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Unbounded generalizations of the Fuglede-Putnam theorem

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Abstract. *In this note, we prove and disprove several generalizations of unbounded versions of the Fuglede-Putnam theorem.*

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1. Essential background

All operators considered here are linear but not necessarily bounded. If an operator is bounded and everywhere defined, then it belongs to $B(H)$ which is the algebra of all bounded linear operators on H (see [14] for its fundamental properties).

Most unbounded operators that we encounter are defined on a subspace (called domain) of a Hilbert space. If the domain is dense, then we say that the operator is densely defined. In such case, the adjoint exists and is unique.

Let us recall a few basic definitions about non-necessarily bounded operators. If S and T are two linear operators with domains $D(S)$ and $D(T)$ respectively, then T is said to be an extension of S, written as $S \subset T$, if $D(S) \subset D(T)$ and S and T coincide on $D(S)$.

An operator T is called closed if its graph is closed in $H \oplus H$. It is called closable if it has a closed extension. The smallest closed extension of it is called its closure and it is denoted by $\overline{T}$ (a standard result states that a densely defined T is closable iff T^* has a dense domain, and in which case $\overline{T} = T^{**}$). If T is closable, then

$$S \subset T \Rightarrow \overline{S} \subset \overline{T}.$$

If T is densely defined, we say that T is self-adjoint when $T = T^*$; symmetric if $T \subset T^*$; normal if T is *closed* and $TT^* = T^*T$.

The product ST and the sum $S + T$ of two operators S and T are defined in the usual fashion on the natural domains:

$$D(ST) = \{x \in D(T) : \ Tx \in D(S)\}$$

and

$$D(S+T)=D(S)\cap D(T).$$

In the event that S, T and ST are densely defined, then

$$T^*S^*\subset(ST)^*,$$

with the equality occurring when $S\in B(H)$. If $S+T$ is densely defined, then

$$S^*+T^*\subset(S+T)^*$$

with the equality occurring when $S\in B(H)$.

Let T be a linear operator (possibly unbounded) with domain $D(T)$ and let $B\in B(H)$. Say that B commutes with T if

$$BT\subset TB.$$

In other words, this means that $D(T)\subset D(TB)$ and

$$BTx=TBx,\ \forall x\in D(T).$$

Let A be an injective operator (not necessarily bounded) from $D(A)$ into H. Then $A^{-1}:\operatorname{ran}(A)\to H$ is called the inverse of A, with $D(A^{-1})=\operatorname{ran}(A)$.

If the inverse of an unbounded operator is bounded and everywhere defined (e.g. if $A:D(A)\to H$ is closed and bijective), then A is said to be boundedly invertible. In other words, such is the case if there is a $B\in B(H)$ such that

$$AB=I\text{ and }BA\subset I.$$

If A is boundedly invertible, then it is closed.

The resolvent set of A, denoted by $\rho(A)$, is defined by

$$\rho(A)=\{\lambda\in\mathbb{C}:\ \lambda I-A\text{ is bijective and }(\lambda I-A)^{-1}\in B(H)\}.$$

The complement of $\rho(A)$, denoted by $\sigma(A)$,

$$\sigma(A)=\mathbb{C}\setminus\rho(A)$$

is called the spectrum of A.

2. Introduction

The aim of this paper is to obtain some generalizations of the Fuglede-Putnam theorem involving unbounded operators.

Recall that the original version of the Fuglede-Putnam theorem reads:

Theorem 2.1 ([6, 20]). *If* $A \in B(H)$ *and if* M *and* N *are normal (non-necessarily bounded) operators, then*

$$AN \subset MA \Longrightarrow AN^* \subset M^*A.$$

The problem leading to the above theorem was first mooted by J. von Neumann in [18] who had already established it in a finite-dimensional setting. B. Fuglede was the first one to prove this theorem in [6] in the case $N = M$, and where $\dim H = \infty$ was allowed. It is important to tell readers that P. R. Halmos obtained in [8] almost simultaneously as B. Fuglede a quite different proof of the theorem above. More precisely, at the end of August 1949, B. Fuglede communicated his proof to P. R. Halmos at the Boulder meeting of the American Mathematical Society. Halmos' proof dealt with the all bounded version, however, P. R. Halmos indicated that only minor modifications were needed to adapt his proof to the more general case of unbounded operators.

