MORE ABOUT TWO PARAMETER SOR METHOD (*)

by SAADAT MOUSSAVI (in Oshkosh) (**)

SOMMARIO. - *Dato un sistema lineare* $Ax = b$, *uno spezzamento* $A = A_0 - A_1$ *porta alla successione iterativa* $x_k = Bx_{k-1} + C$ *con* $B = A_0^{-1}A_1$ *e* $C = A_0^{-1}b$. *Il vettore dell'errore è* $e_k = x_k - x_{soluzione}$ *e fornisce* $e_k = Be_{k-1} = \ldots = B^k e_0$. *Perciò* $||e_k|| = ||B^k e_0|| < ||B^k|| \cdot ||e_0|| \approx C_{k,p}\rho(B)^{k-p} \cdot ||e_0||$. *Dunque la convergenza a breve termine (rispettivamente a lungo termine) può essere migliorata minimizzando le norme di* B *(rispettivamente il raggio spettrale di* B*). In questo lavoro si considerano sia il raggio spettrale che le norme di differenti matrici iterative in competizione fra loro.*

SUMMARY. - *Given linear system* $Ax = b$, *a splitting* $A = A_0 - A_1$ *leads to the iterative sequence* $x_k = Bx_{k-1} + C$ *with* $B = A_0^{-1}A_1$ *and* $C = A_0^{-1}b$. *The error vector is* $e_k = x_k - x_{solution}$ *which yields* $e_k = Be_{k-1} = \ldots = B^k e_0$. *Hence* $||e_k|| = ||B^k e_0|| < ||B^k|| \cdot ||e_0|| \approx C_{k,p}\rho(B)^{k-p} \cdot ||e_0||$. *Therefore the short-term (long-term) convergence may be improved by minimizing norms of* B *(spectral radius of* B*). In this paper we consider both the spectral radius and the norms of competing iteration matrices.*

1. Preliminaries.

The well known "SOR" method is obtained from a one part splitting of the system matrix A, using one parameter ω.

M. Sisler introduced a new method by using one parameter for the lower triangular matrix L. Later he combined the above two methods to get a two parametric method [8],[9] and [10].

D. Young considered yet another two parametric method (MSOR).

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(**) Indirizzo dell'Autore: Mathematics Department – University of Wisconsin Oshkosh – Oshkosh, WI 54901 (U.S.A.).

The two parameters weight the diagonal of a positive-definite and consistently ordered 2-cyclic matrix [7]. For the first time G. Golub and J. dePillis used Singular Value Decomposition (SVD) to improve MSOR for the case that the coefficient matrix A is symmetric [12]. We generalized their results and also consider a special non-symmetric case.

2. Introduction.

To find the solution vector x to the linear system $Ax = b$, where A is a sparse $n \times n$ matrix and b is a given n-vector of complex n-space, usually A is not easy to invert. Therefore, one seeks an easy-to-invert part of A, say A_0. Hence

$$A = A_0 - A_1 \tag{2.1.1}$$

or equivalently,

$$A = A_0(I - A_0^{-1}A_1) = A_0(I - B) \tag{2.1.2}$$

where $B = A_0^{-1}A_1$ is called the *iteration matrix.*

Relation (2.1.1) is called an *additive splitting* which defines the $\{x_k\}$ for an arbitary fixed x_0 via,

$$A_0 x_{k+1} - A_1 x_k = b \quad k = 0, 1, 2, \ldots$$

or equivalently

$$x_{k+1} = A_0^{-1}A_1 x_k + A_0^{-1}b \quad k = 0, 1, 2, \ldots$$

$$x_{k+1} = Bx_k + A_0^{-1}b \quad k = 0, 1, 2, \ldots$$

Looking at relation (2.1.1), it is clear that if $\{x_k\}$ converges at all, it must converge to $x_{sol} = A^{-1}b$ (vector solution), where $Ax_{sol} = b$.

Relation (2.1.2) shows that $\{x_k\}$ converges to $x_{sol} = A^{-1}b$ for each x_0 if and only if $\rho(B) < 1$, where $\rho(B)$ is the spectral radius of B [1]. Use relation (2.1.2) to measure the asymptotic convergence R_∞ of the sequence $\{x_k\}$ where R_∞ is defined by $R_\infty = -\log \rho(B)$ which carries information on how fast the sequence $\{x_k\}$ converges.

In fact $\frac{1}{R_\infty}$ represents, asymptotically the number of iterations that suffice to produce one additional decimal place of accuracy in x_k's.

The above splitting is called *stationary* since there is no altering of parameter from iteration to iteration. It is called *one part splitting* since each x_{k+1} depends only on one previous vector x_k.

Examples of one-part stationary splitting are represented in the following important iteration methods.

JACOBI: Choose

$$A_0 = D, \quad A_1 = L + U$$

then

$$B_{jacobi} = B_j = D^{-1}(L + U)$$

where D is the diagonal part of A and $-L, -U$ are strictly lower and upper triangular parts of A respectively.

S.O.R.: Choose

$$A_0 = \frac{1}{\omega}D - L, \quad A_1 = \left(\frac{1}{\omega} - 1\right) D + U$$

then

$$B = B_\omega = (D - \omega L)^{-1}((1 - \omega)D + \omega U) \ . \tag{2.1.3}$$

Successive Overrelaxation (SOR) method was developed independently by Frankel [2] and Young [3], [4] in 1950.

Modified successive overrelaxation (MSOR) method first considered by Devogelaere [5] in 1958. Here is how it works. Consider the matrix A in the following form

$$A = \begin{bmatrix} D_1 & M \\ N & D_2 \end{bmatrix}$$

where D_1 and D_2 are square non-singular matrices. Use ω for the "red" equations corresponding to D_1 and ω' for the "black" equations corresponding to D_2 then

$$A_0 = \begin{bmatrix} \frac{1}{\omega} D_1 & 0 \\ N & \frac{1}{\omega'} D_2 \end{bmatrix}$$

and

$$A_1 = A_0 - A = \begin{bmatrix} (\frac{1}{\omega} - 1)\ D_1 & -M \\ 0 & (\frac{1}{\omega'} - 1)\ D_2 \end{bmatrix} .$$

Therefore, iteration matrix $B_{(\omega,\omega')}$ is defined by

$$B_{(\omega,\omega')} = A_0^{-1}A_1 = \begin{bmatrix} (1-\omega)\, I_1 & \omega F \\ \omega'(1-\omega)G & \omega\omega' GF + (1-\omega')I_2 \end{bmatrix} \qquad (2.1.4)$$

where $F = -D_1^{-1}M$ and $G = -D_2^{-1}N$.

