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# A Riemann-type Minimal Integral for the Classical Problem of Primitives

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SUMMARY. - *Properties of the C-integral are given. In [3] it is proved that the C-integral is the minimal integral which includes Lebesgue integrable functions and derivatives. The proof of the result in the general case is very involved and technical. In this note it is sketched in a particular case.*

Let $F : [a,b] \to \mathbf{R}$ be a differentiable function. The problem of recovering $F$ from its derivative by integration is called *problem of primitives.*

To this end the Lebesgue integral is a good tool, when it works. In fact the function $F(x) = x^2 sin\frac{1}{x^2}$, if $0 < x \leq 1$, $F(x) = 0$ if $x = 0$, is differentiable at each point of $[0,1]$, but its derivative is not Lebesgue integrable.

In 1912 A. Denjoy provided a first solution of the problem of primitives, based on an integration process, called *totalization.* A second solution, based on the notion of major and minor functions was given in 1914 by O. Perron. A third solution, based on generalized Riemann sums was obtained indipendently by J. Kurzweil in 1957 and by R. Henstock in 1963. These three integration methods are equivalent.

In 1986 A.M. Bruckner, R.J. Fleissner and J. Foran (see [4]) observed that the previous solutions possess a generality which is not needed for this purpose. Indeed the function $F(x) = x\, sin\frac{1}{x^2}$, if $0 < x \leq 1$, $F(x) = 0$ if $x = 0$, is a primitive for the Denjoy-Perron-

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Kurzweil-Henstock integral, but it is neither a Lebesgue primitive, nor a differentiable function, or a sum of a Lebesgue primitive and a differentiable function.

Therefore the following question arises: *Is there a minimal integral which includes Lebesgue integrable functions and derivatives?*

The descriptive version of this problem has been treated by Bruckner, Fleissner and Foran still in [4]. Their approach is based on the simple observation that for the required minimal integral, a function $F$ is the indefinite integral of $f$ if and only if $F' = f$ almost everywhere and there exist a differentiable function $H_1$ and an absolutely continuous function $H_2$ such that $F = H_1 + H_2$.

In 1996 B. Bongiorno (see [1]) introduced a Riemann-type integration process, called *C-integration*, which integrates the derivatives and falls exactly in between the Lebesgue integral and the Denjoy integral.

In [3], in cooperation with B. Bongiorno and D. Preiss, I prove that the C-integral integrates precisely the functions which are sum of a derivative and of a Lebesgue integrable function. Then the C-integral is the constructive minimal integral which includes derivatives and Lebesgue integrable functions.

The proof of the result in [3] in the general case is very involved and technical. In this note I sketch it in a particular case.

## 1. The C-integral

A "gage" on the interval $[a, b] \subset \mathbf{R}$ is a positive function defined on $[a, b]$. Given a gage $\delta$, a "$\delta$-fine partition" of $[a, b]$ is a collection $\{(I_i, x_i) : i = 1, \ldots, p\}$ of pairwise nonoverlapping intervals $I_i \subset [a, b]$ such that $\bigcup_{i=1}^{p} I_i = [a, b]$ and $I_i \subset (x_i - \delta(x_i), x_i + \delta(x_i))$. If $\bigcup_{i=1}^{p} I_i \subset [a, b]$, the collection $\{(I_i, x_i) : i = 1, \ldots, p\}$ is called a "$\delta$-fine partial partition" of $[a, b]$.

Definition 1.1. *(see [1]) Let $f: [a, b] \to \mathbf{R}$. It is said that $f$ is C-integrable on $[a, b]$ if there exists a constant $A$ such that given $\varepsilon > 0$ there is a gage $\delta$ with*

$$\left| \sum_{i=1}^{p} f(x_i)|I_i| - A \right| < \varepsilon ,$$

*for each $\delta$-fine partition $\{(I_i, x_i) : i = 1, \dots, p\}$ of $[a, b]$ satisfying the condition*

$$\sum_{i=1}^{p} \operatorname{dist}(x_i, I_i) < 1/\varepsilon. \tag{1}$$

*The number $A$ is called $C$-integral of $f$ on $[a, b]$ and we write $A = (C)\int_a^b f$.*

I note that each Lebesgue integrable function is C-integrable. The easiest way to see this is to recall that the Lebesgue integral is equivalent to the McShane integral and observe that the McShane integral is included in the C-integral. I note also that the C-integrability implies the Henstock-Kurzweil integrability (with the same value of the integral).