Then, C. R. Putnam [20] proved the above version. S. K. Berberian [3] amazingly noted that the two versions were equivalent.

There are different proofs of the Fuglede-Putnam theorem. The most elegant proof perhaps is the one due to M. Rosenblum [23]. For other proofs, see e.g. [21] and [22].

There have been many generalizations of the Fuglede-Putnam theorem since Fuglede's paper. However, most generalizations were devoted to relaxing the normality assumption (see e.g. [12], and the references therein). Apparently, the first generalization of the Fuglede theorem to an unbounded A was established in [19]. Then, the first generalization involving unbounded operators of the Fuglede-*Putnam* theorem is:

Theorem 2.2. *Let* A *be a closed symmetric operator and let* N *be an unbounded normal operator. If* $D(N) \subset D(A)$, *then*

$$AN \subset N^*A \Longrightarrow AN^* \subset NA.$$

In fact, the previous result was established in [10] under the assumption of the self-adjointness of A. However, and by scrutinizing the proof in [10] or [11], it is seen that only the closedness and the symmetricity of A were needed. Other unbounded generalizations may be consulted in [1], [2], and [13], as well as some of the references therein. In the end, readers may wish to consult the survey [16] exclusively devoted to the Fuglede-Putnam theorem and its applications.

3. Generalizations of the Fuglede-Putnam theorem

If a densely defined operator N is normal, then so is its adjoint. However, if N^* is normal, then N^{**} does not have to be normal (unless N itself is closed). A simple counterexample is to take the identity operator I_D restricted to some unclosed dense domain $D \subset H$. Then I_D cannot be normal for it is not closed. But, $(I_D)^* = I$ which is the full identity on the entire H, is obviously normal. Notice in the end that if N is a densely defined closable operator, then N^* is normal if and only if $\overline{N}$ is.

The first improvement is that in the very first version by B. Fuglede, the normality of the operator is not needed as only the normality of its closure will do. This observation has already appeared in [4], but we reproduce the proof here.

Theorem 3.1. *Let $B \in B(H)$ and let A be a densely defined and closable operator such that $\overline{A}$ is normal. If $BA \subset AB$, then*

$$BA^* \subset A^*B.$$

Proof. Since $\overline{A}$ is normal, $\overline{A}^* = A^*$ remains normal. Now,

$$\begin{aligned} BA \subset AB \Longrightarrow & B^*A^* \subset A^*B^* \text{ (by taking adjoints)} \\ \Longrightarrow & B^*\overline{A} \subset \overline{A}B^* \text{ (by using the classical Fuglede theorem)} \\ \Longrightarrow & BA^* \subset A^*B \text{ (by taking adjoints again),} \end{aligned}$$

establishing the result. □

Remark 3.2. Notice that $BA^* \subset A^*B$ does not yield $BA \subset AB$ even in the event of the normality of A^* (see [15]).

Let us now turn to the extension of the Fuglede-Putnam version. A similar argument to the above one could be applied.

Theorem 3.3. *Let $B \in B(H)$ and let N, M be densely defined closable operators such that $\overline{N}$ and $\overline{M}$ are normal. If $BN \subset MB$, then*

$$BN^* \subset M^*B.$$

Proof. Since $BN \subset MB$, it ensues that $B^*M^* \subset N^*B^*$. Taking adjoints again gives $B\overline{N} \subset \overline{M}B$. Now, apply the Fuglede-Putnam theorem to the normal $\overline{N}$ and $\overline{M}$ to get the desired conclusion $BN^* \subset M^*B$. □

Jabłoński et al. obtained in [9] the following version.

Theorem 3.4. *If N is a normal (bounded) operator and if A is a closed densely defined operator with $\sigma(A) \neq \mathbb{C}$, then:*

$$NA \subset AN \Longrightarrow g(N)A \subset Ag(N)$$

for any bounded complex Borel function g on $\sigma(N)$. In particular, we have $N^*A \subset AN^*$.