Young [6] has proved that if A is positive definite then

$$\rho(B)_{\omega_b} < \bar{\rho}(B_{(\omega,\omega')})$$

where $\bar{\rho}(B_{\omega,\omega'})$ is virtual spectral radius of $B_{(\omega,\omega')}$.

Golub & dePillis [12] considered the matrix $A = \begin{bmatrix} I_p & M \\ M^t & I_q \end{bmatrix}$, they used the singular value decomposition of the corner matrix M

$$M = U\Sigma V^t \qquad (2.1.5)$$

where $p \times p$ matrix U and $q \times q$ matrix V are orthogonal and Σ is the $p \times q$ "diagonal matrix" defined by

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 & \cdot & & \cdot & 0 & \cdot & \cdot & 0 \\ 0 & \Sigma_2 & 0 & & \cdot & 0 & \cdot & \cdot & 0 \\ \cdot & & \cdot & & & & & & \\ 0 & \cdot & \cdot & \ddots & & 0 & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \Sigma_p & 0 & \cdot & \cdot & 0 \end{bmatrix} \qquad (2.1.6)$$
$$\underbrace{\hspace{6em}}_{p\times p}\quad\underbrace{\hspace{4em}}_{p\times(q-p)}$$

where

$$\Sigma_1 \geq \Sigma_2 \geq \ldots \geq \Sigma_p \geq 0\ .$$

From (2.1.5), it is clear that Σ_i^2 the eigenvalues of Matrix MM^t (and of M^tM) are the squares of the singular values of M. The number of non-zero singular values Σ_i of M equals the rank of matrix M.

They showed that the eigenvalues and 2-norms of matrices $B_{(\omega,\omega')}$ and $\Delta(\omega,\omega')$ are related as follows:

a) $\sigma(B_{(\omega,\omega')}) = \sigma(\Delta(\omega,\omega'))$ (2.1.7)

b) $\rho(B_{(\omega,\omega')}) = \rho(\Delta(\omega,\omega')) = \max\limits_i ||\rho(\Delta_i(\omega,\omega'))||$ (2.1.8)

c) $||B^k_{(\omega,\omega')}||_2 = ||\Delta^k(\omega,\omega')||_2 = \max\limits_i ||\Delta_i^k(\omega,\omega')||_2$ for all k . (2.1.9) where $\Delta(\omega, \omega')$ is the following matrix

$$\Delta(\omega,\omega') = \begin{bmatrix} \Delta_1(\omega,\omega') & 0 & & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \Delta_p(\omega,\omega') & \\ 0 & & 0 & (1-\omega')I_{q-p} \end{bmatrix} \tag{2.1.10}$$

where each 2×2 matrix $\Delta_i(\omega, \omega')$ is given by

$$\Delta_i(\omega,\omega') = \begin{bmatrix} (1-\omega) & \omega\Sigma_i \\ \omega'(1-\omega)\Sigma_i & (1-\omega') + \omega\omega'\Sigma_i^2 \end{bmatrix}, \; i = 1,2,3,\dots,p \tag{2.1.11}$$

where Σ_i are the singular values of (2.1.6).

3. Three parameter SOR method.

LEMMA 3.1. *If* $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ *is a square matrix with square diagonal submatrices* A_{11} *and* A_{22}*, then*

$$\det \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{cases} \det A_{11} \cdot \det(A_{22} - A_{21}A_{11}^{-1}A_{12}) \; if \; A_{11}^{-1} \; exists, \\ \det A_{22} \cdot \det(A_{11} - A_{12}A_{22}^{-1}A_{21}) \; if \; A_{22}^{-1} \; exists. \end{cases}$$

Proof. Without loss of generality let A_{22} be non-singular

$$\begin{bmatrix} I & -A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ A_{22}^{-1}A_{21} & I \end{bmatrix}. \tag{3.1.1}$$

Hence

$$\det\left(\begin{bmatrix} I & -A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \right) =$$

$$= \det\left(\begin{bmatrix} I & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ A_{22}^{-1}A_{21} & I \end{bmatrix} \right).$$

Which implies that

$$1 \cdot \det \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \det A_{22} \cdot \det(A_{11} - A_{21}A_{22}^{-1}A_{21})$$

LEMMA 3.2. *Let* $A = \begin{bmatrix} D_1 & M \\ N & D_2 \end{bmatrix}$ *where* D_1 *and* D_2 *are non-singular matrices.* $\mu \in \sigma(B_j)$ *if and only if* $\mu^2 \in \sigma(GF)$ *where* $F = -D_1^{-1}M$ *and* $G = -D_2^{-1}N$.

Proof. For Jacobi iteration matrix B_j we have the following splitting.

$$A_0 = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & -M \\ -N & 0 \end{bmatrix}$$

hence,

$$B_j = A_0^{-1}A_1 = \begin{bmatrix} 0 & -D_1^{-1}M \\ -D_2^{-1}N & 0 \end{bmatrix} = \begin{bmatrix} 0 & F \\ G & 0 \end{bmatrix}.$$

Clearly $\mu \in \sigma(B)$ if and only if

$$\det(B_j - \mu I) = \det \begin{bmatrix} -\mu I_1 & F \\ G & -\mu I_2 \end{bmatrix} = 0\,. \qquad (3.2.2)$$

By Lemma 3.1 and relation (3.2.2)

$$|-\mu I_1| \cdot |-\mu I_2 - G(-\mu I_2)^{-1}F| = 0 \qquad (3.2.3)$$

relation (3.2.3) holds if and only if $\mu \in \sigma(\frac{1}{\mu}GF)$ or equivalently $\mu^2 \in \sigma(GF)$.

THEOREM 3.3. *Suppose that* $A = \begin{bmatrix} D_1 & M \\ N & D_2 \end{bmatrix}$, *where* D_1 *and* D_2 *are non-singular matrices and the easy to invert part of matrix* A *is given by*

$$A_0 = \begin{bmatrix} \frac{1}{\omega}D_1 & 0 \\ \alpha N & \frac{1}{\omega'}D_2 \end{bmatrix}.$$

Let $\mu \in \sigma(B_j)$. *If* λ *satisfies*

$$(\lambda + \omega - 1)(\lambda + \omega' - 1) = (\alpha\lambda + (1 - \alpha))\omega\omega'\mu^2 \tag{3.3.4}$$

then λ *is an eigenvalue of* $B_{(\omega,\omega',\alpha)}$.

