* K_i *of* L_2*, we have* $\psi_2([K_i]) = 1$*, and hence* $\psi_2([\nu_i]) = 1$*, where* ν_i *is the framing of* K_i *in* $\mathcal{L}_2$*.*

Proof. Let μ' be the meridian of the knot Σ_2 which is oriented so that $\psi_2([\mu']) = t^{-1}$. Suppose $\psi([K_i]) = t^c$ for some c with $0 < c < n$. Let K_i' be the knot in $\mathcal{O}_2 - (L_0^* \cup L_1^* \cup (L_2 - K_i))$ obtained by a band sum of K_i and c parallel copies of μ', where the bands are also disjoint from $L_0^* \cup L_1^* \cup (L_2 - K_i)$. Then $[K_i'] = [K_i] + c[\mu'] \in H_1(\mathcal{O}_2 - \Sigma_2)$ and hence $\psi_2([K_i']) = 1$. Furthermore, K_i is (smoothly) isotopic to K_i' in $\mathcal{O}_2 - (L_2 - K_i)$. Hence, we can replace the component K_i of L_2 by K_i' without changing the isotopy type of L_2 in $\mathcal{O}_2$. By repeating this procedure, we obtain the desired framed link $\mathcal{L}_2$. □

Step 4. *We show that there is a framed link* $\mathcal{L}_3 = (L_3, \boldsymbol{\nu}_3)$ *for* $(\mathcal{O}_3, \Sigma_3, \psi_3)$ *which satisfies the following conditions.*

- *L_3 is disjoint from the dual link $L_0^* \cup L_1^* \cup L_2^* = L_0^* \cup \Sigma_3 \cup L_2^*$.*
- *Let $(\mathcal{O}_4, \Sigma_4, \psi_4)$ be the $\boldsymbol{Z}_n$-orbifold obtained from $(\mathcal{O}_3, \Sigma_3, \psi_3)$ by surgery on $\mathcal{L}_3$. Then $\mathcal{O}_4 \cong S^3$, Σ_4 is equal to the image of the knot Σ_3 in $\mathcal{O}_4$ and is a trivial knot, and ψ_4 sends the meridian of Σ_4 to t. In particular, $(\mathcal{O}_4, \Sigma_4, \psi_4)$ is equivalent to (S^3, O, ψ_O).*

This step is performed by using the following fact as in the final step of [19, Proof of Theorem 1.1]: The knot Σ_3 in $\mathcal{O}_3 \cong S^3$ can be transformed into a trivial knot by a sequence of crossing changes, and each of them can be realized by ± 1-surgery on a trivial knot whose linking number with Σ_3 is 0.

Now the proof of the weaker version of Theorem 2.1 is completed as follows. Let $\mathcal{L}$ be the framed link in $\mathcal{O}_4 \cong S^3$ obtained as the union of (the images of) the dual framed links to $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$. Then, by the construction, $\mathcal{L}$ is a framed link for the $\boldsymbol{Z}_n$-orbifold $(\mathcal{O}_4, \Sigma_4, \psi_4) \cong (S^3, O, \psi_O)$, and $(\mathcal{O}, \Sigma, \psi)$ is obtained by surgery on $\mathcal{L}$. Let $\tilde{\mathcal{L}}$ be the covering framed link of $\mathcal{L}$ in the covering manifold $\tilde{\chi}(S^3, O, \psi_O) \cong S^3$, and let $\varphi_n := \phi(S^3, O, \psi_O)$ be the periodic diffeomorphism on $\tilde{\chi}(S^3, O, \psi_O) \cong S^3$ determined by (S^3, O, ψ_O). Then, by Lemma 3.6, $(\tilde{\mathcal{L}}, \varphi_n)$ is an (augmented) periodic framed link of period n, and gives a surgery description of the periodic diffeomorphism f on $M \cong \tilde{\chi}(\mathcal{O}, \Sigma, \psi)$. This completes the proof of the weaker version of Theorem 2.1 (1).

The “only if” part of the weaker version of Theorem 2.1 (2) is obvious. To prove the “if” part, suppose that there is a component C of $\Sigma(f)$ as in Theorem 2.1 (2). Let K_i be the component of Σ obtained as the image of C. Then we replace the framed link $\mathcal{L}_1$ in Step 1 with $\mathcal{L}_1 - (K_i, \nu_i)$, and replace the framed knot $\mathcal{L}_2$ with an empty framed link. Then we see the resulting framed link $\tilde{\mathcal{L}}$ is a non-augmented periodic framed link describing f.