Remark 3.5. It is worth noticing that B. Fuglede obtained, long ago, in [7] a unitary $U \in B(H)$ and a closed and symmetric T with domain $D(T) \subset H$ such that $UT \subset TU$ but $U^*T \not\subset TU^*$.

Next, we give a generalization of Theorem 3.4 to an unbounded N, and as above, only the normality of $\overline{N}$ is needed.

Theorem 3.6. *Let p be a one variable complex polynomial. If N is a densely defined closable operator such that $\overline{N}$ is normal and if A is a densely defined operator with $\sigma[p(A)] \neq \mathbb{C}$, then*

$$NA \subset AN \Longrightarrow N^*A \subset AN^*$$

whenever $D(A) \subset D(N)$.

Remark 3.7. This is indeed a generalization of the bounded version of the Fuglede theorem. Observe that when $A, N \in B(H)$, then $\overline{N} = N$, $D(A) = D(N) = H$, and $\sigma[p(A)]$ is a compact set.

Proof of Theorem 3.6. First, we claim that $\sigma(A) \neq \mathbb{C}$, whereby A is closed. Let λ be in $\mathbb{C} \setminus \sigma[p(A)]$. Then, and as in [5], we obtain

$$p(A) - \lambda I = (A - \mu_1 I)(A - \mu_2 I) \cdots (A - \mu_n I)$$

for some complex numbers μ_1, μ_2, $\cdots$, μ_n. By consulting again [5], readers see that $\sigma(A) \neq \mathbb{C}$.

Now, let $\lambda \in \rho(A)$. Then

$$NA \subset AN \Longrightarrow NA - \lambda N \subset AN - \lambda N = (A - \lambda I)N.$$

Since $D(A) \subset D(N)$, it is seen that $NA - \lambda N = N(A - \lambda I)$. So

$$N(A - \lambda I) \subset (A - \lambda I)N \Longrightarrow (A - \lambda I)^{-1}N \subset N(A - \lambda I)^{-1}.$$

Since $\overline{N}$ is normal, we may now apply Theorem 3.1 to get

$$(A - \lambda I)^{-1}N^* \subset N^*(A - \lambda I)^{-1}$$

because $(A - \lambda I)^{-1} \in B(H)$. Hence

$$N^*A - \lambda N^* \subset N^*(A - \lambda I) \subset (A - \lambda I)N^* = AN^* - \lambda N^*.$$

But

$$D(AN^*) \subset D(N^*) \text{ and } D(N^*A) \subset D(A) \subset D(N) \subset D(\overline{N}) = D(N^*).$$

Thus, $D(N^*A) \subset D(AN^*)$, and so

$$N^*A \subset AN^*,$$

as needed. □

Now, we present a few consequences of the preceding result. The first one is given without proof.

Corollary 3.8. *If N is a densely defined closable operator such that $\overline{N}$ is normal and if A is an unbounded self-adjoint operator with $D(A) \subset D(N)$, then*

$$NA \subset AN \Longrightarrow N^*A \subset AN^*.$$

Corollary 3.9. *If N is a densely defined closable operator such that $\overline{N}$ is normal and if A is a boundedly invertible operator, then*

$$NA \subset AN \Longrightarrow N^*A \subset AN^*.$$

Proof. We may write

$$NA \subset AN \Longrightarrow NAA^{-1} \subset ANA^{-1} \Longrightarrow A^{-1}N \subset NA^{-1}.$$

Since $A^{-1} \in B(H)$ and $\overline{N}$ is normal, Theorem 3.1 gives

$$A^{-1}N^* \subset N^*A^{-1} \text{ and so } N^*A \subset AN^*,$$

as needed. □

A Putnam's version seems impossible to obtain unless strong conditions are imposed. However, the following special case of a possible Putnam's version is worth stating and proving. Besides, it is somewhat linked to the important notion of anti-commutativity.