Conversely, let $\lambda \in \sigma(B_{(\omega,\omega',\alpha)})$, *then every* μ *satisfying* (3.3.4) *is an eigenvalue of* B_j.

Proof.

$$B_{(\omega,\omega',\alpha)} = A_0^{-1}A_1 = \begin{bmatrix} \omega D_1^{-1} & 0 \\ -\alpha\omega\omega' D_2^{-1} N D_1^{-1} & \omega' D_2^{-1} \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} (\frac{1}{\omega} - 1) D_1 & -M \\ (\alpha - 1) N & (\frac{1}{\omega'} - 1) D_2 \end{bmatrix}$$

or equivalently

$$B_{(\omega,\omega',\alpha)} = \begin{bmatrix} (1 - \omega) I_1 & \omega F \\ \omega'(1 - \alpha\omega) G & \alpha\omega\omega' GF + (1 - \omega') I_2 \end{bmatrix} \tag{3.3.5}$$

where $F = D_1^{-1}M$ and $G = D_2^{-1}N$.

λ is an eigenvalue of $B_{(\omega,\omega',\alpha)}$ if and only if $\det(B_{(\omega,\omega',\alpha)} - \lambda I) = 0$

$$B_{(\omega,\omega',\alpha)} - \lambda I = \begin{bmatrix} ((1 - \omega) - \lambda) I_1 & \omega F \\ \omega'(1 - \alpha\omega) G & \alpha\omega\omega' GF + ((1 - \omega') - \lambda) I_2 \end{bmatrix} .$$

By Lemma 3.1

$$\det(B_{(\omega,\omega',\alpha)} - \lambda I) = (1 - \omega - \lambda)^p \cdot$$

$$\cdot \det \left[(1 - \omega' - \lambda) I_2 - \frac{(1 + \alpha\lambda - \alpha)\omega\omega'}{(1 - \omega - \lambda)} GF) \right] \tag{3.3.6}$$

where p is the size of I_1. By relation (3.3.6) $\det(B_{(\omega,\omega',\alpha)} - \lambda I) = 0$ if and only if

$$\det \left[(1 - \omega' - \lambda) I_2 - \frac{(1 + \alpha\lambda - \alpha)\omega\omega'}{(1 - \omega - \lambda)} GF \right] = 0 . \tag{3.3.7}$$

Relation (3.3.7) holds if and only if

$$(1-\omega'-\lambda)\in\sigma\left[\frac{(1+\alpha\lambda-\alpha)\omega\omega'}{(1-\omega-\lambda)}GF\right]$$

or equivalently

$$\frac{(1-\omega-\lambda)(1-\omega'-\lambda)}{(1+\alpha\lambda-\alpha)\omega\omega'}\in\sigma(GF)\;. \qquad (3.3.8)$$

On the other hand, by Lemma 3.2, $\mu^2\in\sigma(GF)$, then

$$(\lambda+\omega-1)(\lambda+\omega'-1)=(\alpha\lambda+(1-\alpha))\omega\omega'\mu^2\;.$$

REMARK

(1) If $\alpha=1$ and $\omega=\omega'$ then (3.3.4) reduces to SOR Method.

(2) If $\alpha=1$ and $\omega\neq\omega'$ then (3.3.4) reduces to MSOR Method.

(3) If $\omega=\omega'$ and $\alpha=\frac{r}{\omega}$ then (3.3.4) reduces to AOR Method [13].

4. Singular value decomposition and Jacobi method.

Suppose $A=\begin{bmatrix} I_p & -M\\ -N & I_q\end{bmatrix}$, and let $M=U\Sigma V^t$, $N=QSR^t$ be the singular value decompositions of M and N respectively. Where $p\times p$ matrices U, R and $q\times q$ matrices V, Q are orthogonal, and $p\times q$ matrix Σ, $q\times p$ matrix S are "diagonal matrices" defined by

$$\Sigma=\underbrace{\begin{bmatrix} \Sigma_1 & 0 & \cdot & & \cdot & 0 & \cdot & \cdot & 0\\ 0 & \Sigma_2 & 0 & & \cdot & 0 & \cdot & \cdot & 0\\ \cdot & & \ddots & & & & & & \\ 0 & \cdot & \cdot & \cdot & & 0 & \cdot & \cdot & 0\\ 0 & \cdot & \cdot & \cdot & \Sigma_p & 0 & \cdot & \cdot & 0 \end{bmatrix}}_{p\times q}$$

$$S = \underbrace{\begin{bmatrix} s_1 & 0 & \cdot & \cdot & 0 & 0 & 0 \\ 0 & s_2 & 0 & \cdot & & \cdot & \cdot \\ \cdot & & \cdot & & & & \\ \cdot & & & \cdot & 0 & 0 & 0 \\ 0 & & & & & s_{p-1} & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & 0 & s_p \\ 0 & & & & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & & & \cdot & \cdot \\ 0 & 0 & & & 0 & 0 & 0 \end{bmatrix}}_{q \times p} \tag{4.1.0}$$

The Jacobi iteration matrix B_j for matrix A is

$$B_j = \begin{bmatrix} 0 & M \\ N & 0 \end{bmatrix} . \tag{4.1.1}$$

Substitute $M = U\Sigma V^t$, $N = QSR^t$ for M and N respectively in (4.1.1), and "factor out" the orthogonal matrices

$$B_j = \begin{bmatrix} 0 & M \\ N & 0 \end{bmatrix} = \begin{bmatrix} 0 & U\Sigma V^t \\ QSR^t & 0 \end{bmatrix}$$

$$B_j = \underbrace{\begin{bmatrix} U & 0 \\ 0 & Q \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} 0 & \Sigma \\ S & 0 \end{bmatrix}}_{\Gamma_j} \underbrace{\begin{bmatrix} R^t & 0 \\ 0 & V^t \end{bmatrix}}_{L^t} .$$

Hence $B_j = KT_jL^t$.