## 2. Properties of the C-integral

(see [1, 2])

- The $C$-integral is linear, in particular the space of $C$-integrable functions is a vector space.
- $C$-integrability on an interval implies $C$-integrability on each subinterval.
- Henstock's lemma: $f$ is $C$-integrable if and only if for each $\varepsilon > 0$ there exists a gage $\delta$ such that

  $$\sum_{i=1}^{p} \left| f(x_i)|I_i| - (C)\int_{I_i} f \right| < \varepsilon\ ,$$

  for each $\delta$-fine partial partition $\{(I_i, x_i)\}$, $i = 1, \cdots, p$, satisfying the condition $\sum_{i=1}^{p} \operatorname{dist}(x_i, I_i) < 1/\varepsilon$.
- All derivatives are $C$-integrable

  *Proof.* Let $f(x) = F'(x)$ for each $x \in [a, b]$. Given $\varepsilon > 0$ we set

  $$\tau = \frac{\varepsilon^2}{(1 + \varepsilon(b - a))}.$$

By the definition of derivative, for each $x \in [a,b]$ there exists $\delta(x) > 0$ such that

$$\left| \frac{F(y) - F(x)}{y - x} - f(x) \right| < \frac{\tau}{2},$$

for all $y \in [a,b]$ with $|y - x| < \delta(x)$.

Let $I = (\alpha, \beta) \subset (x - \delta(x), x + \delta(x))$ and let $F(I) = F(\beta) - F(\alpha)$. Then

$$\begin{aligned} | f(x)|I| - F(I) | &= | [F(\beta) - F(\alpha)] - f(x)(\beta - \alpha) | \\ &\leq |F(\beta) - F(x) - f(x)(\beta - x)| \\ &\quad + |F(\alpha) - F(x) - f(x)(\alpha - x)| \\ &< \frac{\tau}{2}|\beta - x| + \frac{\tau}{2}|\alpha - x| \leq \tau(\mathrm{dist}(x, I) + |I|). \end{aligned}$$

So, if $\{(I_i, x_i) : i = 1, \ldots, p\}$ is a $\delta$-fine partition of $[a,b]$ satisfying the condition

$$\sum_{i=1}^{p} \mathrm{dist}(x_i, I_i) < 1/\varepsilon,$$

we have

$$\begin{aligned} &\left| \sum_{i=1}^{p} f(x_i)|I_i| - (F(b) - F(a)) \right| \\ &\qquad \leq \sum_{i=1}^{p} |f(x_i)|I_i| - F(I_i)| \\ &\qquad < \tau \sum_{i=1}^{p} (\mathrm{dist}(x_i,\ I_i) + |I_i|) \\ &\qquad < \tau \left( \frac{1}{\varepsilon} + (b - a) \right) \\ &\qquad = \frac{\varepsilon^2}{1 + \varepsilon(b-a)} \frac{1 + \varepsilon(b-a)}{\varepsilon} = \varepsilon. \end{aligned}$$

□

- THE $C$-PRIMITIVES

DEFINITION 2.1. *A function $F$ is said to be $AC_c$ on a set $E \subset [a,b]$ if $\forall \varepsilon > 0$ there exist a gage $\delta$ and a constant $\eta > 0$ such that*

$$\sum_i |F(I_i)| < \varepsilon$$

*for each $\delta$-fine partial partition $\{(I_i, x_i) : \ i = 1, \dots, p\}$, satisfying the conditions*

$$x_i \in E, \quad i = 1, 2, \cdots p; \qquad \sum_{i=1}^{p} |I_i| < \eta;$$

$$\sum_{i=1}^{p} \operatorname{dist}(x_i, I_i) < 1/\varepsilon.$$

*$F$ is said to be $ACG_c$ if it is continuous and there exists a sequence $(E_n)$ of measurable subsets of $[a,b]$ such that $[a,b] = \cup_n E_n$ and $F$ is $AC_c$ on each $E_n$.*