Proposition 3.10. *If N is a densely defined closable operator such that $\overline{N}$ is normal and if A is a densely defined operator with $\sigma(A) \neq \mathbb{C}$, then*

$$NA \subset -AN \Longrightarrow N^*A \subset -AN^*$$

whenever $D(A) \subset D(N)$.

Proof. Consider

$$\widetilde{N} = \begin{pmatrix} N & 0 \\ 0 & -N \end{pmatrix} \quad \text{and} \quad \widetilde{A} = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$$

where $D(\widetilde{N}) = D(N) \oplus D(N)$ and $D(\widetilde{A}) = D(A) \oplus D(A)$. Then $\overline{\widetilde{N}}$ is normal and $\widetilde{A}$ is closed. Besides $\sigma(\widetilde{A}) \neq \mathbb{C}$. Now

$$\widetilde{N}\widetilde{A} = \begin{pmatrix} 0 & NA \\ -NA & 0 \end{pmatrix} \subset \begin{pmatrix} 0 & -AN \\ AN & 0 \end{pmatrix} = \widetilde{A}\widetilde{N}$$

for $NA \subset -AN$. Since $D(\widetilde{A}) \subset D(\widetilde{N})$, Theorem 3.6 applies, i.e. it gives $\widetilde{N}^*\widetilde{A} \subset \widetilde{A}\widetilde{N}^*$ which, upon examining their entries, yields the required result. □

We finish this section by giving counterexamples to some "generalizations".

EXAMPLE 3.11 ([13]). Consider the unbounded linear operators A and N which are defined by

$$Af(x) = (1+|x|)f(x) \text{ and } Nf(x) = -i(1+|x|)f'(x)$$

(with $i^2=-1$) on the domains

$$D(A) = \{f \in L^2(\mathbb{R}) : (1+|x|)f \in L^2(\mathbb{R})\}$$

and

$$D(N) = \{f \in L^2(\mathbb{R}) : (1+|x|)f' \in L^2(\mathbb{R})\}$$

respectively, and where the derivative is taken in the distributional sense. Then A is a boundedly invertible, positive, self-adjoint unbounded operator. As for N, it is an unbounded normal operator N (details may consulted in [13]). It was shown that such that

$$AN^* = NA \text{ but } AN \not\subset N^*A \text{ and } N^*A \not\subset AN$$

(in fact $ANf \neq N^*Af$ for all $f \neq 0$).

So, what this example is telling us is that $NA = AN^*$ (and not just an "inclusion"), that N and N^* are both normal, $\sigma(A) \neq \mathbb{C}$ (as A is self-adjoint), but $NA \not\subset AN^*$.

This example can further be beefed up to refute certain possible generalizations.

EXAMPLE 3.12 (Cf. [17]). There exist a closed operator T and a normal M such that $TM \subset MT$ but $TM^* \not\subset M^*T$ and $M^*T \not\subset TM^*$. Indeed, consider

$$M = \begin{pmatrix} N^* & 0 \\ 0 & N \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}$$

where N is normal with domain $D(N)$ and A is closed with domain $D(A)$ and such that $AN^* = NA$ but $AN \not\subset N^*A$ and $N^*A \not\subset AN$ (as defined above). Clearly, M is normal and T is closed. Observe that $D(M) = D(N^*) \oplus D(N)$ and $D(T) = D(A) \oplus L^2(\mathbb{R})$. Now,

$$TM = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}\begin{pmatrix} N^* & 0 \\ 0 & N \end{pmatrix} = \begin{pmatrix} 0_{D(N^*)} & 0_{D(N)} \\ AN^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0_{D(N)} \\ AN^* & 0 \end{pmatrix}$$

where e.g. $0_{D(N)}$ is the zero operator restricted to $D(N)$. Likewise

$$MT = \begin{pmatrix} N^* & 0 \\ 0 & N \end{pmatrix}\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ NA & 0 \end{pmatrix}.$$

Since $D(TM) = D(AN^*) \oplus D(N) \subset D(NA) \oplus L^2(\mathbb{R}) = D(MT)$, it ensues that $TM \subset MT$. Now, it is seen that

$$TM^* = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \begin{pmatrix} N & 0 \\ 0 & N^* \end{pmatrix} = \begin{pmatrix} 0 & 0_{D(N^*)} \\ AN & 0 \end{pmatrix}$$

and

$$M^*T = \begin{pmatrix} N & 0 \\ 0 & N^* \end{pmatrix} \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ N^*A & 0 \end{pmatrix}.$$

Since $ANf \neq N^*Af$ for any $f \neq 0$, we infer that $TM^* \not\subset M^*T$ and $M^*T \not\subset TM^*$.