Now

$$B_jB_j^t = (K\Gamma_jL^t)(K\Gamma_jL^t)^t$$

$$B_jB_j^t = K\Gamma_j\Gamma_j^tK^t . \tag{4.1.2}$$

Equivalence relation (4.1.2) implies that the eigenvalues and 2-norms of $B_jB_j^t$ and $\Gamma_j\Gamma_j^t$ are agreed, i.e.

$$\sigma(B_jB_j^t) = \sigma(\Gamma_j\Gamma_j^t) = \left\{ s_i^2, \Sigma_k^2 \,\middle|\, \begin{array}{l} i = 1,2,\ldots,p \\ k = 1,2,\ldots,p \end{array} \right\}$$

and

$$||B_j^k||_2 = ||\Gamma_j^k||_2 \quad \text{for all } k .$$

Since $||\Gamma||_2 = \rho[\Gamma^t\Gamma]^{\frac{1}{2}}$ [6],

$$||B_j||_2 = ||\Gamma_j||_2 = \rho\left(\begin{bmatrix} 0 & S \\ \Sigma & 0 \end{bmatrix}\begin{bmatrix} 0 & \Sigma \\ S & 0 \end{bmatrix}\right)^{\frac{1}{2}}$$
$$= \rho\left(\begin{bmatrix} S^2 & 0 \\ 0 & \Sigma^2 \end{bmatrix}\right)^{\frac{1}{2}} = max\{s_1, \Sigma_1\}\,.$$

Furthermore, since $\rho(\Gamma) \leq ||\Gamma||_2$ [6],

$$\rho(B_j) = \rho(\Gamma_j) \leq max\{s_1, \Sigma_1\}\,. \tag{4.1.3}$$

On the other hand by Lemma 3.2

$$[\rho(B_j)]^2 = \rho(NM) = \rho(QSR^tU\Sigma V^t)$$
$$\leq ||QSR^t||_2||U\Sigma V^t||_2 = ||S||_2||\Sigma||_2 = s_1\Sigma_1$$

Therefore

$$\rho(B_j) \leq \sqrt{s_1\Sigma_1}\,. \tag{4.1.4}$$

By relations (4.1.3) and (4.1.4) one could conclude that

$$\rho(B_j) = \rho(\Gamma_j) \leq min\{max\{s_1, \Sigma_1\}, \sqrt{s_1\Sigma_1}\}\,.$$

The above argument give us the following theorem:

THEOREM 4.1. *Suppose* $A = \begin{bmatrix} I_p & -M \\ -N & I_q \end{bmatrix}$, *and let* $M = U\Sigma V^t$, $N = QSR^t$ *be the singular value decompositions of M and N respectively. Then*

(a) $||B_j^k||_2 = ||\Gamma_j^k||_2$ for all k, where $\Gamma_j = \begin{bmatrix} 0 & \Sigma \\ S & 0 \end{bmatrix}$

(b) $||B_j||_2 = max\{s_1, \Sigma_1\}$

(c) $\rho(B_j) = \rho(\Gamma_j) \leq min\{max\{s_1, \Sigma_1\}, \sqrt{s_1\Sigma_1}\}\,.$

5. SVD and three parameter SOR method.

Suppose $A = \begin{bmatrix} I_p & -M \\ -M^t & I_q \end{bmatrix}$, and let $M = U\Sigma V^t$ be the singular value decompositions of M. Where $p \times p$ matrix U and $q \times q$ matrix V are orthogonal, and $p \times q$ matrix Σ is diagonal matrix defined by (2.1.6). If the easy to invert part of A is given by

$$A_0 = \begin{bmatrix} \frac{1}{\omega} I_p & 0 \\ \alpha M^t & \frac{1}{\omega'} I_q \end{bmatrix}$$

then, the iteration matrix for this method is

$$B_{(\omega,\omega',\alpha)} = A_0^{-1} A_1 = \begin{bmatrix} (1-\omega) I_p & \omega M \\ \omega'(1-\alpha\omega) M^t & \alpha\omega\omega' M^t M + (1-\omega') I_q \end{bmatrix}. \tag{5.1.1}$$

Substitute $M = U\Sigma V^t$ for M in (5.1.1)

$$B_{(\omega,\omega',\alpha)} = \begin{bmatrix} (1-\omega) I_p & \omega U\Sigma V^t \\ \omega'(1-\alpha\omega) V\Sigma^t U^t & \alpha\omega\omega' V\Sigma^t \underbrace{U^t U}_{I} \Sigma V^t + (1-\omega') I_q \end{bmatrix}$$

"Factor out" V and U

$$B_{(\omega,\omega',\alpha)} = \underbrace{\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} (1-\omega) I_p & \omega\Sigma \\ \omega'(1-\alpha\omega)\Sigma^t & \alpha\omega\omega'\Sigma^t\Sigma + (1-\omega') I_q \end{bmatrix}}_{\Gamma_{(\omega,\omega',\alpha)}} \cdot$$

$$\cdot \underbrace{\begin{bmatrix} U^t & 0 \\ 0 & V^t \end{bmatrix}}_{Q^t}$$

Hence, $B_{(\omega,\omega',\alpha)} = Q\Gamma_{(\omega,\omega',\alpha)} Q^t$ where matrix Q is a unitary matrix.

There is a permutation matrix [12] P such that

$$\Delta(\omega,\omega',\alpha) = P\Gamma_{(\omega,\omega',\alpha)} P^t =$$

$$= \begin{bmatrix} \Delta_1(\omega,\omega',\alpha) & 0 & & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \Delta_p(\omega,\omega',\alpha) & \\ 0 & & 0 & (1-\omega') I_{q-p} \end{bmatrix} \tag{5.1.2}$$

where each 2×2 matrix $\Delta(\omega,\omega',\alpha)$ is given by

$$\Delta_i(\omega,\omega',\alpha) = \begin{bmatrix} (1-\omega) & \omega\Sigma_i \\ \omega'(1-\alpha\omega)\Sigma_i & (1-\omega')+\alpha\omega\omega'\Sigma_i^2 \end{bmatrix},$$

$$i = 1,2,3,\ldots,p$$

where Σ_i are the singular values of (2.1.6).