Note that the framed links $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ are integral and disjoint from the singular set. This implies that the components of the framed link $\tilde{\mathcal{L}}$ covering the dual framed links to $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ are integral (cf. Remark 4.2). Hence, $\tilde{\mathcal{L}}$ is integral if and only if the covering framed link to $\mathcal{L}_0$ is integral. In the next section, we show that we can choose $\mathcal{L}_0$ so that this condition is satisfied.

4. Lifting to an integral framed link

Throughout this section, the symbol $|\cdot|$ denotes the order of a group, the order of an element of a group, or the number of the connected components of a topological space, according as "·" is a group, a group element, or a topological space.

LEMMA 4.1. *Let (K,ν) be a framed knot for a $\boldsymbol{Z}_n$-orbifold $(\mathcal{O},\Sigma,\psi)$, and let $(\tilde{K},\tilde{\nu})$ be a component of the covering framed link of (K,ν). Then $(\tilde{K},\tilde{\nu})$ is integral, if and only if*

$$|\mathrm{int}([\nu],[\mu])| = \frac{|\psi(H_1(\partial N(K)))|}{|\psi([\nu])|\cdot|\psi([\mu])|},$$

where μ is the meridian of K, int *denotes the intersection pairing on $H_1(\partial N(K))$, and the symbol ψ denotes the homomorphism $H_1(\partial N(K)) \to \boldsymbol{Z}_n$ obtained as the composition of the natural homomorphism $H_1(\partial N(K)) \to H_1(\mathcal{O}-\Sigma)$ and the homomorphism $\psi : H_1(\mathcal{O}-\Sigma) \to \boldsymbol{Z}_n$.*

Proof. Note that $\tilde{\nu}$ is integral if and only if the intersection number $\mathrm{int}([\tilde{\nu}],[\tilde{\mu}])$ is equal to ± 1, where $\tilde{\mu}$ be the meridian of $\tilde{K}$. To calculate the intersection number, put $n' = |\psi(H_1(\partial N(K)))|$, and let π denote the restriction of the covering projection to $\partial N(\tilde{K})$ (instead of the covering projection itself). Then $\pi^{-1}(\nu)$ and $\pi^{-1}(\mu)$, respectively, consist of $n'/|\psi([\nu])|$ parallel copies of $\tilde{\nu}$ and $n'/|\psi([\mu])|$ parallel copies of $\tilde{\mu}$. So, we have

$$\mathrm{int}([\pi^{-1}(\nu)],[\pi^{-1}(\mu)]) = \frac{(n')^2}{|\psi([\nu])|\cdot|\psi([\mu])|}\cdot \mathrm{int}([\tilde{\nu}],[\tilde{\mu}]).$$

On the other hand, since $\partial N(\tilde{K})$ is an n'-fold covering of $\partial N(K)$, we have

$$\mathrm{int}([\pi^{-1}(\nu)],[\pi^{-1}(\mu)]) = n'\cdot \mathrm{int}([\nu],[\mu]).$$

Thus we have,

$$\mathrm{int}([\nu],[\mu]) = \frac{n'}{|\psi([\nu])|\cdot|\psi([\mu])|}\cdot \mathrm{int}([\tilde{\nu}],[\tilde{\mu}]).$$

Hence $\mathrm{int}([\tilde{\nu}],[\tilde{\mu}]) = 1$ if and only if the identity in the lemma holds. This completes the proof. □

REMARK 4.2. In Lemma 4.1, if K is disjoint from Σ, then $(\tilde{K}, \tilde{\nu})$ is integral if and only if (K, ν) is integral. However, if K is a component of Σ, then this does not hold.

