A TOPOLOGICAL INTERPRETATION OF CELLULAR MAPS (*)

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SOMMARIO. - *In questo lavoro diamo una definizione combinatoria rigorosa della categoria $\mathcal{A}$ delle mappe cellulari e classifichiamo le mappe cellulari con spazi topologici, o meglio, definiamo una categoria $\mathcal{T}$ di spazi topologici con funzioni continue come morfismi, che è equivalente ad $\mathcal{A}$, così che le mappe cellulari classificano gli elementi di $\mathcal{T}$.*

SUMMARY. - *In this paper we give a rigorous combinatorial definition of the category $\mathcal{A}$ of cellular maps and we classify the cellular maps with topological spaces, or, better, we find a category $\mathcal{T}$ of topological spaces with continuous functions as morphisms, which is equivalent to $\mathcal{A}$, so that the cellular maps classify the elements of $\mathcal{T}$.*

1. Introduction.

Let $\mathbf{C}$, $\mathbf{R}$ and $\mathbf{Z}$ be the set of complex, real and integer numbers, $Re(z)$ the real part of $z \in \mathbf{C}$.

Let

$$\Gamma = \{z \in \mathbf{C} : \mid z \mid \leq 1\};$$

$$\Omega = fr(\Gamma) = \{z \in \mathbf{C} : \mid z \mid = 1\}.$$

$$Q_n = \begin{cases} \Gamma \setminus \{0\}, & \text{if } n \geq 1 \text{ finite} \\ \{z \in \mathbf{C} : Re(z) \leq 0\}, & \text{if } n = \infty. \end{cases}$$

$$fr(Q_n) = \begin{cases} \Omega, & \text{if } n \geq 1 \text{ finite} \\ i\,\mathbf{R}, & \text{if } n = \infty. \end{cases}$$

Call punctured polygons the topological spaces:

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$$P_n = \begin{cases} Q_n \setminus \{z \in \mathbf{C} : z^n = 1\}, & \text{if } n \geq 1 \text{ finite} \\ Q_\infty \setminus \mathbf{Z} , & \text{if } n = \infty. \end{cases}$$

Put

$$fr(P_n) = fr(Q_n) \cap P_n .$$

Call side of P_n a connected component (i.e. a maximal open arcs) of $fr(P_n)$.

In [D] Damphousse introduces the category $\mathcal{D}$ (that he call the category of cellular maps as we'll call the category $\mathcal{A}$) having:

i) as *objects* the connected topological spaces that are given by taking (with repetitions) a disjointed union of punctured polygons and identifying all the sides two by two and considering the quotient topology;

ii) as *morphisms* the local homeomorphisms that are particular rolls on the polygons, i.e. of the type $z \longmapsto z^k$ between polygons P_{kn} and P_n or $z \longmapsto e^{2\pi z/n}$ between polygons P_∞ and P_n.

This definition is interesting for the fact that the objects constructed are surfaces.

But, though the objects of $\mathcal{D}$ are topological things, the morphisms are defined on the various polygons that form the objects of $\mathcal{D}$, considering the algebraic structure that they inherit from $\mathbf{C}$.

In a first moment we have tried to consider the category having the same objects as $\mathcal{D}$, but the functions, which are local homeomorphisms and send side into side, as morphisms.

Clearly, with this definition, a morphism can fail to be onto and so the new category is not equivalent to $\mathcal{D}$, and the things are not better if we consider only onto maps.

In this paper we give, through the cartographic group $\mathbf{G}$, which is the free product of the ciclic group with two elements and the Klein group and was introduced by Grothendieck, the definition of a category $\mathcal{A}$ whose objects are omogeneous spaces on $\mathbf{G}$.

We'll see easily that $\mathcal{A}$ is equivalent to $\mathcal{D}$ and we call it the category of cellular maps, too.

However the main goal of this paper is to find a "pure topological" category $\mathcal{T}$, equivalent to the "combinatorial" category $\mathcal{A}$ of cellular maps. Its objects are subsets of surfaces and the morphisms are continuous functions, which are local homeomorphisms, excluding discrete sets.