Since $\Delta(\omega,\omega',\alpha) = P\Gamma_{(\omega,\omega',\alpha)}P^t$ hence,

$$B_{(\omega,\omega',\alpha)} = QP^t\Delta(\omega,\omega',\alpha)PQ^t \quad \text{for unitary} \quad QP^t . \tag{5.1.3}$$

Equivalence relation (5.1.3) implies that the eigenvalues and the 2–norms of $B_{(\omega,\omega',\alpha)}$ and $\Delta(\omega,\omega',\alpha)$ are agreed. Hence, if λ is an eigenvalue of $B_{(\omega,\omega',\alpha)}$ it must be one of the eigenvalues of $\Delta_i(\omega,\omega',\alpha)$. Therefore,

$$\begin{aligned} \lambda \in \sigma(B_{(\omega,\omega',\alpha)}) &\Leftrightarrow \lambda \in \sigma\Delta(\omega,\omega',\alpha)) \\ &\Leftrightarrow \det\begin{bmatrix} \lambda-(1-\omega) & \omega\Sigma_i \\ \omega'(1-\alpha\omega)\Sigma_i & \lambda-(1-\omega')+\alpha\omega\omega'\Sigma_i^2 \end{bmatrix} = 0 \\ &\Leftrightarrow (\lambda+\omega-1)(\lambda+\omega'-1-\alpha\omega\omega'\Sigma_i^2) \\ &\quad -(1-\alpha\omega)\omega\omega'\Sigma_i^2 = 0 \end{aligned}$$

Or equivalently

$$(\lambda+\omega-1)(\lambda+\omega'-1) = (\alpha\lambda+(1-\alpha))\omega\omega'\Sigma_i^2$$

By Lemma 3.1 and singular value decomposition properties

$$\{\mu_i^2\} = \sigma[[(B_j)]^2 = \sigma(MM^t) = \{\Sigma_i^2\}$$

Therefore

$$(\lambda+\omega-1)(\lambda+\omega'-1) = (\alpha\lambda+(1-\alpha))\omega\omega'\mu_i^2 .$$

Let us summarize the above arguments in the following theorem:

THEOREM 5.1. *Suppose* $A = \begin{bmatrix} I_p & -M \\ -M^t & I_q \end{bmatrix}$, *and let* $M = U\Sigma V^t$ *be the singular value decompositions of* M. *If the easy to invert part of* A *is given by*

$$A_0 = \begin{bmatrix} \frac{1}{\omega}D_1 & 0 \\ \alpha M^t & \frac{1}{\omega'}D_2 \end{bmatrix}$$

then the eigenvalues $\mu_i \in \sigma(B_j)$, $\lambda_i \in \sigma(B_{(\omega,\omega',\alpha)})$ *are related by the following functional relation:*

$$(\lambda + \omega - 1)(\lambda + \omega' - 1) = (\alpha\lambda + (1-\alpha))\omega\omega'\mu_i^2 .$$

Moreover, eigenvalues and 2-norms of matrices $\Delta(\omega, \omega', \alpha)$ *of (5.1.2) and* $B_{(\omega,\omega',\alpha)}$ *are related as follows:*
(a) $\sigma(B_{(\omega,\omega',\alpha)}) = \sigma(\Delta(\omega,\omega',\alpha))$
(b) $\rho(B_{(\omega,\omega',\alpha)}) = \rho(\Delta(\omega,\omega',\alpha)) = \max_i ||\rho(\Delta_i(\omega,\omega',\alpha))||$
(c) $||B^k_{(\omega,\omega',\alpha)}||_2 = ||\Delta^k(\omega,\omega',\alpha)||_2 = \max_i ||\Delta_i^k(\omega,\omega',\alpha)||^2$ for all k.

6. A special non-symmetric case.

Suppose $A = \begin{bmatrix} I_p & -M \\ -N & I_1 \end{bmatrix}$, and let $M = U\Sigma V^t$, $N = QSR^t$ be the singular value decompositions of M and N respectively. Let $U^tR = D_1 = diag(d_1, d_2, \ldots, d_p)$ and $Q^tV = D_2 = diag(c_1, c_2, \ldots, c_q)$ be diagonal matrices.

Jacobi iteration matrix for the matrix A is given by matrix (4.1.1) $B_j = \begin{bmatrix} 0 & M \\ N & 0 \end{bmatrix}$.

Substitute $M = U\Sigma V^t$, $N = QSR^t$ for M and N respectively in B_j and "factor out" the orthogonal matrices.

$$B_j = \underbrace{\begin{bmatrix} U & 0 \\ 0 & Q \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 0 & \Sigma \\ S & 0 \end{bmatrix}}_{\Gamma_j} \underbrace{\begin{bmatrix} R^t & 0 \\ 0 & V^t \end{bmatrix}}_{K}$$

Hence

$$B_j = L\Gamma_j K . \tag{6.1.1}$$

Notice that

$$KL = \begin{bmatrix} R^t & 0 \\ 0 & V^t \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} R^tU & 0 \\ 0 & V^tQ \end{bmatrix} = \begin{bmatrix} D_1^* & 0 \\ 0 & D_2^* \end{bmatrix} = D \tag{6.1.2}$$

where D_1^* and D_2^* are the complex conjugate of D_1 and D_2 respectively.

Multiply (6.1.1) by K and K^t from left and right respectively.

$$KB_jK^t = KL\Gamma_jKK^t \,.$$

By relation (6.1.2), we have

$$KB_jK^t = D\Gamma_j \,. \tag{6.1.3}$$

Unitary equivalence relation (6.1.3) implies that both the eigenvalues and the 2–norms agree for both matrices B_j and $D\Gamma_j$. Matrix $D\Gamma_j$ has the following form

$$D\Gamma_j = \left[\begin{array}{ccc|cccc}
\bar{d}_1 & 0 & 0 & 0 & 0 & \cdot & 0 \\
0 & \bar{d}_2 & & & \cdot & & \cdot \\
 & & \cdot & & & & \\
0 & & \bar{d}_p & 0 & 0 & \cdot & 0 \\
\hline
0 & \cdot\ \cdot & 0 & \bar{c}_1 & 0 & \cdot & 0 \\
0 & \cdot\ \cdot & 0 & 0 & \cdot & & 0 \\
\cdot & & \cdot & \cdot & & \bar{c}_p & \\
\cdot & \cdot & \cdot & & & \cdot & \\
0 & 0 & 0 & 0 & & 0 & \bar{c}_q
\end{array}\right] .$$

$$\begin{array}{c}
\cdot \left[\begin{array}{ccc|cccccc}
0 & 0 & 0 & \Sigma_1 & \cdot & 0 & 0 & \cdot & 0 \\
0 & & & & \cdot\ \cdot & & & & \cdot \\
0 & 0 & 0 & & & \Sigma_p & 0 & \cdot & 0 \\
\hline
s_1 & \cdot\ \cdot & 0 & 0 & 0 & \cdot & 0 & \cdot & 0 \\
0 & s_2\ \cdot & 0 & 0 & 0 & \cdot & & & 0 \\
\cdot & & & & \cdot & & & & \\
\cdot & \cdot & s_p & \cdot & & & & & 0 \\
0 & & 0 & & & & & & \\
0 & & 0 & 0 & 0 & & 0 & \cdot & 0
\end{array}\right]
\begin{array}{l}
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\} p \\
\left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right\} q
\end{array} \\
\underbrace{\hspace{12em}}_{p}\quad\underbrace{\hspace{12em}}_{q}
\end{array}$$