LEMMA 4.3. *Let $(\mathcal{O}, \Sigma, \psi)$ be a $\boldsymbol{Z}_n$-orbifold and K a component of Σ. Then there is a slope ν of K, such that $\psi([\nu]) = 1$ and the covering framed link of (K, ν) is integral.*

Proof. Let λ and μ be a longitude and the meridian of K, and put $n' = |\psi(H_1(\partial N(K)))|$, $n_0 = |\langle\psi([\lambda])\rangle \cap \langle\psi([\mu])\rangle|$, $n_\lambda = n'/|\psi([\lambda])|$, and $n_\mu = n'/|\psi([\mu])|$. Then we have $n' = n_0 n_\lambda n_\mu$, because

$$n' = |\psi(H_1(\partial N(K)))| = \frac{|\psi([\lambda])| \cdot |\psi([\mu])|}{|\langle\psi([\lambda])\rangle \cap \langle\psi([\mu])\rangle|} = \frac{(n'/n_\lambda)(n'/n_\mu)}{n_0}.$$

Furthermore, we have $\gcd(n_\lambda, n_\mu) = 1$, because

$$\begin{aligned} \boldsymbol{Z}/n_\lambda\boldsymbol{Z} \times \boldsymbol{Z}/n_\mu\boldsymbol{Z} &\cong \frac{\langle\psi([\lambda])\rangle}{\langle\psi([\lambda])\rangle \cap \langle\psi([\mu])\rangle} \times \frac{\langle\psi([\mu])\rangle}{\langle\psi([\lambda])\rangle \cap \langle\psi([\mu])\rangle} \\ &\cong \frac{\psi(H_1(\partial N(K)))}{\langle\psi([\lambda])\rangle \cap \langle\psi([\mu])\rangle} \\ &\cong \boldsymbol{Z}/(n_\lambda n_\mu)\boldsymbol{Z}. \end{aligned}$$

Choose a generator s of $\psi(H_1(\partial N(K)))$, such that $\psi([\lambda]) = s^{n_\lambda}$. Then we have $\psi([\mu]) = s^{n_\mu u}$ for some integer u, such that $\gcd(u, n_0 n_\lambda) = 1$. Note that $\gcd(n_\lambda, n_\mu u) = 1$, and hence there are integers b_1 and b_2, such that

$$\begin{vmatrix} n_\lambda & n_\mu u \\ b_1 & b_2 \end{vmatrix} = 1.$$

Let α and β be the slopes of K, such that

$$[\alpha] = n_\mu u[\lambda] - n_\lambda[\mu], \quad [\beta] = b_2[\lambda] - b_1[\mu] \quad \in H_1(\partial N(K)).$$

Then we have

$$\psi([\alpha]) = 1, \quad \psi([\beta]) = s.$$

Hence, $\ker(\psi : H_1(\partial N(K)) \to \boldsymbol{Z}_n)$ is freely generated by $[\alpha]$ and $n'[\beta]$. Let ν be a slope of K, such that $[\nu] \in \ker\psi$. Then $[\nu] =$

$p[\alpha]+qn'[\beta]$ for some integers p and q with $\gcd(p,qn')=1$. Note that

$$\mathrm{int}([\nu],[\mu])=pn_\mu u+qn'b_2=n_\mu(pu+qv),\quad \text{where}\quad v=n_0n_\lambda b_2.$$

By Lemma 4.1, the covering framed link of (K,ν) is integral if and only if

$$|\mathrm{int}([\nu],[\mu])|=\frac{|\psi(H_1(\partial N(K)))|}{|\psi([\nu])|\cdot|\psi([\mu])|}=\frac{n'}{1\cdot(n'/n_\mu)}=n_\mu.$$

Hence, ν satisfies the desired condition if and only if $pu+qv=\pm1$. Thus the proof of the lemma is reduced to showing the existence of integers p and q, such that

$$pu+qv=\pm1,\quad \gcd(p,qn')=\gcd(p,qn_0n_\lambda n_\mu)=1.$$

The second condition is replaced with the condition $\gcd(p,n_\mu)=1$, because the first condition implies $\gcd(p,qn_0n_\lambda b_2)=1$.

To prove the existence of such integers, recall that $\gcd(u,n_0n_\lambda)=1$ and $\gcd(u,b_2)=1$ (because $n_\lambda b_2-n_\mu ub_1=1$), and hence $\gcd(u,v)=\gcd(u,n_0n_\lambda b_2)=1$. Thus there are integers p_0 and q_0, such that $p_0u+q_0v=1$. For each integer k, put $p_k=p_0+kv$ and $q_k=q_0-ku$. Then $p_ku+q_kv=1$. Thus we have only to show that $\gcd(p_k,n_\mu)=1$ for some k. To this end, note that if r is a prime factor of n_μ dividing v, then r does not divide p_k for any k. (This follows from the fact that $\gcd(p_0,v)=1$.) Let $r_1,\cdots,r_m$ be the prime factors of n_μ which does not divide v. By the observation above, we have only to show that there is an integer k, such that $p_k\not\equiv 0 \pmod{r_i}$ $(i=1,\cdots,m)$. To this end, we use the following natural isomorphism

$$\boldsymbol{Z}/(r_1\cdots r_m)\boldsymbol{Z}\cong\boldsymbol{Z}/r_1\boldsymbol{Z}\times\cdots\times\boldsymbol{Z}/r_m\boldsymbol{Z}.$$