2. The Categories $\mathcal{A}$.

The cartographic group is the group

$$\mathbf{G} =< \sigma_0, \sigma_1, \sigma_2 : {\sigma_1}^2 = {\sigma_0}^2 = {\sigma_2}^2 = (\sigma_0\sigma_2)^2 = 1 >,$$

the free product of the cyclic group with two elements $\{1, \sigma_1\}$ and the Klein group $K =< \sigma_0, \sigma_2 : {\sigma_0}^2 = {\sigma_2}^2 = (\sigma_0\sigma_2)^2 = 1 >$.

A homogeneous space X of $\mathbf{G}$ is a set X with a transitive $\mathbf{G}$ -action on it.

We define *cellular map* a homogeneous space X of $\mathbf{G}$, with the property:

(*) every $\sigma_0, \sigma_1, \sigma_2, \sigma_0\sigma_2$ has no fixed point in X .

Given two cellular maps X and Y , a morphism is a function $f : X \longrightarrow Y$ so that

$$f * g = g * f, \quad \forall g \in \mathbf{G}.$$

Being the action of $\mathbf{G}$ transitive, f is suriective and determined by its image on a point of X .

An isomorphism will be an iniective homomorphism.

Call $\mathcal{A}$ the category of cellular maps.

For the property (*) , Kx is a set of four distinct points, $\forall x \in X$.

Hence σ_0 and σ_2 give a partition of X with sets of four points collected in squares and σ_1 is a partition of X in sets of two points, that connects all the squares.

We call:

vertex = a minimal set of X closed for σ_1, σ_2.

face = a minimal set of X closed for σ_1, σ_0.

side = a minimal set of X closed for σ_0, σ_2.

Note that sides are the sets Kx.

Call

$$\alpha = \sigma_1\sigma_0,$$

$$\beta = \sigma_1\sigma_2.$$

Half of the cardinality of the points of a vertex V (risp. a face V) gives us the number m_V (risp.n_F) of sides of V (risp. F) (i.e.that intersect V (resp. F)).

If $p \in V$, then m_V (risp. n_F) is the minimal number ,if finite, so that $\beta^{m_V}(p) = p$ (risp. $\alpha^{n_F}(p) = p$).

Proposition 1. $\mathcal{A} \simeq \mathcal{D}$.

Proof. Every element $X \in \mathcal{A}$ corresponds to an element $\Phi(X) \in \mathcal{D}$ in the way that we'll show.

We associate to every face F of X the spaces $Q(F) = Q_{n_F} \times \{F\}$ and $P(F) = P_{n_F} \times \{F\}$.

It is easy to show that there is a function

$$g_F : F \longrightarrow Q(F) \setminus P(F),$$

so that

$\sigma_1(x) = y \iff g_F(x) = g_F(y)$,

$\sigma_0(x) = y \iff (g_F(x), g_F(y))$ is a side of $P(F)$.

Put

$$g = \bigcup_{F \ face \ of \ X} g_F$$

$\Phi(X)$ will be the space of $\mathcal{D}$ defined by:

$$\Phi(X) = \bigcup_{F \ face \ of \ X} P(F)/\sim$$

where the oriented arc $(g(x), g(\sigma_0 x))$ is identified to the oriented arc $(g(\sigma_2 x), g(\sigma_2\sigma_0 x))$ and the topology is the quotient one.

Clearly, Φ is a functor and gives an equivalence of $\mathcal{A}$ and $\mathcal{D}$. $\diamondsuit$

3. The Topological Interpretation $\mathcal{T}$ of the Category $\mathcal{A}$.

Consider the category $\mathcal{T}$ that has:

i) as objects the triples (A, B, C) of first-countable topological spaces with:

O1. $C \subset B \subset A$;

O2. $A \setminus C$ is a surface (i.e. a triangulable manifold);

O3. $B \setminus C$ is a discrete (in A) non-void set of open arcs each one with a compact closure in B ;

O4. C is a discrete set of A ;

O5. $A \setminus B$ is a disjointed union of open discs $\{ \Lambda : \Lambda \in \mathcal{L} \}$;

O6. Each $F_\Lambda = cl_A \Lambda$ has a one point compactification homeomorphic to a disc.