Or equivalently

$$D\Gamma_j = \left[\begin{array}{cccc|cccc}
0 & 0 & & 0 & \Sigma_1\bar{d}_1 & 0\ \cdot & 0 & \cdot\ 0 \\
0 & & & & & \cdot & & \cdot \\
0 & 0 & & 0 & & & \Sigma_p\bar{d}_p & \cdot\ 0 \\
\hline
s_1\bar{c}_1 & & \cdot\ \cdot & 0 & 0 & \cdot & 0 & 0 \\
0 & s_2\bar{c}_2 & \cdot & 0 & 0 & \cdot & & 0 \\
\cdot & & & \cdot & \cdot & & & 0 \\
0 & & & s_p\bar{c}_p & & & & \cdot \\
0 & 0 & & 0 & 0 & 0 & & 0
\end{array}\right] \tag{6.1.4}$$

There is a permutation matrix P such that

$$\Delta = PD\Gamma_jP^t = \begin{bmatrix}
\Delta_1 & 0 & & & 0 & 0 \\
0 & \Delta_2 & & & \cdot & \\
\cdot & & \cdot & & \cdot & \\
\cdot & & & \Delta_p & 0 & \\
0 & & & 0 & 0 & 0 \\
\cdot & & & \cdot & \cdot & \cdot \\
0 & & & 0 & 0 & 0
\end{bmatrix} \tag{6.1.5}$$

where each 2×2 matrix Δ_i is given by

$$\Delta_i = \begin{bmatrix} 0 & \bar{d}_i\Sigma_i \\ \bar{c}_is_i & 0 \end{bmatrix} .$$

Obviously by (6.1.4) and (6.1.5) $\mu \in \sigma(D\Gamma_j)$ if and only if $\mu \in \sigma(\Delta_i)$, i.e.

$$\det \begin{bmatrix} -\mu & \bar{d}_i\Sigma_i \\ \bar{c}_is_i & -\mu \end{bmatrix} = 0 .$$

Or equivalently

$$\mu^2 - \overline{c_id_i}s_i\Sigma_i = 0 .$$

Hence,

$$\mu^2 = \overline{c_id_i}s_i\Sigma_i \tag{6.1.6}$$

which gives us the following lemma:

LEMMA 6.1. *Suppose* $A = \begin{bmatrix} I_p & -M \\ -N & I_q \end{bmatrix}$, *and let* $M = U\Sigma V^t, N = QSR^t$ *be the singular value decompositions of* M *and* N *respectively. If* $U^tR = D_1 = \operatorname{diag}(d_1, d_2, \ldots, d_p)$ *and* $Q^tV = D_2 = \operatorname{diag}(c_1, c_2, \ldots, c_q)$ *are diagonal matrices, then* $\mu_i \in \sigma(B_j)$ *if and only if* $\mu_i^2 = \overline{c_i d_i} s_i \Sigma_i$. *(Where* Σ_i *and* s_i *are the singular values of* M *and* N *respectively).*

Moreover, eigenvalues and 2–norms of matrices B_j *and* Δ *of (6.1.5) are related as follows:*

(a) $\sigma(B_j) = \sigma(\Delta)$

(b) $\rho(B_j) = \rho(\Delta) = \max_i ||\rho(\Delta_i)||$

(c) $||B_j^k||_2 = ||\Delta^k||_2 = \max_i ||\Delta_i^k||_2$ *for all k.*

Under the assumption of Lemma 6.1 if the easy to invert part of A is given by

$$A_0 = \begin{bmatrix} \frac{1}{\omega} I_p & 0 \\ aN & \frac{1}{\omega'} I_q \end{bmatrix}$$

then the iteration matrix corresponding to A_0 is given by

$$B_{(\omega,\omega',\alpha)} = A_0^{-1} A_1 = \begin{bmatrix} (1-\omega) I_p & \omega M \\ \omega'(1-\alpha\omega) N & \alpha\omega\omega' NM + (1-\omega') I_q \end{bmatrix} . \tag{6.1.7}$$

Substitute $M = U\Sigma V^t$, $N = QSR^t$ for M and N respectively in matrix (6.1.7)

$$B_{(\omega,\omega',\alpha)} = \begin{bmatrix} (1-\omega) I_p & \omega U\Sigma V^t \\ \omega'(1-\alpha\omega) QSR^t & \alpha\omega\omega' QSR^tU\Sigma V^t + (1-\omega') I_q \end{bmatrix} .$$

Factor out U, V, Q and R.

$$B_{(\omega,\omega',\alpha)} = \underbrace{\begin{bmatrix} U & 0 \\ 0 & Q \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} (1-\omega) U^tR & \omega\Sigma \\ \omega'(1-\alpha\omega) S & \alpha\omega\omega' SR^tU\Sigma + (1-\omega') Q^tV \end{bmatrix}}_{\Gamma_{(\omega,\omega',\alpha)}} \cdot$$

$$\cdot \underbrace{\begin{bmatrix} R^t & 0 \\ 0 & V^t \end{bmatrix}}_{K}$$

Hence

$$B_{(\omega,\omega',\alpha)} = L\Gamma_{(\omega,\omega',\alpha)} K . \tag{6.1.8}$$

Multiply (6.1.8) by K and K^t from left and right respectively.

$$KB_{(\omega,\omega',\alpha)}K^t = KL\Gamma_{(\omega,\omega',\alpha)}KK^t\,.$$

By relation (6.1.2),

$$KB_{(\omega,\omega',\alpha)}K^t = D\Gamma_{(\omega,\omega',\alpha)}\,. \tag{6.1.9}$$

Unitary equivalence relation (6.1.9) implies that both the eigenvalues and the 2-norms agree for both matrices $B_{(\omega,\omega',\alpha)}$ and $D\Gamma_{(\omega,\omega'\alpha)}$.