For each $i=1,\cdots,m$, choose an integer c_i, such that $c_i\not\equiv -p_0$ $(\mathrm{mod}\ r_i)$. By the isomorphism above, there is an integer c, such that $c\equiv c_i\not\equiv -p_0 \pmod{r_i}$ $(i=1,\cdots,m)$. Since $\gcd(v,r_1\cdots r_m)=1$, there is an integer k such that $kv\equiv c \pmod{r_1\cdots r_m}$. Then $p_k=p_0+kv\equiv p_0+c_i\not\equiv 0 \pmod{r_i}$ for each $i=1,\cdots,m$. This completes the proof. □

By Lemma 4.3, we can choose the framed link $\mathcal{L}_0$ in the previous section so that its covering framed link is integral. Together with the comment at the end of the last section, this completes the proof of Theorem 2.1.

In the remainder of this section, we prove Corollaries 2.2 and 2.4. To prove Corollary 2.2, let $(\mathcal{L}, \varphi_n)$ be an integral surgery description of an orientation-preserving periodic diffeomorphism on a closed 3-manifold M. Since $\mathcal{L}$ is integral, surgery on $\mathcal{L}$ is realized by attaching 2-handles to $S^3 = \partial B^4$ along $\mathcal{L}$, and hence $M = \chi(\mathcal{L})$ is equal to the boundary of the smooth 4-manifold obtained in this way (with straightning of the angle). Furthermore, the periodic diffeomorphism φ_n extends to that on W. This completes the proof of Corollary 2.2.

Corollary 2.4 is proved as follows. Let M and f be as in Corollary 2.4. Then there is a closed orientable 3-manifold $\hat{M}$ with an orientation-preserving periodic diffeomorphism $\hat{f}$, satisfying the following conditions:

1. $\hat{M} = M \cup V$ and $M \cap V = \partial M = \partial V$, where V is a disjoint union of handlebodies.
2. The restriction of $\hat{f}$ to M is equal to f.
3. The singular set $\Sigma(\hat{f})$ is disjoint from V.

Let $(\mathcal{O}, \Sigma, \psi)$ be the $\boldsymbol{Z}_n$-orbifold corresponding to the periodic diffeomorphism $\hat{f}$ on $\hat{M}$. By the last condition, there is a graph Γ_0 in $\mathcal{O} - \Sigma$, such that $V/\hat{f} = N(\Gamma_0)$. We can choose the framed links $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ in Section 3 in such a way that they are disjoint from (the image of) Γ_0. Let $(\tilde{\mathcal{L}}, \varphi_n)$ be the periodic framed link as in Section 3 constructed from the above framed links $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$, and let Γ be the inverse image of the graph Γ_0 in $\mathcal{O}_4$, where $\mathcal{O}_4$ is as in Section 3. Then we see that the triple $(\tilde{L}, \Gamma, \varphi_n)$ gives a surgery description of f.

5. Visualizaton of the isometry groups of 2-component hyperbolic links with ≤ 9 crossings

In [9], Henry and Weeks have determined the isometry groups of the complements of the hyperbolic knots and links in the table of Rolfsen

[20] by improving the computer program SnapPea [24]. By virtue of the knot complementary theorem [7], we see that, for any hyperbolic knot K in S^3, the isometry group $\mathrm{Isom}(S^3 - K)$ of the complete hyperbolic manifold $S^3 - K$ is isomorphic to the *symmetry group* $\mathrm{Sym}(S^3, K)$, the group of the diffeomorphisms of the pair (S^3, K) modulo pairwise isotopy. However, this does not hold for hyperbolic links. Namely, for a hyperbolic link L, the natural homomorphism $\mathrm{Sym}(S^3, L) \to \mathrm{Isom}(S^3 - L)$ is not necessarily surjective, though it is always injective. In fact, the following follows from the table of [9].