THEOREM 2.2. *$F$ is a $C$-primitive if and only if it is $ACG_c$.*

- THE $C$-INTEGRAL DOES NOT INCLUDE THE RIEMANN IMPROPER INTEGRAL

The function $F(x) = x\, sin\frac{1}{x^2}$ if $0 < x \leq 1$, $F(x) = 0$ if $x = 0$, is a Riemann improper primitive, but it is not $ACG_c$.
Indeed, let $a_h = (\pi + 2h\pi)^{-1/2}$ and $b_h = (\pi/2 + 2h\pi)^{-1/2}$. Then $F(a_h) = 0$, $F(b_h) = b_h$ and $\sum_h a_h = \sum_h b_h = \infty$. Moreover the intervals $(a_h, b_h)$ are pairwise disjoint, so $\sum_h (b_h - a_h) < \infty$. Therefore, given $0 < \varepsilon < 1$, for each gage $\delta(x)$ and each $\eta > 0$ we can find $n, p$ such that

$$(a_{n+i}, b_{n+i}) \subset (0, \delta(0)), \ i = 1, \cdots, p\,,$$

$$\varepsilon < \sum_{n+1}^{n+p} a_h < \frac{1}{\varepsilon} \ \text{ and } \ \sum_{n+1}^{n+p} (b_h - a_h) < \eta.$$

Hence $\sum_{n+1}^{n+p} b_h > \sum_{n+1}^{n+p} a_h > \varepsilon$.
Now let $I_1 = (a_{n+1}, b_{n+1}), \cdots, I_p = (a_{n+p}, b_{n+p})$. Then $\{(I_1, 0), \dots, (I_p, 0)\}$ is a $\delta$-fine partial partition with

$$\sum_{i=1}^{p} |I_i| < \eta, \ \sum_{i=1}^{p} \operatorname{dist}(0, I_i) = \sum_{n+1}^{n+p} a_i < 1/\varepsilon$$

$$\text{and} \quad \sum_{i=1}^{p} |F(b_{n+i}) - F(a_{n+i})| = \sum_{i=1}^{p} b_{n+i} > \varepsilon.$$

- MONOTONE CONVERGENCE THEOREM:
  Let
  $$f_1 \leq f_2 \leq \cdots \leq f_n \leq \cdots \to f$$
  If $f_n$ is $C$-integrable on $[a, b]$, $n = 1, 2, \ldots$, and if $(C) \int_a^b f_n \to l$ then $f$ is $C$-integrable on $[a, b]$ and
  $$(C) \int_a^b f = \lim_n (C) \int_a^b f_n.$$

- DOMINATED CONVERGENCE THEOREM:
  Let $f_1, f_2, \ldots, f_n, \ldots$ be a sequence of Lebesgue measurable functions and let $g, h$ be $C$-integrable on $[a, b]$. If
  - $f_n(x) \to f(x)$ a.e. in $[a, b]$,
  - $g(x) \leq f_n(x) \leq h(x)$ a.e. in $[a, b]$,

  then $f$ is $C$-integrable on $[a, b]$ and
  $$(C) \int_a^b f = \lim_n (C) \int_a^b f_n.$$

- VITALI'S CONVERGENCE THEOREM:

  Let $f_1, f_2, \ldots, f_n, \ldots$ be a sequence of $C$-integrable functions such that
  - $f_n(x) \to f(x)$ a.e. in $[a, b]$,
  - the sequence $F_1, F_2, \ldots, F_n, \ldots$ of $C$-primitives is uniformly $ACG_c$.

  Then $f$ is $C$-integrable and
  $$(C) \int_a^b f = \lim_n \, (C) \int_a^b f_n.$$

- THE MULTIPLIERS

  If $f$ is $C$-integrable in $[a,b]$ and if $g$ is a function of bounded variation, then $fg$ is $C$-integrable in $[a,b]$ and

$$(C)\int_a^b f(t)g(t)dt = [Fg]_a^b - (L)\int_a^b Fdg \ ,$$

  where $F$ is the $C$-primitive of $f$ and the second integral is the Lebesgue-Stieltjes integral of $F$ with respect to $g$.