Now let us investigate all four submatrices of $\Gamma_{(\omega,\omega',\alpha)}$.

(i)

$$(1-\omega)U^tR = (1-\omega)D_1 = \begin{bmatrix} (1-\omega)d_1 & 0\cdot & \cdot & 0 \\ 0 & (1-\omega)d_2 & \cdot & 0 \\ \cdot & & \ddots & \\ \cdot & & & \\ 0 & & \cdots & 0 \\ 0 & & \cdots & (1-\omega)d_p \end{bmatrix}$$

where $U^tR = D_1 = \text{diag}(d_1, d_2, \ldots, d_p)$

(ii)

$$\omega\Sigma = \omega \underbrace{\begin{bmatrix} \Sigma_1 & 0 & \cdot & & \cdot & 0\cdot\cdot 0 \\ 0 & \Sigma_2 & 0 & & \cdot & 0\cdot\cdot 0 \\ \cdot & & \ddots & & & \\ 0 & \cdot & \cdot & & & 0\cdot\cdot 0 \\ 0 & \cdot & \cdot & & \Sigma_p & 0\cdot\cdot 0 \end{bmatrix}}_{p\times q}$$

(iii)

$$\omega'(1-\alpha\omega)S = \omega'(1-\alpha\omega) \begin{array}{c} \left[\begin{array}{cccccccc} s_1 & 0 & \cdot & \cdot & 0 & 0 & 0 \\ 0 & s_2 & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & 0 & 0 & 0 \\ 0 & & & & & s_{p-1} & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & 0 & s_p \\ 0 & & & & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & & & \cdot & \cdot \\ 0 & 0 & & & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\} p\times p \\ \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} (q-p)\times p \end{array},$$

(iv)

$$SR^tU\Sigma = SD_1^*\Sigma = \begin{array}{c}\left[\begin{array}{ccccc|cccc}
s_1\bar{d}_1\Sigma_1 & 0 & \cdot & \cdot & 0 & 0 & \cdot & & 0 \\
0 & s_2\bar{d}_2\Sigma_2 & \cdot & & & \cdot & & & \cdot \\
\cdot & & & \cdot & & 0 & 0 & \cdot & 0 \\
\cdot & & & & & & & & \\
0 & & & & s_p\bar{d}_p\Sigma_p & 0 & \cdot & \cdot & 0 \\
\cdot & \cdot & & & & \cdot & & & \cdot \\
0 & 0 & \cdot & \cdot & 0 & 0 & \cdot & \cdot & 0
\end{array}\right] \\
\underbrace{\hspace{16em}}_{q\times p}\underbrace{\hspace{7em}}_{q\times(q-p)}\end{array}$$

Hence, $\alpha\omega\omega' SR^tU\Sigma + (1-\omega')Q^tV = \alpha\omega\omega'(SD_1^*\Sigma) + (1-\omega')D_2 =$

$$= \begin{bmatrix}
\alpha\omega\omega' s_1\bar{d}_1\Sigma_1 + (1-\omega')c_1 & 0 \cdot & \cdot & \cdot & 0 & 0 & \cdot & 0 \\
 & & & & & & & 0 \\
0 & & \cdot & \cdot & & & & \cdot \\
\cdot & & 0 & \cdot & \cdot & 0 & 0 & 0 \\
0 & & & & \alpha\omega\omega' s_p\bar{d}_p\Sigma_p + (1-\omega')c_p & 0 & \cdot & 0 \\
0 & & \cdot & \cdot\ 0 & 0 & (1-\omega')c_{p+1} & & 0 \\
\cdot & & \cdot & \cdot\ \cdot & \cdot & \cdot & \ddots & \\
0 & 0 & 0 & 0 & 0 & 0 & \cdot & (1-\omega')c_q
\end{bmatrix}$$

where $Q^tV = D_2 = \mathrm{diag}(c_1, c_2, \ldots, c_q)$.

Therefore, the matrix $D\Gamma$ is given by the following

$$\left[\begin{array}{ccc|cccccc}
(1-\omega)|d_1|^2 & \cdot\ \cdot & 0 & \omega\Sigma_1\bar{d}_1 & \cdot\ \cdot & 0 & 0 & \cdot\ \cdot & 0 \\
\cdot & & \cdot & \cdot & & \cdot\ \cdot & & & \cdot \\
\cdot & & \cdot & \cdot & & \cdot\ \cdot & & & \cdot \\
0 & \cdot\ \cdot & (1-\omega)|d_p|^2 & 0 & \cdot\ \cdot & \omega\Sigma_p\bar{d}_p & 0 & \cdot\ \cdot & 0 \\
\hline
\omega'(1-\alpha\omega)s_1\bar{c}_1 & & 0 & \alpha\omega\omega' s_1\overline{d_1c_1}\Sigma_1 + (1-\omega')|c_1|^2 & \cdot\ \cdot & 0 & 0 & \cdot\ \cdot & 0 \\
\cdot & & \cdot & \cdot & & \cdot\ \cdot & & & \cdot \\
\cdot & & \cdot & \cdot & & \cdot\ \cdot & & & \cdot \\
0 & & \omega'(1-\alpha\omega)s_p\bar{c}_p & 0 & \cdot\ \cdot & \alpha\omega\omega' s_p\overline{d_pc_p}\Sigma_p + (1-\omega')|c_p|^2 & 0 & \cdot\ \cdot & 0 \\
0 & \cdot\ \cdot & 0 & 0 & \cdot\ \cdot & 0 & (1-\omega')|c_{p+1}|^2 & \cdot\ \cdot & 0 \\
\cdot & & \cdot & \cdot & & \cdot\ \cdot & & & \cdot \\
\cdot & & \cdot & \cdot & & \cdot\ \cdot & & & \cdot \\
0 & \cdot\ \cdot & 0 & 0 & \cdot\ \cdot & 0 & 0 & \cdot\ \cdot & (1-\omega')|c_q|^2
\end{array}\right]$$

There is a permutation matrix P [12] such that $PD\Gamma P^t$ has only 2×2 and 1×1 submatrices

$$\Delta(\omega,\omega',\alpha) = PD\Gamma P^t =$$

$$= \begin{bmatrix} \Delta_1(\omega,\omega',\alpha) & 0 & & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \Delta_p(\omega,\omega',\alpha) & \\ 0 & & 0 & (1-\omega')|c_k|^2 I_{p-q} \end{bmatrix} \qquad (6.1.10)$$

$$\text{for } k = p+1,\ p+2,\dots,q$$

where each 2×2 matrix $\Delta_i(\omega,\omega',\alpha)$ is given by

$$\begin{bmatrix} (1-\omega)|d_i|^2 & \omega\varepsilon_i\bar{d}_i \\ \omega'(1-\alpha\omega)s_i\bar{c}_i & (1-\omega')|c_i|^2 + \alpha\omega\omega' s_i\overline{d_i s_i}\Sigma_i \end{bmatrix}$$

$$i = 1,2,3,\dots,p$$

where Σ_i and s_i are the singular values of M and N respectively.