PROPOSITION 5.1. *The following is the list of 2-component hyperbolic links with* ≤ 9 *crossings, such that* $\mathrm{Sym}(S^3, L)$ *is a proper subgroup of* $\mathrm{Isom}(S^3 - L)$. *Here* D_n *denotes the dihedral group of order* $2n$.

L	7^2_8	8^2_{15}	8^2_{16}	9^2_{45}	9^2_{46}	9^2_{47}	9^2_{50}
Isom	D_4	D_4	$(\boldsymbol{Z}_2)^3$	$(\boldsymbol{Z}_2)^3$	D_4	D_4	$(\boldsymbol{Z}_2)^2$
Sym	$(\boldsymbol{Z}_2)^2$	$(\boldsymbol{Z}_2)^2$	$(\boldsymbol{Z}_2)^2$	$(\boldsymbol{Z}_2)^2$	$(\boldsymbol{Z}_2)^2$	$(\boldsymbol{Z}_2)^2$	$(\boldsymbol{Z}_2)$

REMARK 5.2. Though the isometry groups of the complements of the links 9^2_{24} and 9^2_{40} were not determined in [9], the current version of SnapPea [24] can calculate them: According to SnapPea, they are isomorphic to D_6 and D_3, respectively, and are isomorphic to the symmetry groups.

In this section, we give "visualizations" of the full isometry groups of the links in the table above. The following proposition enables us to see the full isometry groups of the complements of the links in the table, except 9^2_{50}.

PROPOSITION 5.3. *For non-zero integers* a *and* b, *let* $L(a, b)$ *be the Montesinos link* $M(0; (2, 1), (2, -1), (2ab + 1, b))$ *and* $L(b)$ *be the 2-bridge link* $S(4b, 2b - 1)$. *Then we have the following:*

(1) The complement of $L(a, b)$ *is diffeomorphic to that of* $L(b)$.

(2) The complement of $L(a, b)$ *is hyperbolic, (i.e., it admits a complete hyperbolic structure of finite volume) if and only if* $2b - 1 \neq \pm 1$.

(3) Suppose $2b - 1 \neq \pm 1$. *Then we have the following:*

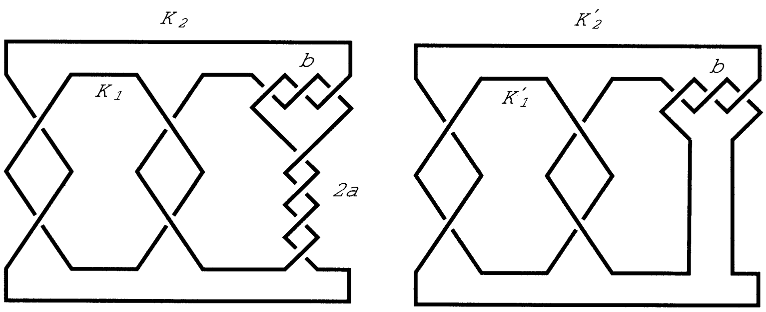
- $\mathrm{Sym}(S^3, L(a, b)) \cong (\boldsymbol{Z}_2)^2$.

- $\mathrm{Isom}(S^3 - L(a,b)) \cong \mathrm{Isom}(S^3 - L(b)) \cong \mathrm{Sym}(S^3, L(b))$, *and they are isomorphic to* D_4 *or* $(\boldsymbol{Z}_2)^3$ *according as* b *is even or odd.*

Proof. (1) Let K_1 and K_2 [resp. K_1' and K_2'] be the components of $L(a,b)$ [resp. $L(b)$] depicted in Figure 1. Then, by twisting the pair $(S^3 - K_1, K_2)$ along the interior of the disk bounded by K_1, we see it is diffeomorphic to the pair $(S^3 - K_1', K_2')$. This implies $S^3 - L(a,b) \cong S^3 - L(b)$.

(2) This follows from the well-known fact that a 2-bridge knot/link $S(p,q)$ is hyperbolic if and only if $q \not\equiv \pm 1 \pmod{p}$.