## 3. Main Property of the C-integral

THEOREM 3.1. *(see [3]). A function $f : [a,b] :\rightarrow \mathbf{R}$ is $C$-integrable if and only if there exist a derivative $f_1$ and a Lebesgue integrable function $f_2$ such that*

$$f = f_1 + f_2$$

*or, equivalently, $f : [a,b] \rightarrow \mathbf{R}$ is $C$-integrable if and only if there exist a differentiable function $H_1$ and an absolutely continuous function $H_2$ such that $(H_1 + H_2)'(x) = f(x)$, a.e. in $[a,b]$.*

*Sketch of the proof:* The C-integral integrates the derivatives and the Lebesgue integrable functions and is linear. Then the "if" part follows immediately. □

Let $f$ be $C$-integrable on $[a,b]$. For the proof of the "only if" part we need the following notion and lemma.
An interval $[\alpha,\beta] \subset [a,b]$ is said to be $f$-*regular* if there exists a derivative $h$ such that $f - h$ is Lebesgue integrable in $[\alpha,\beta]$.
Denote by $G$ the union of the interior of all $f$-regular intervals and let $P = [a,b] \setminus G$. From now on, in order to avoid the use of many integrals, every integral will be understood as Denjoy integral.

LEMMA 3.2. *If $\alpha,\beta \in P$ and*

*1) $f$ is Lebesgue integrable on $P \cap [\alpha,\beta]$,*

*2)* $\sum_n \sup\{|\int_J f| : J \subset \mathrm{n}^{\mathrm{th}}$ connected component of $(\alpha,\beta)\setminus P\} < \infty$

*then $[\alpha,\beta]$ is $f$-regular.*

Assume $P \cap (a,b) \neq \emptyset$. Then, being $f$ Denjoy integrable, there exist $\alpha, \beta \in P$ such that $P \cap (\alpha, \beta) \neq \emptyset$ and conditions 1), 2) hold. Therefore $[\alpha, \beta]$ is $f$-regular, which is in contradiction with $P \cap (\alpha, \beta) \neq \emptyset$. Therefore $P = \{a, b\}$ and, by a final application of Lemma, the interval $[a, b]$ is $f$-regular.

The proof of the Lemma in the general case is very involved and technical. Therefore I prefer to report it only in a special case.

*Lemma's sketch of proof in the special case $P = \{a, b\}$.* Assume $f(a) = f(b) = 0$. Take

$$0 < \varepsilon_n < \frac{1}{2} \text{ with } \sum_{n=1}^{\infty} \varepsilon_n < +\infty, \tag{2}$$

and let $\delta_n$ be a gage corresponding to $\varepsilon = \varepsilon_n$ in the Henstock's Lemma. Assume also

$$\delta_n(x) \leq \delta_{n-1}(x) \leq \frac{1}{2} \text{ for all } n \text{ and } x. \tag{3}$$

Set

$$I_0 = [a + \delta_1(a), b - \delta_1(b)].$$

Then, by induction, define a family $I_n = [\gamma_n, \gamma_{n+1}]$, $n = \pm 1, \pm 2, \ldots$, of nonoverlapping intervals such that

- $(a, b) = \cup_n I_n$,
- $\lim_{n \to -\infty} \gamma_n = a, \quad \lim_{n \to +\infty} \gamma_n = b$,

and such that there exists $k_n$ with

$$\begin{aligned} &I_n \subset (a, a + \delta_{k_n}(a)], \quad \text{if } n < 0, \\ &I_n \subset [b - \delta_{k_n}(b), b), \quad \text{if } n > 0, \end{aligned} \tag{4}$$

$$\left| \int_{J_n} f \right| = \varepsilon_{k_n} \operatorname{dist}(I_n, \{a, b\}), \tag{5}$$

and

$$\operatorname{dist}(J_n, \{a, b\}) = \operatorname{dist}(I_n, \{a, b\}), \tag{6}$$

for a suitable interval $J_n \subset I_n$.

Now, since $I_n$ is $f$-regular there exists a derivative $h_n$ such that

$$\int_{I_n} |f - h_n| < \varepsilon_{k_n} \mathrm{dist}(I_n, \{a, b\}), \text{ and } h_n(x) = 0, \quad \forall x \notin (I_n)^0.$$

So

$$h = \sum_{n=-\infty}^{\infty} h_n$$

is a derivative. Then, to finish the proof, it needs to show that $f - h$ is Lebesgue integrable in $[a, b]$. Set $T_k = \{n : k_n = k\}$. Therefore

$$\begin{aligned} \int_a^b |f - h| &= \sum_{n=-\infty}^{\infty} \int_{I_n} |f - h_n| \\ &= \sum_{k=1}^{\infty} \sum_{n \in T_k} \int_{I_n} |f - h_n| \\ &< \sum_{k=1}^{\infty} \varepsilon_k \sum_{n \in T_k} \mathrm{dist}(I_n, \{a, b\}). \end{aligned}$$