O7. Each point $x \in A$ has such a base of neighbourhoods U_n that $U_n \setminus \{x\}$ is connected $\forall n$.

ii) as morphisms the continuous functions $f : (A, B, C) \longrightarrow (A', B', C')$ so that:

M1. $f (B \setminus C) \subset B' \setminus C'$;

M2. $f (A \setminus B) \subset A' \setminus B'$;

M3. $f (C) \subset C'$;

M4. f is a local homeomorphism for each point of A excluding a discrete subset of $(A \setminus B) \cup C$ (i.e. $\exists D$ discrete subset of A not intersecting $B \setminus C$, so that $\forall x \in D$, $\exists\, U$ neighbourhood of x , so that $f\,|_U$ is a homeomorfism).

Note that a morphism f has not to be onto.
For $n \geq 1, n$ finite, put

$$A_n = \Gamma,$$

$$B_n = \Omega,$$

$$C_n = \{p_1, p_2, ..., p_n\},$$

with $p_1, p_2, ..., p_n$ distinct points of Ω , so that $(p_n, p_1), (p_1, p_2), ..., (p_{n-1}, p_n)$ are arcs contained in $\Omega \setminus C_n$.

For $n = \infty$, put

$$A_\infty = \Gamma \setminus \{p\},$$

$$B_\infty = \Omega \setminus \{p\},$$

$$C_\infty = \{p_k : k \in \mathbf{Z}\}$$

with p_k (distinct) points of Ω, so that

$$\lim_{k \to \pm\infty} p_k = p,$$

and $\forall k$, (p_k, p_{k+1}) is an open arc joining p_k with p_{k+1} contained in $B_\infty \setminus C_\infty$.

Proposition 2. $\mathcal{A} \simeq \mathcal{T}$.

Proof. Every element X $\in \mathcal{A}$ corresponds to an element $\Psi(X) \in \mathcal{T}$ in the way that we'll show.

To every face F of X we associate the spaces

$$A(F) = A_{n_F} \times \{F\},$$

$$B(F) = B_{n_F} \times \{F\},$$

$$C(F) = C_{n_F} \times \{F\}.$$

It is easy to show that there is a function

$$f_F : F \longrightarrow C(F)$$

so that

$\sigma_1(x) = y \iff f_F(x) = f_F(y)$,

$\sigma_0(x) = y \iff (f_F(x), f_F(y))$ is an arc in $B(F) \setminus C(F)$.

Put

$$f = \bigcup_{F\ face\ of\ X} f_F$$

Call $\Psi(X)$ the space $(A, B, C) \in \mathcal{T}$ defined by:

$$A = \bigcup_{F\ face\ of\ X} A(F)/\sim$$

$$B = \bigcup_{F\ face\ of\ X} B(F)/\sim$$

$$C = \bigcup_{F\ face\ of\ X} C(F)/\sim$$

where the arc $[f(x), f(\sigma_0 x)]$ is identified to the arc $[f(\sigma_2 x), f(\sigma_2\sigma_0 x)]$ with $f(x) \sim f(\sigma_2 x)$ and $f(\sigma_0 x) \sim f(\sigma_2\sigma_0 x)$.

The topology is not the quotient one because every point has to have a countable neighbourhood base.

We have to define a neighbourhood base for each point $c \in C$, so that $f^{-1}(c) = V$ (that is a vertex of X) is not finite.

Let $f^{-1}(c) = \{x_k, y_k : k \in \mathbf{Z}\}$ distinct points of X with $x_k, y_k \in F_k$, $\sigma_2(x_k) = y_k$, $\sigma_1(x_k) = y_{k+1}$, and F_k not necessarily distinct faces of X.

Define by induction on $| k |$ an immersion of the face $A(F_k)$ in the sector of A_∞ delimited by the arc $[p_k, p_{k+1}]$ of B_∞, so that:

a) the sides $[f(x_k), f(\sigma_0(x_k))] \subset A(F_k)$ and $[f(y_{k+1}), f(\sigma_0(y_{k+1}))] \subset A(F_{k+1})$ are identified with the ray $[0, p_{k+1}]$, with $f(x_k) = f(y_{k+1})$ corresponding to 0,

b) $A(F_k) \setminus B(F_k)$ goes onto the interior of the sector.

Give to c the topology that cames from 0.