(3) Recalling the fact that the isometry group of a cusped hyperbolic manifold is isomorphic to the combinatorial automorphism group of the canonical triangulation of the manifold (see [9]), we see that the isometry groups of hyperbolic 2-bridge links are determined by [22] and [2]. Because, [22] gives presumably canonical topological ideal triangulations of 2-bridge link complements and calculates their automorphism groups, and [2] proves that the triangulations are actually canonical. The symmetry group of $L(a,b)$ is obtained as the subgroup of $\mathrm{Isom}(S^3 - L(a,b)) \cong \mathrm{Isom}(S^3 - L(b))$ preserving the meridians of the components. Of course, this coincides with that obtained in [21] by assuming the orbifold theorem, which is now established by [1],[3], and [4]. □



L (a;b) = M (0;(2;1);(2, -1), (2ab+1, b)) L (b) = S (4b, 2b-1)
a = 2, b = 3

Figure 1

The links 7_8^2, 8_{15}^2, 8_{16}^2, 9_{45}^2, 9_{46}^2, and 9_{47}^2 are equivalent to the links $L(1,-2)$, $L(1,2)$, $L(1,-3)$, $L(1,3)$, $L(1,-4)$, and $L(2,-2)$, respectively, and the proposition above enables us to "see" their isometry groups.

However, for the remaining link $L = 9_{50}^2$, we cannot expect such a visualization, because there is not a link L' in S^3, such that $S^3 - L' \cong S^3 - L$ and $\mathrm{Sym}(S^3, L') \cong \mathrm{Isom}(S^3 - L') \cong \mathrm{Isom}(S^3 - L)$ by the following proposition.

Proposition 5.4. *Let $L = K_1 \cup K_2$ be the link 9_{50}^2 (see Figure 2), and let $\chi(p_1/q_1, p_2/q_2)$ be the manifold obtained from S^3 by surgery on the framed link $(L, (p_1/q_1, p_2/q_2))$, where p_i and q_i are relatively prime integers for $i = 1, 2$. Then $\mathrm{Isom}(S^3 - L)$ extends to an action on $\chi(p_1/q_1, p_2/q_2)$ if and only if $p_2/q_2 = (2p_1 - q_1)/p_1$. In particular, $\mathrm{Isom}(S^3 - L)$ does not extend to an action on a homology 3-sphere obtained by surgery on L.*

To prove this proposition, we need the following lemma.

Lemma 5.5. *Let $L = K_1 \cup K_2$ be as in Proposition 5.4. Let f be an isometry of $S^3 - L$ which does not extend to a homeomorphism of the pair (S^3, L). Then f interchanges the two cusps and $f_* : H_1(\partial N(K_1)) \to H_1(\partial N(K_2))$ is given by*

$$f_* \begin{pmatrix} \lambda_1 \\ \mu_1 \end{pmatrix} = \pm \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \lambda_2 \\ \mu_2 \end{pmatrix}$$

where λ_i and μ_i are the meridian and the preferred longitude of K_i, respectively.

Proof. Though this lemma can be obtained by using SnapPea, we give an elementary proof by using the Alexander polynomial. Let $\Delta_L(t_1, t_2)$ be the Alexander polynomial of L. Then we have

$$\Delta_L(t_1, t_2) = t_1^{-1} t_2^{-1} - t_2^{-1} + 1 - t_2 + t_1 t_2,$$

where t_i corresponds to $\mu_i \in H_1(S^3 - L)$. (This polynomial is different from that in the table of [20] because of the difference choice of the orientation of L.) Note that we have $t_1 = \mu_1 = \lambda_2$ and $t_2 = \mu_2 = \lambda_1$ as elements of $H_1(S^3 - L)$, because the linking number of K_1 and K_2 is equal to 1. Suppose that the automorphism f_* of $H_1(S^3 - L)$ is given by

$$f_* \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}.$$

Then $\Delta_L(t_1^a t_2^b, t_1^c t_2^d)$ is equal to $\Delta_L(t_1, t_2)$ modulo trivial units of the integral group ring of the free abelian group $H_1(S^3 - L)$. By using this fact and the assumption that f does not extend to a homeomorphism of (S^3, K), we can see that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}.$$

Suppose that f preserves each of the two cusps. Then we have $f_*(\mu_1) = \mu_1 + 2\lambda_1 \in H_1(\partial N(K_1))$ and $f_*(\mu_2) = -\mu_2 \in H_1(\partial N(K_2))$. Thus f extends to a homeomorphism from S^3 to the manifold obtained from S^3 by 1/2-surgery on the trefoil knot K_1, a contradiction. Hence f interchanges the two cusps and we obtain the desired identity. □