Assume, by contradiction, that

$$\sum_{n \in T_k} \mathrm{dist}(I_n, \{a, b\}) > 2.$$

Then there exist a finite set $\tilde{T}_k \subset T_k$ and $\tilde{n} \in T_k \setminus \tilde{T}_k$ such that

$$\sum_{n \in \tilde{T}_k} \mathrm{dist}(I_n, \{a, b\}) \leq 2 \tag{7}$$

and

$$\sum_{n \in \tilde{T}_k} \mathrm{dist}(I_n, \{a, b\}) + \mathrm{dist}(I_{\tilde{n}}, \{a, b\}) > 2. \tag{8}$$

Moreover, for each $n \in \tilde{T}_k$ by (5) there exists $J_n \subset I_n$ such that $\mathrm{dist}(J_n, \{a, b\}) = \mathrm{dist}(I_n, \{a, b\})$ and

$$\left| \int_{J_n} f \right| = \varepsilon_k \mathrm{dist}(I_n, \{a, b\}). \tag{9}$$

Therefore, by (4) the collection

$$\{(J_n, a)\}_{n \in \tilde{T}_k, n<0} \cup \{(J_n, b)\}_{n \in \tilde{T}_k, n>0}$$

is a $\delta_k$-fine partial partition and by (2) and (7) it satisfies the condition

$$\sum_{n\in\tilde{T}_k} \operatorname{dist}(J_n, \{a,b\}) < \frac{1}{\varepsilon_k}.$$

So, by (9), by Henstock's lemma and by condition $f(a) = f(b) = 0$, it follows

$$\varepsilon_k \sum_{n\in\tilde{T}_k} \operatorname{dist}(I_n, \{a,b\}) = \sum_{n\in\tilde{T}_k} \left| \int_{J_n} f \right| \leq \varepsilon_k.$$

Hence

$$\sum_{n\in\tilde{T}_k} \operatorname{dist}(I_n, \{a,b\}) \leq 1.$$

Then, by (3)

$$\sum_{n\in\tilde{T}_k} \operatorname{dist}(I_n, \{a,b\}) + \operatorname{dist}(I_{\tilde{n}}, \{a,b\}) \leq 1 + \frac{1}{2} < 2,$$

which is in contradiction with (8). Therefore

$$\sum_{n\in T_k} \operatorname{dist}(I_n, \{a,b\}) \leq 2.$$

Thus

$$\int_a^b |f-h| \leq 2\sum_{k=1}^{\infty} \varepsilon_k < \infty.$$

□

**Final remark.** A careful reading of the previous Theorem's proof in [3], shows that replacing in the requeriment (1) of the definition of C-integral the function $1/\varepsilon$ with any positive function of $\varepsilon$ one obtains an integral equivalent to the C-integral.

## References

[1] B. Bongiorno, *Un nuovo integrale per il problema delle primitive*, Le Matematiche **51** (1996), no. 2, 299–313.

[2] B. Bongiorno, *On the minimal solution of the problem of primitives*, J. Math. Anal. Appl. **251** (2000), no. 2, 479–487.

[3] B. BONGIORNO, L. DI PIAZZA, AND D. PREISS, *A constructive minimal integral which includes Lebesgue integrable functions and derivatives*, J. London Math. Soc. **62** (2000), no. 1, 117–126.
[4] A.M. BRUCKNER, R.J. FLEISSNER, AND J. FORAN, *The minimal integral which includes Lebesgue integrable functions and derivatives*, Coll. Math. (1986), 289–293.
[5] R. HENSTOCK, *The general theory of integration*, Clarendon Press, Oxford, 1991.
[6] E.J. MCSHANE, *A unified theory of integration*, Amer. Math. Monthly **80** (1973), 349–359.
[7] E.J. MCSHANE, *Unified integration*, Academic Press, New York, 1983.
[8] W.F. PFEFFER, *The Riemann approach to integration*, Cambridge University Press, Cambridge, 1993.
[9] S. SAKS, *Theory of the integral*, Dover, New York, 1964.

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