Note that in this case c has a neighbourhood homeomorphic to $\Gamma \setminus (0, 1]$ with c corresponding to 0.

So A is a subset of a surface.

Now it is an exercise to see that $(A, B, C) \in \mathcal{T}$ and that the correspondence Ψ is functorial.

Viceversa, let $(A, B, C) \in \mathcal{T}$.

Note that,by (O6), each $F = F_\Lambda$ is locally compact.

So, also by (O6), F will be homeomorphic either to the closed disc Γ or to the closed disc Γ without a point p of Ω.

In the first case, $F \cap C$ correspond in this homeomorphism to a finite number n_F of points of the circumference Ω.

In the second case, $F \cap C$ correspond to infinite distinct points $\{p_k : k \in \mathbf{Z}\}$ of Ω with

$$\lim_{k \to \pm\infty} p_k = p,$$

in this case we say that $n_F = \infty$.

So, in both cases, we can assume that

$$B \cap F = \bigcup_{k \in \mathbf{Z}} [q_k, q_{k+1}]$$

with $[q_k, q_{k+1}]$ arcs and $q_k \in C \; \forall k \in \mathbf{Z}$ (exactly we had to write q_k^Λ insted of q_k).

If $n_F = \infty$, the arcs $[q_k, q_{k+1}]$ are all distinct and $\lim_{k \to \pm\infty} q_k = q$; if n_F is finite we have $q_{k+n_F} = q_k$ and $[q_1, q_2], \ldots, [q_{n_F}, q_{n_F+1}]$ distinct.

Let X_Λ be the subset of $C \times C$:

$$X_\Lambda = \{(q_k, q_{k+1}) : k \in \mathbf{Z}\} \cup \{(q_{k+1}, q_k) : k \in \mathbf{Z}\}$$

Put

$$X = \bigcup_{\Lambda \in \mathcal{L}} X_\Lambda \times \{\Lambda\}.$$

X will be an homogeneous space of $\mathbf{G}$, if we define:

$$\sigma_0(q_k, q_{k+1}, \Lambda) = (q_{k+1}, q_k, \Lambda)$$
$$\sigma_1(q_k, q_{k+1}, \Lambda) = (q_k, q_{k-1}, \Lambda)$$
$$\sigma_1(q_k, q_{k+1}, \Lambda) = (q_k, q_{k+1}, \Lambda')$$

if $q_k, q_{k+1} \in cl(\Lambda') = F_{\Lambda'}$ (exactly $\sigma_1(q_k^\Lambda, q_{k+1}^\Lambda, \Lambda) = (q_k^\Lambda, q_{k+1}^\Lambda, \Lambda'))$.

It is an exercise to see that (O7) implies that the action of $\mathbf{G}$ is transitive on X .

It is clear that $\Psi(X) = (A, B, C)$.

Finally, it is necessary and sufficient to see that each morphism of $\mathcal{T}$ is determined, within isomorphisms, by the image of a disc $\Lambda \in \mathcal{L}$ and an oriented arc $[q_1, q_2]$ with $(q_1, q_2) \in X_\Lambda$,i.e. a morphism h of (A, B, C) that send a $\Lambda \in \mathcal{L}$ into itself and the oriented arc $[q_1, q_2]$ into itself, is isomorphic to the identity.

It is clear that it is enough to show that h is the identity on C and doesn't exchange the elements of $\mathcal{L}$.

In fact, in this case, h is, by (M4), an homeomorphism on each arc of $B \setminus C$ and we can suppose that on each F_Λ, h is the identity on the frontier. So it has to be iniective and hence isomorphic to the identity.

Now by (M2) and (M4) and connectedness $h(\Lambda) \subset \Lambda \; \forall \Lambda$.

Moreover if $(q_1, q_2), (q_2, q_3)$ are two arcs of Λ with $h(q_2) = q_2$, $h(q_1) = q_1, h(q_3) = q_1$, it is clear that there is an arc in Λ starting from q_2 of double points contradicting the discreteness of D in (M4).

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References

[D] Damphousse P., *Fondement de la description combinatoire des cartes cellulares*, Ann. Sc. Math. Quebec **11**, $n^o 2$ (1987), 279-294.