We now prove to Proposition 5.4. By Lemma 5.5, we have

$$f_*(p_1\mu_1 + q_1\lambda_1) = (2p_1 - q_1)\mu_2 + p_1\lambda_2 \in H_1(\partial N(K_2)).$$

The first assertion follows from this identity. The second assertion follows from the fact that the order of $H_1(\chi(p_1/q_1, (2p_1 - q_1)/p_1))$ is

$$\begin{vmatrix} p_1 & q_1 \\ p_1 & 2p_1 - q_1 \end{vmatrix} = 2p_1(p_1 - q_1) \neq \pm 1.$$

Finally, we give a visualization of the group $\mathrm{Isom}(S^3 - L)$ for $L = 9^2_{50}$. The "simplest" closed manifold obtained by Dehn surgery on L, satisfying the condition of Proposition 5.4, is the manifold $\chi(0/1, 1/0)$. Then the following hold.

1. $\chi(0/1, 1/0)$ is obtained by 0-surgery on the trefoil knot K_1.
2. The dual knot K_1^* is isotopic to the knot in $S^3 - \mathrm{int}N(K_1) \subset \chi(0/1, 1/0)$ represented by a meridian of K_1 and the dual knot K_2^* is identified with $K_2 \subset S^3 - \mathrm{int}N(K_1) \subset \chi(0/1, 1/0)$.

3. $S^3 - (K_1 \cup K_2) \cong \chi(0/1, 1/0) - (K_1^* \cup K_2^*)$

We perform a crossing change between $K_2 = K_2^*$ and the core of the sewed back solid torus $N(K_1)$ as illustrated in Figure 2. Then further isotopy shows that the pair $(\chi(0/1, 1/0), K_1^* \cup K_2^*)$ is homeomorphic to the symmetric pair illustrated in Figure 2. This figure enables us to see $\mathrm{Isom}(S^3 - L)$, and gives a surgery description of the "exotic" isometry of $S^3 - L$.

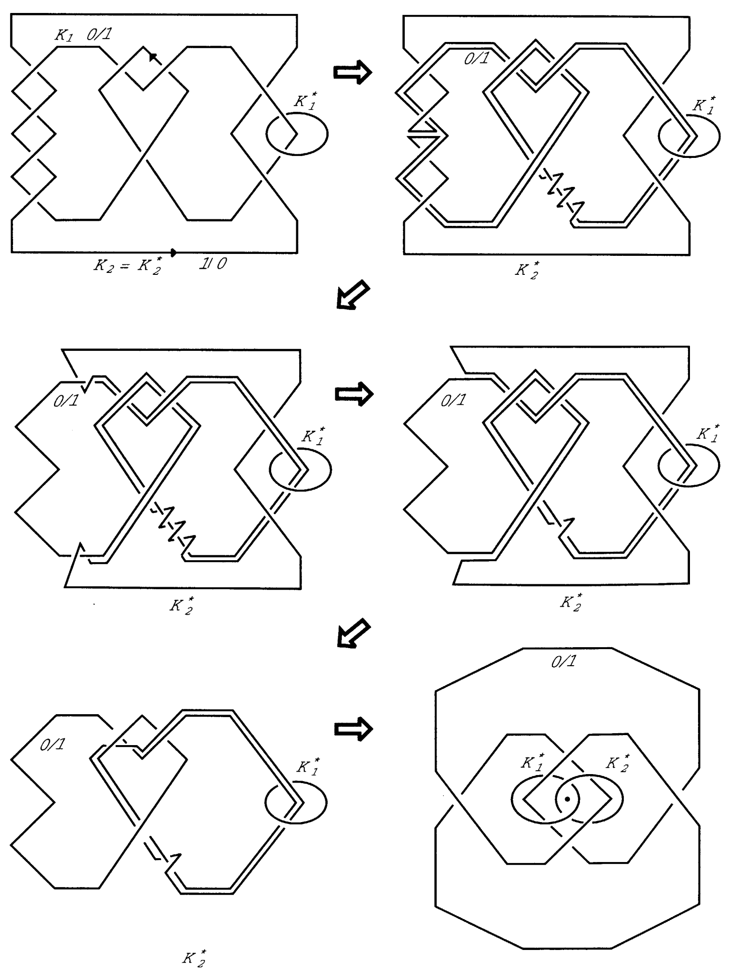


Figure 2

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