Hence, if λ is an eigenvalue of $B_{(\omega,\omega',\alpha)}$ it must be one of the eigenvalues of $\Delta_i(\omega,\omega',\alpha)$. Therefore

$$\lambda \in \sigma(B_{(\omega,\omega'\alpha)}) \Leftrightarrow \lambda \in \sigma(\Delta(\omega,\omega',\alpha))$$

$$[(1-\omega)|d_i|^2 - \lambda[\alpha\omega\omega' s_i\overline{d_i c_i}\Sigma_i + (1-\omega')|c_i|^2 - \lambda] = \omega\omega'(1-\alpha\omega)s_i\Sigma\,\overline{d_i c_i}$$

Or equivalently

$$[(1-\omega)|d_i|^2 - \lambda][(1-\omega')|c_i|^2 - \lambda] = \omega\omega' s_i\Sigma_i\overline{d_i c_i}$$
$$[1-\alpha(\omega+\lambda+|d_i|^2(1-\omega))]\ . \qquad (6.1.11)$$

This argument results the following theorem:

THEOREM 6.2. *Suppose* $A = \begin{bmatrix} I_p & -M \\ -N & I_q \end{bmatrix}$, *and let* $M = U\Sigma V^t$, $N = QSR^t$ *be the singular value decompositions of M and N respectively. If the easy to invert part of A is given by*

$$A_0 = \begin{bmatrix} \frac{1}{\omega}I_p & 0 \\ \alpha N & \frac{1}{\omega'}Iq \end{bmatrix}$$

and $U^tR = D_1 = \mathrm{diag}(d_1, d_2, \dots, d_p)$, $Q^tV = D_2 = \mathrm{diag}(c_1, c_2, \dots, c_q)$ *are diagonal matrices. Then the eigenvalues* $\mu_i \in \sigma(B_j)$, $\lambda_i \in \sigma(B_{(\omega,\omega',\alpha)})$ *are related by the following functional relation:*

$$(\lambda + \omega - 1)(\lambda + \omega' - 1) = (\alpha\lambda + (1 - \alpha))\omega\omega'\mu_i^2 \ .$$

Moreover, eigenvalues and 2–norms of matrices $\Delta(\omega, \omega', \alpha)$ *of (6.1.10) and* $B_{(\omega,\omega',\alpha)}$ *are related as follows:*

(a) $\sigma(B_{(\omega,\omega',\alpha)}) = \sigma(\Delta(\omega, \omega', \alpha))$

(b) $\rho(B_{(\omega,\omega',\alpha)}) = \rho(\Delta(\omega, \omega', \alpha)) = \max\limits_i ||\rho(\Delta_i(\omega, \omega', \alpha))||$

(c) $||B^k_{(\omega,\omega',\alpha)}||_2 = ||\Delta^k(\omega, \omega', \alpha)||_2 = \max\limits_i ||\Delta_i^k(\omega, \omega', \alpha)||_2$ *for all* k.

Proof. Since Q^tV and U^tR are diagonal orthogonal matrices then, absolute value of each diagonal elements of these two matrices is identity, i.e.

$$|d_i| = 1 \text{ for all } i = 1, 2, \dots, p$$

$$|c_i| = 1 \text{ for all } j = 1, 2, \dots, q \ .$$

Hence relation (6.1.11) becomes

$$[(1 - \omega) - \lambda][(1 - \omega') - \lambda] = \omega\omega' s_i \Sigma_i \overline{d_i c_i} [1 - \alpha(\omega + \lambda + (1 - \omega))] \ .$$

Since by Lemma 6.1 $\mu^2 = \overline{c_1 d_1} s_i \Sigma_i$, therefore

$$(\lambda + \omega - 1)(\lambda + \omega' - 1) = (\alpha\lambda + (1 - \alpha))\omega\omega'\mu_i^2 \ .$$

REFERENCES

[1] PLEMMONS R. and BERMAN A., *Nonnegative matrices in the mathematical science*, Academic Press, New York, London (1979).

[2] FRANKEL S.P., *Convergence rate of iterative treatments of partial differential equation*, Math. Tables Aids Compute *4*, pp. 65-75; 94, 488, 493 (1950).

[3] YOUNG D., *Iterative methods for solving partial differential equations of elliptic type*, Doctoral thesis, Harvard Univ., Cambridge, Massachusetts (1950).

[4] YOUNG D., *Iterative methods for solving partial differential equations of elliptic type*, Trans. Amer. Math. Soc. *76*, pp. 92-111; 54, 94, 95, 163, 227, 234, 265, 365 (1954).

[5] DEVOGELAERE R., *Over-relaxation, Abstract*, Amer. Math. Soc. Notices *5*, pp. 147-273 (1950).

[6] YOUNG D., *Iterative solution of large linear system of equations*, Academic Press, New York, London (1971).

[7] VARGA R., *Matrix iterative analysis*, Prentice Hall, Englewood Cliffs (1962).

[8] SISLER M., *Uberein iterationsverfahren fur zyklische matrizen*, Apl. Mat. *17*, pp. 225-233 (1972).

[9] SISLER M., *Uber die knvergenenz eines gewissen iterationsverfahren fur zyklische matrizen,*, Apl. Mat. *18*, pp. 89-98 (1973).

[10] SISLER M., *Uber die optimierung eines zweiparametrigen Iterationsverfahren*, Apl. Mat. *20*, pp. 126-142 (1975).

[12] GOLUB G. and DEPILLIS J., *Toward an effective two-parameter SOR method*, Academic Press, New York, London (1988).

[13] HADJIDIMOS A., *Accelerated overrelaxation method*, Math Com. *32*, pp. 149-157 (1978).