References

[1] I. F. Z. Bensaid, S. Dehimi, B. Fuglede, and M. H. Mortad, *The Fuglede theorem and some intertwining relations*, Adv. Oper. Theory **6** (2021), Paper No. 9, 8 pp.

[2] A. Ber, V. Chilin, F. Sukochev, and D. Zanin, *Fuglede-Putnam theorem for locally measurable operators*, Proc. Amer. Math. Soc. **146** (2018), 1681–1692.

[3] S. K. Berberian, *Note on a theorem of Fuglede and Putnam*, Proc. Amer. Math. Soc. **10** (1959), 175–182.

[4] I. Boucif, S. Dehimi, and M. H. Mortad, *On the absolute value of unbounded operators*, J. Operator Theory **82** (2019), 285–306.

[5] S. Dehimi and M. H. Mortad, *Unbounded operators having self-adjoint, subnormal or hyponormal powers*, Math. Nachr., to appear.

[6] B. Fuglede, *A commutativity theorem for normal operators*, Proc. Nati. Acad. Sci. **36** (1950), 35–40.

[7] B. Fuglede, *Solution to Problem 3, Math. Scand.* **2** (1954), 346–347.

[8] P. R. Halmos, *Commutativity and spectral properties of normal operators*, Acta Sci. Math. Szeged **12** (1950), 153–156.

[9] Z. J. Jabłoński, Il B. Jung, and J. Stochel, *Unbounded quasinormal operators revisited*, Integral Equations Operator Theory **79** (2014), 135–149.

[10] M. H. Mortad, *An application of the Putnam-Fuglede theorem to normal products of self-adjoint operators*, Proc. Amer. Math. Soc. **131** (2003), 3135–3141.

[11] M. H. Mortad, *Normal products of self-adjoint operators and self-adjointness of the perturbed wave operator on $L^2(\mathbb{R}^n)$*, Thesis (Ph.D.)-The University of Edinburgh (United Kingdom). ProQuest LLC, Ann Arbor, MI, 2003.

[12] M. H. Mortad, *Yet more versions of the Fuglede-Putnam theorem*, Glasg. Math. J. **51** (2009), 473–480.

[13] M. H. Mortad, *An all-unbounded-operator version of the Fuglede-Putnam theorem*, Complex Anal. Oper. Theory, **6** (2012), 1269–1273.

[14] M. H. Mortad, *An operator theory problem book*, World Scientific Publishing Co., 2018.

[15] M. H. Mortad, *Counterexamples in operator theory*, Birkhäuser/Springer, 2022.

[16] M. H. Mortad, *The Fuglede-Putnam theory*, submitted monograph.
[17] M. H. Mortad, *Yet another generalization of the Fuglede-Putnam theorem to unbounded operators*, arXiv:2003.00339
[18] J. von Neumann, *Approximative properties of matrices of high finite order*, Portugaliae Math. **3** (1942), 1–62.
[19] A. E. Nussbaum, *A commutativity theorem for unbounded operators in Hilbert space*, Trans. Amer. Math. Soc., **140** (1969), 485–491.
[20] C. R. Putnam, *On Normal Operators in Hilbert Space*, Amer. J. Math. **73** (1951), 357–362.
[21] H. Radjavi and P. Rosenthal, *Hyperinvariant subspaces for spectral and n-normal operators*, Acta Sci. Math. (Szeged) **32** (1971), 121–126.
[22] W. Rehder, *On the adjoints of normal operators*, Arch. Math. (Basel) **37** (1981), 169–172.
[23] M. Rosenblum, *On a theorem of Fuglede and Putnam*, J. London Math. Soc. **33** (1958), 376–377.

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